Lecture 19

Monday, November 24, 2014

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Schumacher Compression:

Consider an iid quantum source $\sigma^{A^n} = \rho^{\otimes n}$.

(e.g. pure state $|\psi_{x^n}\rangle^{A^n} = |\psi_{x_1}\rangle^{A_1} \otimes |\psi_{x_2}\rangle^{A_2} \otimes \dots \otimes |\psi_{x_n}\rangle^{A_n}$

where $x_i \sim \text{iid } p(x)$

Source is $\{(p(x), |\psi_x\rangle)\}_x$ with density $\rho = \sum_x p(x) |\psi_x\rangle\langle\psi_x|$)

Encoding Task:

Encoder operates at rate R.

$\mathcal{E}^{A^n \to M}$ where $\dim(M) \leq 2^{nR}$

Decoder

$\mathcal{D}^{M \to \hat{A}^n}$

Error Tolerance:

What is the goal?

Reproducing the density is not good enough.

This can be done with $R=0$. Just prepare $\rho^{\otimes n}$ in the $\hat{A}^n$ system.

(Quantum state and source distribution are combined into one concept.)

Consider the purification of ρ as $|\varphi_\rho\rangle^{RA}$.

Then the purification of $\sigma^{A^n} = \rho^{\otimes n}$ is $|\varphi_\rho\rangle^{\otimes n}$ (Density $(\varphi_\rho^{RA})^{\otimes n} = \varphi_\rho^{R^n A^n}$)

$$\left\| \left(\mathcal{I}^R \otimes \mathcal{I}^{A^n \to \hat{A}^n}\right) \varphi_\rho^{R^n A^n} - \mathcal{D}^{M \to \hat{A}^n} \circ \mathcal{E}^{A^n \to M}\left(\varphi_\rho^{R^n A^n}\right) \right\|_1 \leq \varepsilon \quad \text{"error"}$$

This is similar to the difference between fidelity of a channel and entanglement fidelity

Achievability: Rate R is achievable if $\forall \varepsilon > 0$

$\exists\, n, \mathcal{E}, \mathcal{D}$ at rate R s.t. error is less than ε.

Theorem: $\inf\{R : \text{achievable}\} = H(A)_\rho$

Proof: **Achievability:**

Encoder measure if the source is in the typical subspace then maps isometrically to M.

Assume $R > H(A)_\rho$

Encoder: For simplicity of analysis, two steps:

1.) Measure if typical: ← strong or weak

$$\mathcal{E}_1^{A^n \to YA^n}\left(\sigma^{A^n}\right) = |0\rangle\langle 0|^Y \otimes \overbrace{\left(I - \Pi_\delta^{A^n}\right)\sigma^{A^n}\left(I - \Pi_\delta^{A^n}\right)}^{\sigma_0^{A^n}}$$
$$+ |1\rangle\langle 1|^Y \otimes \underbrace{\Pi_\delta^{A^n}\,\sigma^{A^n}\,\Pi_\delta^{A^n}}_{\sigma_1^{A^n}}$$

"No, not typical"

"Yes, typical"

Projection onto typical subspace.

2.) Map to M:

$$\mathcal{E}_2^{YA^n \to YM'}\left(\sigma^{YA^n}\right) = |0\rangle\langle 0|^Y \otimes \mathrm{Tr}\left(\sigma_0^{A^n}\right)\, |e\rangle\langle e|^{M'}$$
$$+ |1\rangle\langle 0|^Y \otimes U_f\, \sigma_1^{A^n}\, U_f^\dagger$$

where $U_f = \sum_{z^n \in T_\delta^{(n)}} |f(z^n)\rangle^{M'} \langle z^n|^{A^n}$

↑ enumeration of typical set.

Let $M = (Y, M')$

Dimension of $M \leq 2^{n(H(A)+\delta)+1} \leq 2^{nR}$ for δ small enough and n large enough.

