Lecture 18

Tuesday, November 18, 2014

12:25 PM

Strongly Typical Set:

$$T_\varepsilon^{(n)} = \{ x^n : \forall x \in \mathcal{X}, \ |t_{x^n}(x) - p(x)| \leq \varepsilon p(x) \}$$

$$\boxed{\text{Definition in text: } \quad T_\varepsilon^{(n)} = \left\{ x^n : \forall x \in \mathcal{X} \ |t_{x^n}(x) - p(x)| \leq \varepsilon \text{ and } t_{x^n}(x) = 0 \text{ if } p(x) = 0 \right\}}$$

Properties:

1.) $P(T_\varepsilon^{(n)}) \to 1$ LLN

2.) $\forall x^n \in T_\varepsilon^{(n)}: \quad (1-\varepsilon) E_p g(X) \leq E_{t_{x^n}} g(X) \leq (1+\varepsilon) E_p g(X)$

3.) $T_\varepsilon^{(n)} \in A^{(n)}_{\varepsilon H(X)}$ i.e. inherets all properties of weak typicality

Property 2 Discussion:

Consider
$$E_{t_{x^n}} g(X) = \sum_x t_{x^n}(x) g(x) = \frac{1}{n} \sum_x \sum_{i=1}^n 1(x_i = x) g(x)$$
$$= \frac{1}{n} \sum_{i=1}^n g(x_i)$$
$$= \text{Sample average.}$$

Proof of Prop. 2:
$$|E_{t_{x^n}} g(X) - E_p g(X)| = \left| \sum_x (t_{x^n}(x) - p_x(x)) g(x) \right|$$
$$\leq \sum_x |t_{x^n}(x) - p_x(x)| g(x)$$
$$\leq \sum_x \varepsilon p_x(x) g(x)$$
$$= \varepsilon E_p g(X). \qquad \square$$

Property 3: Proof:
$$\forall x^n \in T_\varepsilon^{(n)}, \quad \left| \frac{1}{n} \log \frac{1}{p(x^n)} - H(X) \right| = \left| E_{t_{x^n}} \log \frac{1}{p(x)} - E_p \log \frac{1}{p(x)} \right| \leq \varepsilon H(X)$$
$$\Rightarrow x^n \in A^{(n)}_{\varepsilon H(x)} \qquad \square$$

Strong Jointly typical Set:

$$T_\varepsilon^{(n)}(X,Y) = \{ x^n, y^n : \forall x, y, \ |t_{x^n y^n}(x,y) - p(x,y)| \leq \varepsilon p(x,y) \}$$

Notice: $x^n, y^n \in T_\varepsilon^{(n)} \Rightarrow x^n \in T_\varepsilon^{(n)},\ y^n \in T_\varepsilon^{(n)}$

Proof:
$$\underbrace{\left| \sum_y \left(t_{x^n y^n}(x,y) - p(x,y)\right)\right|}_{t_{x^n}(x) - p(x)} \le \sum_y \left|t_{x^n y^n}(x,y) - p(x,y)\right| \le \sum_y \varepsilon p(x,y) = \varepsilon p(x)$$
by assumption

- Same properties as strong typical set.
- Same properties as weak jointly typical set $\left(T_\varepsilon^{(n)}(X,Y) \in A_{\varepsilon H(X,Y)}^{(n)}\right)$

Conditionally Typical Set.

Weak: $A_\varepsilon^{(n)}(Y|x^n) = \{y^n : (x^n, y^n) \in A_\varepsilon^{(n)}(X,Y)\}$

Strong: $T_\varepsilon^{(n)}(Y|x^n) = \{y^n : (x^n, y^n) \in T_\varepsilon^{(n)}(X,Y)\}$

Text:
$A_\varepsilon^{(n)}(Y|x^n) = \{y^n : |\bar{H}(y^n|x^n) - H(Y|X)| \le \varepsilon\}$
$T_\varepsilon^{(n)}(Y|x^n) = \{y^n : |t_{y^n|x^n}(y|x) - p(y|x)| \le \varepsilon \ \forall x,y\}$

Properties:

1.) $|A_\varepsilon^{(n)}(Y|x^n)| \le 2^{n(H(Y|X) + 2\varepsilon)} \quad \forall x^n$
$|T_\varepsilon^{(n)}(Y|x^n)| \le 2^{n(H(Y|X) + 2\varepsilon H(X,Y))} \quad \forall x^n$

Only strong typicality.

