ELE 530 February 28, 2011
Theory of Detection and Estimation Handout #6

**Homework #3**
Due March 7

**Notice:** Please bring your completed homework with you to your appointment.

1. *Power Spectral Density.* Show that the second moment of the (random) Fourier transform of a wide-sense stationary process $X(t)$ corresponds to the power spectral density. Some care is needed to address this correctly. The Fourier transform of a wide-sense stationary process will generally not converge. Consider, for example, an i.i.d. discrete process. The sum of i.i.d. random variables tends to blow up, causing the Fourier integral not to converge. We must properly normalize to retain an interesting result. Assume that $X(t)$ is integrable (over finite intervals) with probability one. Let the partial Fourier transform $X_T(f)$ of the process $X(t)$ be given by

$$X_T(f) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} X(t)\, e^{-j2\pi ft}\, dt.$$

The partial Fourier transforms will generally not converge to a limit as $T$ goes to infinity, but that is not necessary for our purpose. The distribution, however, will converge to a limit under the appropriate conditions (non-periodic, zero-mean, etc.).

Show that

$$\lim_{T\to\infty} \mathbf{E}\, |X_T(f)|^2 = S_X(f),$$

where $S_X(f)$ is the power spectral density. Feel free to assume that the power spectral density is finite (no delta functions, implying both zero-mean and non-periodic). The last step in this analysis can be a bit detailed, so rigor is not required.

2. *Minimum Absolute Error.* We have discussed the Bayesian estimators that minimize the squared-error and the probability of error. What estimator minimizes the absolute-error, $\mathbf{E}\,|X - \hat{X}|$?

For simplicity, do not worry about any observed data $Y$. Just assume that $p_X(x)$ is known, and find the choice of $\hat{X}$ (constant) that minimizes the expected absolute-error. We know how to extent this when observations are available.

Additionally, you may assume that $X$ is a continuous or discrete random variable—whatever helps you arrive at the answer.

3. *Shifted Periodic Process.* Let $\{X(t)\}$ be a random process that is periodic with period $P$ with probability one. Let $T$ be a random variable independent of the process $\{X(t)\}$ and uniformly distributed on the set $[0, P]$. Show that the shifted process $Y(t) = X(t+T)$ is stationary.

4. *Optimal Linear Filtering.*

   Let $\{W_n\}$ and $\{Z_n\}$ be two uncorrelated zero-mean, unit-variance, white noise processes, and let the processes $\{X_n\}$ and $\{Y_n\}$ be defined by

$$\begin{aligned} X_n &= \frac{1}{2}X_{n-1} + W_n + 3W_{n-1}, \\ Y_n &= X_n + 2Z_n. \end{aligned}$$

   a. Derive the power spectral densities and cross power spectral density $S_X(f)$, $S_Y(f)$, and $S_{X,Y}(f)$. Feel free to substitute $z = e^{j2\pi f}$ to simplify notation.
   b. Derive the Wiener smoothing filter for $\{X_n\}$ given $\{Y_n\}$—the optimal (MMSE) non-causal linear filter, $\hat{X}_n = \sum_{i=-\infty}^{\infty} h_i Y_{n-i}$—specified by either the frequency-domain transfer function $H(f)$ or the time-domain impulse response $h_n$. Also, find the MSE.
   c. Derive the causal Wiener filter for $\{X_n\}$ given $\{Y_n\}$—the optimal (MMSE) causal linear filter, $\hat{X}_n = \sum_{i=0}^{\infty} h_i Y_{n-i}$—specified by either the frequency-domain transfer function $H(f)$ or the time-domain impulse response $h_n$. Also, find the MSE.
   d. Derive the optimal one-step predictor, $\hat{X}_n = \sum_{i=1}^{\infty} h_i X_{n-i}$. What is the MSE?
   e. Derive the optimal predictor using only two samples from the past, $\hat{X}_n = h_1 X_{n-1} + h_2 X_{n-2}$. What is the MSE?
   f. Repeat part a. if the process $\{Y_n\}$ were redefined as follows:

$$Y_n = \frac{1}{2}Y_{n-1} + W_n.$$

5. *LMS Filter.* Implement the LMS Filter using Matlab, and use it to predict one step ahead for the following process $\{X[n] : n = 1, 2, ...\}$:

   Generate mutually independent random random variables, integer $K$ in the set $\{1, 2, ..., 1000\}$, $X[1] \sim \mathcal{N}(0, 1)$, and $W[i] \sim \mathcal{N}(0, 1)$.

   For $i = 2, ..., K$, let $X[i] = \frac{1}{\sqrt{2}}X[i-1] + \frac{1}{\sqrt{2}}W[i-1]$.

   For $i = K + 1, ..., K + 1000$, let $X[i] = -\frac{1}{\sqrt{2}}X[i-1] + \frac{1}{\sqrt{2}}W[i-1]$.

   For $i = K + 1001, ..., K + 2000$, let $X[i] = \frac{1}{\sqrt{2}}X[i-1] + \frac{1}{\sqrt{2}}W[i-1]$.

   ...and continue on in this cycle (notice that the negative sign is present for every other 1,000 iterations).

   The memory of this process is short. Implement the LMS filter using a small order (about 5 taps). Plot the performance of the filter as a function of time (plot the squared-error, with a little bit of smoothing over time so that it's easier to view).

   Now implement the best LTI predictor of the same order that is not adaptive. You will need to calculate the autocorrelation and cross-correlation functions for this. Feel free to just count statistics of the process rather than calculate this analytically if you'd like. This is the filter that the LMS filter is attempting to mimic (in theory). How well does it perform?