ELE 530 — Feb. 2, 2011
Theory of Detection and Estimation — Handout #2

Homework #1
Due Feb. 11

Notice: Please bring your completed homework with you to your appointment.

1. *Geometric pairs.* Consider a probability space consisting of the sample space

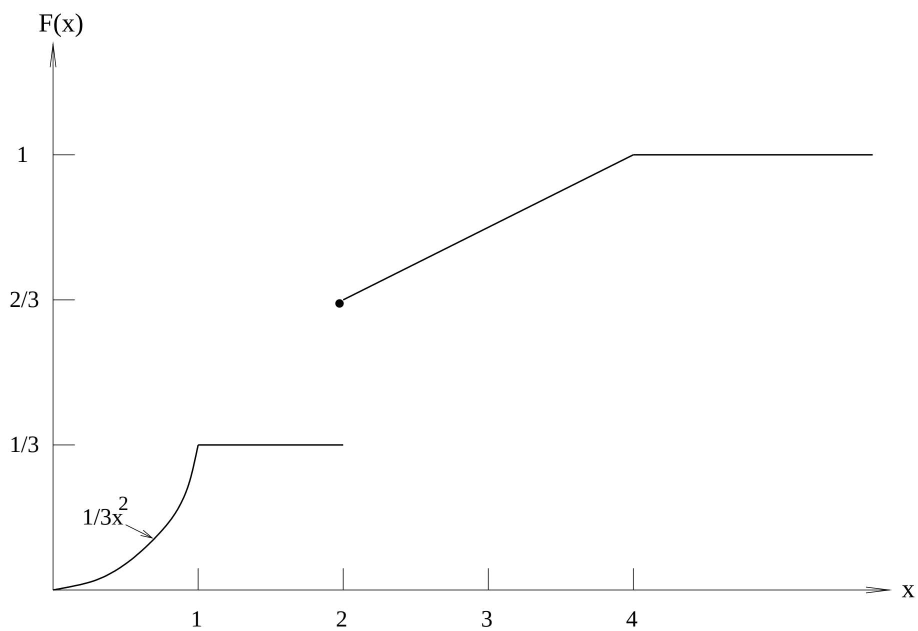
$$\Omega = \{1,2,3,\ldots\}^2 = \{(i,j) :\ i,j \in Z^+\},$$

i.e., all pairs of positive integers, where the set of events is the power set of Ω and the probability measure on points in the sample space is

$$\mathrm{P}((i,j)) = p^2(1-p)^{i+j-2}, \quad 0 < p < 1.$$

a. Find $\mathrm{P}(\{(i,j) :\ i \ge j\})$.
b. Find $\mathrm{P}(\{(i,j) :\ i + j = k\})$.
c. Find $\mathrm{P}(\{(i,j) :\ i \text{ is an odd number}\})$.

2. *Probabilities from cdf.* Let X be a random variable with the cdf shown below.



a. Find the probability of the following events.
  i. $\{X = 2\}$
  ii. $\{X < 2\}$
  iii. $\{X = 2\} \cup \{0.5 \le X \le 1.5\}$
  iv. $\{X = 2\} \cup \{0.5 \le X \le 3\}$
b. Does X have a pdf?

3. *Unions and intersections.* A number x is selected at random in the interval $[-1, 1]$. Consider the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 1\}$, and $C = \{x > 0.75\}$.
   a. Find the probabilities of B, $A \cap B$, and $A \cap C$.
   b. Find the probabilities of $A \cup B$, $A \cup C$, and $A \cup B \cup C$.

4. *Intersection of events bound.* Let A and B be events with $\mathrm{P}(A) \geq 0.9$ and $\mathrm{P}(B) \geq 0.8$. Show that $\mathrm{P}(A \cap B) \geq 0.7$.

5. *Negative evidence.* Suppose that the occurrence of an event B increases the probability that an accused person is guilty; that is, if A is the event that the defendant is guilty then $\mathrm{P}(A \mid B) \geq \mathrm{P}(A)$. The prosecutor finds that B did *not* occur. What can you say about the defendant's conditional probability of being guilty?

6. *Ternary channel.* The probability transition matrix for a ternary communication channel is shown in the following figure.

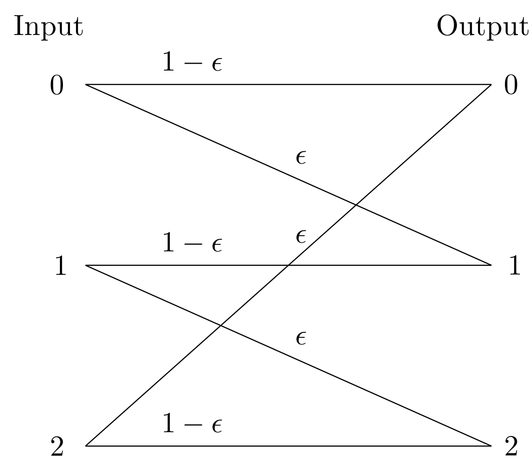


Figure 1: Ternary Communication Channel

The input symbols 0, 1, and 2 occur with probabilities $1/2$, $1/4$, and $1/4$, respectively.

   a. Find the probabilities of the output symbols.
   b. Suppose the observed output is 1. Find the probabilities that the input was 0, 1, or 2.