2.) Fix $\varepsilon' > \varepsilon$, $x^n \in T_\varepsilon^{(n)} \Rightarrow P_{Y^n|x^n}\left(T_{\varepsilon'}^{(n)}(Y|x^n)\right) \ge 1 - \delta_n$ where $\delta_n \to 0$ as $n \to \infty$

3.) Fix $\varepsilon' > \varepsilon$, $x^n \in T_\varepsilon^{(n)} \Rightarrow |T_{\varepsilon'}^{(n)}(Y|x^n)| \ge 2^{n(H(Y|X) - 3\varepsilon' H(X,Y))}$ for n large enough.

Proofs: First notice a property of jointly typical sequences.

$$x^n, y^n \in A_\varepsilon^{(n)} \Rightarrow p(y^n|x^n) = \frac{p(x^n, y^n)}{p(x^n)} \le 2^{-n(H(X,Y) - \varepsilon)}\, 2^{n(H(X) + \varepsilon)} = 2^{-n(H(Y|X) - 2\varepsilon)}$$

Also $p(y^n|x^n) \ge 2^{-n(H(Y|X) + 2\varepsilon)}$

$x^n, y^n \in T_\varepsilon^{(n)} \Rightarrow x^n y^n \in A_{\varepsilon H(X,Y)}^{(n)}$.

Prop. 1: Same proof as before:
$$1 \ge P\left(A_\varepsilon^{(n)}(Y|x^n)\right) = \sum_{y^n \in A_\varepsilon^{(n)}(Y|x^n)} p(y^n|x^n) \ge |A_\varepsilon^{(n)}(Y|x^n)|\, 2^{-n(H(Y|X) + 2\varepsilon)}$$

$$\geq |A_\varepsilon^{(n)}(Y|x^n)|\, 2^{-n[\dots]}$$

Prop. 2: Note that $t_{x^n y^n}(x,y) = t_{x^n y^n}(y|x)\, t_{x^n}(x)$ ← Assumed close to P_X by def. of $T_\varepsilon^{(n)}$

↑ Converge (↑ uniformly) by LLN to $P_{Y|X}$

Type Class:

Let $T_t = \{x^n : t_{x^n} = t\}$.

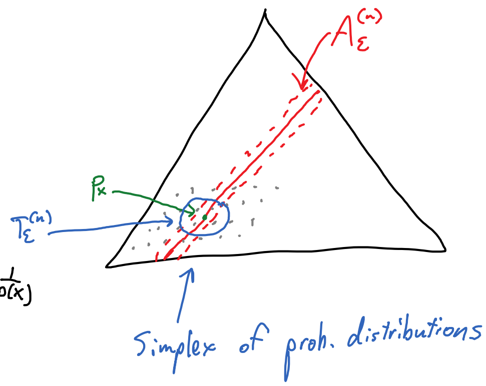
(t ↗ distribution)

Notice that $\{T_t\}$ form a partition of the x^n sequences.

There are only $\binom{n+|\mathcal{X}|-1}{|\mathcal{X}|-1} \leq (n+1)^{|\mathcal{X}|}$ type classes

Strong Typical Set is $T_\varepsilon^{(n)} = \bigcup_{t \text{ close to } P_X} T_t$

Weak Typical Set is $A_\varepsilon^{(n)} = \bigcup_{t:\ E_t \log \frac{1}{p(x)} \approx E_p \log \frac{1}{p(x)}} T_t$



Useful Properties:

