

# Lecture 9

## ELE 301: Signals and Systems

Prof. Paul Cuff

Princeton University

Fall 2011-12

## Discrete-time Fourier Transform

- Represent a Discrete-time signal using  $\delta$  functions
- Properties of the Discrete-time Fourier Transform
  - ▶ Periodicity
  - ▶ Time Scaling Property
  - ▶ Multiplication Property
- Periodic Discrete Duality
- DFT
- Constant-Coefficient Difference Equations

# Fourier Transform for Discrete-time Signals

$$x[n] = \int_1 X(f) e^{i2\pi fn} df,$$
$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi fn}.$$

$X(f)$  is always periodic with period 1.

## Continuous-representation of a discrete-time signal

$$x(t) \triangleq \sum_{k=-\infty}^{\infty} x[k] \delta(t - k).$$

Notice that

$$\begin{aligned} \mathcal{F}[x(t)] &= \int_{-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x[k] \delta(t - k) \right) e^{-i2\pi ft} dt \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x[k] \delta(t - k) e^{-i2\pi ft} dt \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-i2\pi fk} dt. \end{aligned}$$

# Properties of the Discrete-time Fourier Transform

Inherits properties from continuous-time.

Easy Properties:

- Linearity
- Conjugation
- Convolution = Multiplication in frequency domain
- Parseval's Theorem (integrate over one period)
- Time shift

Properties that require care:

- Time-scaling
- Multiplication (circular convolution in frequency)

## Time-scaling

In continuous time we can scale by an arbitrary real number. In discrete-time we scale only by integers.

For an integer  $k$ , define

$$x_k[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k, \\ 0 & \text{if } n \text{ is not a multiple of } k. \end{cases}$$

$$x_k[n] \Leftrightarrow X(kf).$$

## Compressing in time

Compressing in time requires decimation.

## Discrete Difference

What is the Fourier transform of  $y[n] = x[n] - x[n - 1]$ ?

# Dual Derivative Formula

The dual to the continuous-time differentiation formula still holds.

$$nx[n] \Leftrightarrow \frac{i}{2\pi} X'(f).$$

## Accumulation

What is the Fourier transform of  $y[n] = \sum_{m=-\infty}^n x[m]$ ?

(Hint: Inverse of discrete difference)

# Parseval's Theorem

*Theorem:*

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-1/2}^{1/2} |X(f)|^2 df.$$

## Multiplication Property

Multiplication in time equates to circular convolution in frequency.

Notice that multiplying  $\delta$  functions is not well defined.

This is dual to what we saw in the Fourier series.

# Discrete Periodic Duality

periodic signal = discrete transform (though not integer  $f$ )

discrete signal = periodic transform

## Discrete Fourier Transform

Notice that a discrete and periodic signal will have a discrete and periodic transform. This is convenient for numerical computation (computers and digital systems).

The *DFT* is (almost) equivalent to the discrete-time Fourier series of the periodic extension.

For period  $N$ , let

$$x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Then

$$DFT[x] = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

# The DFT Matrix

Since the Fourier transform is linear, the DFT can be encompassed in a matrix.

$$DFT[x] = Fx.$$

Matlab uses the fast-Fourier-transform algorithm to compute the DFT (using the *fft* command).

## Constant-Coefficient Difference Equations

$$\sum_{k=0}^n a_k y[n-k] = \sum_{k=0}^M b_k x[n-k].$$

Find the Fourier Transform of the impulse response (the transfer function of the system,  $H(f)$ ) in the frequency domain.