Your answers should be in terms of the conditional error probability ϵ.

7. *Distance to nearest star.* Let the random variable N be the number of stars in a region of space of volume V. Assume that N is a Poisson random variable with pmf

$$p_N(n) = \frac{e^{-\rho V}(\rho V)^n}{n!}, \quad n = 0, 1, 2, \ldots,$$

where ρ is the "density" of stars in space. We choose an arbitrary point in space and define the random variable X to be the distance from the chosen point to the nearest star. Find the pdf of X in terms ρ.

8. *Marginally Gaussian.* Can a collection (vector) of random variables each be individually Gaussian but not be jointly Gaussian? Provide justification or an example to make your case.

9. *Bias.* Consider an estimator for X based on Y, given by $\hat{X} = f(Y)$. There are two natural ways to define *bias*, so to avoid confusion we will call one of them *average bias.*

   a. *Unbiased.* $\mathbf{E}\,(\hat{X}|X) = X$.

   b. *Unbiased on Average.* $\mathbf{E}\,(\hat{X}) = \mathbf{E}\,X$.

   Definition 1 is used for non-Bayesian estimation, where a distribution on X is not specified.

   Is the MMSE Estimator, $\hat{X} = \mathbf{E}\,(X|Y)$, unbiased according to definition 1? Is it unbiased on average according to definition 2? Either show that it satisfies the property in general, or give justification, such as a counter-example, to show that it does not satisfy the definition in general.

Extra Problems

The following problems are provided to help you review basic probability theory. Do **not** turn in solutions to these problems.

1. Two dice, one white with black dots and the other black with white dots, are tossed. The number of dots facing up on each die is counted and noted, recording the number of dots on the black die first and the number of dots on the white die second.
   a. Find the sample space.
   b. Find the set A corresponding to the event "the total number of dots showing is even."
   c. Find the set B corresponding to the event "both dice are even."
   d. Does A imply B or does B imply A? Find $A \cap B^c$ and describe this event in words.
   e. Let C be the event "the number of dots on the two dice differ by one." Find $A \cap C$.
2. Find $\mathrm{P}(A \mid B)$ if
   a. $A \cap B = \emptyset$.
   b. $A \subseteq B$.
   c. $B \subseteq A$.
3. Let $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B^c) = 0.4$, and $\mathrm{P}(A \cup B) = 0.7$. Find
   a. $\mathrm{P}(A^c \mid B^c)$.
   b. $\mathrm{P}(B^c \mid A)$.
4. Show that events A and B are independent if and only if $\mathrm{P}(A \mid B) = \mathrm{P}(A \mid B^c)$.
5. Consider a binary communication channel with $p(1|0) = 0.1$ and $p(0|1) = 0.2$. Assume that the inputs are equiprobable.
   a. Find the probability that the output is 0.
   b. Find the probability that the input was 0 given that the output is 1.
6. *Gaussian probabilities.* Let $X \sim \mathcal{N}(1000, 400)$. Use the attached table of $Q(x)$ to find:
   a. $\mathrm{P}\{990 < X < 1020\}$
   b. $\mathrm{P}\{X < 1020 \mid X > 990\}$

Table of $Q(x)$

The $Q(\cdot)$ function is the area beneath the righthand tail of the Gaussian pdf $\mathcal{N}(0,1)$:

$$Q(x) \;=\; \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx\,.$$

Two obvious and useful properties of $Q(\cdot)$:

$$Q(x) \;=\; 1 - \Phi(x)\,, \qquad Q(-x) \;=\; 1 - Q(x)$$

The following table lists values of $Q(x)$ for $0 \le x \le 4$.

x	$Q(x)$	x	$Q(x)$
0.0	0.50000	2.0	2.2750×10^{-2}
0.1	0.46017	2.1	1.7864×10^{-2}
0.2	0.42074	2.2	1.3903×10^{-2}
0.3	0.38209	2.3	1.0724×10^{-2}
0.4	0.34458	2.4	8.1975×10^{-3}
0.5	0.30854	2.5	6.2097×10^{-3}
0.6	0.27425	2.6	4.6612×10^{-3}
0.7	0.24196	2.7	3.4670×10^{-3}
0.8	0.21186	2.8	2.5551×10^{-3}
0.9	0.18406	2.9	1.8658×10^{-3}
1.0	0.15866	3.0	1.3499×10^{-3}
1.1	0.13567	3.1	9.6760×10^{-4}
1.2	0.11507	3.2	6.8714×10^{-4}
1.3	0.09680	3.3	4.8342×10^{-4}
1.4	0.08076	3.4	3.3693×10^{-4}
1.5	0.06681	3.5	2.3263×10^{-4}
1.6	0.05480	3.6	1.5911×10^{-4}
1.7	0.04457	3.7	1.0780×10^{-4}
1.8	0.03593	3.8	7.2348×10^{-5}
1.9	0.02872	3.9	4.8096×10^{-5}
2.0	0.02275	4.0	3.1671×10^{-5}

For $x > 4$, the function can be approximated by

$$Q(x) \;\approx\; \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,.$$