1.) $\forall x^n \in T_t$, $p(x^n) = \prod_{i=1}^n p(x_i) = 2^{\sum_{i=1}^n \log p(x_i)}$

$$= 2^{-n\, E_t \log \frac{1}{p(x)}}$$

2.) $|T_t| \leq 2^{nH(t)}$

$|T_t| \approx 2^{nH(t)}$ for valid t.

$$\Rightarrow P_{p(x)}(T_t) \approx 2^{-n D(t\|p)}$$

Quantum Typicality:

Quantum Typicality:

Typical Subspace:

Given a density $\rho^X = \sum_x p(x) |x\rangle^X \langle x|^X$

$T_\varepsilon^{(n)} \subset \mathcal{X}^n$

$$\mathcal{T}_\varepsilon^{(n)} = \text{span}\left\{ |x^n\rangle^{X^n} : x^n \in T_\varepsilon^{(n)} \right\}$$

Projection onto typical subspace: $\Pi_\varepsilon^{(n)} \triangleq \Pi_{\mathcal{T}_\varepsilon^{(n)}}$

Properties:

1.) Unit Probability:

$$\text{Tr}\left(\Pi_\varepsilon^{(n)} \rho^{\otimes n}\right) \to 1$$

ρ^{X^n}

2.) Exponentially small dimension:

$$\text{Tr}\left(\Pi_\varepsilon^{(n)}\right) \leq 2^{n(H(X) + c\varepsilon)}$$

$c = 1$ if weakly typical
$c = H(X)$ if strongly typical

3.) Equipartition:

$$2^{-n(H(X)+c\varepsilon)} \Pi_\varepsilon^{(n)} \leq \Pi_\varepsilon^{(n)} \rho^{X^n} \Pi_\varepsilon^{(n)} \leq 2^{-n(H(X)-c\varepsilon)} \Pi_\varepsilon^{(n)}$$

2. b.) $\text{Tr}\left(\Pi_\varepsilon^{(n)}\right) \geq 2^{n(H(X) + c\varepsilon)}$ for n large enough

$c > 1$ weak
$c > H(X)$ stronger.

Discussion: Product source: $\rho^{\otimes n}$

w.h.p. $\rho^{\otimes n}$ is typical. ← Can be misleading.

According to gentle operator lemma: $\rho^{X^n} \approx \underbrace{\Pi_\varepsilon^{(n)} \rho^{X^n} \Pi_\varepsilon^{(n)}}_{\text{Measure that the state is typical.}}$

Joint Typicality:

Use same definition:

$$\rho^{XY} = \sum_z p(z) \, |z^{XY}\rangle\langle z^{XY}|$$

No notion of $p(x,y)$

$$T_\varepsilon^{(n)}(X,Y) = \text{span}\left\{ |z^n\rangle^{XY} : z^n \in T_\varepsilon^{(n)} \right\}$$

$\Leftarrow$ Does not imply marginally typical

How do we combine joint and marginal typicality?

- Intersection doesn't work.
- Projections: $\left(\Pi^{X^n}_{T_\varepsilon^{(n)}(X)} \otimes I^{Y^n}\right) \Pi_{T_\varepsilon^{(n)}(X,Y)} \; \rho \; \Pi_{T_\varepsilon^{(n)}(X,Y)} \left(\Pi^{X^n}_{T_\varepsilon^{(n)}(X)} \otimes I^{Y^n}\right)$

Does not commute.

Classical-Quantum:

$$\rho^{XB} = \sum_x p(x) \, |x\rangle\langle x|^X \otimes \rho^B = \sum_{x,y} p(x) \, p(y|x) \, |x\rangle\langle x| \otimes |y_x\rangle\langle y_x|^B$$

Conditional Typical Set:

$$A_\varepsilon^{B^n|x^n} = \text{span}\left\{ |y^n_{x^n}\rangle^{B^n} : y^n \in A_\varepsilon^{(n)}(Y|x^n) \right\}$$

$$T_\varepsilon^{B^n|x^n} = \text{span}\left\{ |y^n_{x^n}\rangle^{B^n} : y^n \in T_\varepsilon^{(n)}(Y|x^n) \right\}$$