Decoder:

1.) Reverse the isometry:

$$D_1^{YM' \to YA^n}\left(\sigma^{YM'}\right) = |0\rangle\langle 0|^Y \otimes \mathrm{Tr}\left(\sigma_0^{M'}\right) |e\rangle\langle e|^{A^n} + |1\rangle\langle 1|^Y \otimes U_f^\dagger \sigma_1^{YM'} U_f$$

2.) Discard Y:

$$D_2^{YA^n \to A^n} = \mathrm{Tr}_Y\{\cdot\}$$

Analysis:

$$\left\| \left(I^{R^n} \otimes I^{A^n \to \hat{A}^n}\right) \varphi_\rho^{R^n A^n} - D \circ \mathcal{E}\left(\varphi_\rho^{R^n A^n}\right) \right\|_1$$

$$= \left\| \mathrm{Tr}_Y \left(|1\rangle\langle 1|^Y \otimes \left(I \otimes I^{A^n \to \hat{A}^n}\right) \varphi_\rho^{R^n A^n} \right) - D \circ \mathcal{E}\left(\varphi_\rho^{R^n A^n}\right) \right\|_1$$

$$\leq \left\| |1\rangle\langle 1|^Y \otimes (I \otimes I) \varphi_\rho^{R^n A^n} - D_1 \circ \mathcal{E}\left(\varphi_\rho^{R^n A^n}\right) \right\|_1$$

$$= \left\| |1\rangle\langle 1|^Y \otimes \varphi_\rho^{R^n \hat{A}^n} - \left(|0\rangle\langle 0|^Y \otimes \mathrm{Tr}\left(\sigma_0^{R^n A^n}\right) |e\rangle\langle e|^{\hat{A}^n} + |1\rangle\langle 1| \otimes U_f^\dagger U_f \sigma_1^{R^n A^n} U_f^\dagger U_f \right) \right\|_1$$

$$\leq \left\| |1\rangle\langle 1|^Y \otimes \left(\varphi_\rho^{R^n \hat{A}^n} - U_f^\dagger U_f \sigma_1^{R^n A^n} U_f^\dagger U_f \right) \right\|_1 + \left\| |0\rangle\langle 0|^Y \otimes \mathrm{Tr}\left(\sigma_0^{R^n A^n}\right) |e\rangle\langle e|^{\hat{A}^n} \right\|_1$$

$$= \left\| |1\rangle\langle 1|^Y \otimes \left(\varphi_\rho^{R^n A^n} - \Pi_\delta^{A^n} \sigma_1^{R^n A^n} \Pi_\delta^{A^n} \right) \right\|_1 + \mathrm{Tr}\left(\sigma_0^{R^n A^n}\right)$$

$$\leq 2\sqrt{\delta} + \delta \quad \text{for } n \text{ large enough.} \quad \text{(gentle operator lemma)}$$

Converse Proof:

Suppose $\left\| \omega^{R^n \hat{A}^n} - \varphi_\rho^{R^n A^n} \right\|_1 \leq \varepsilon$

where $\omega^{R^n A^n} = D \circ \mathcal{E}\left(\varphi_\rho^{R^n A^n}\right)$

$$2nR \geq H(M)_\omega + \left| H(M|R^n) \right|$$

Let ω^{RM} be the intermediate state after encoding.

$$\geq H(M)_\omega - H(M|R^n)_\omega$$

$$= I(M;R^n)_\omega$$

$$\geq I(\hat{A}^n ; R^n) \qquad \text{Data Processing}$$

$$\geq I(A^n; R^n)_\varphi - n\varepsilon'$$ Alicki-Fannes' inequality.

$$= 2H(A^n)_\rho - n\varepsilon'$$ $\varepsilon' \to 0$ as $\varepsilon \to 0$.

$$= 2nH(A)_\rho - n\varepsilon'$$

True for $\forall \varepsilon > 0 \Rightarrow R > H(A)_\rho$

Entanglement Concentration:

Suppose Nodes A and B have two parts of a pure state

$\left(|\varphi\rangle^{AB}\right)^{\otimes n}$ product

Not a Bell state.

How many maximally entangled e-bits can be produced?

Encoders: $\mathcal{E}^{A^n \to \hat{A}}$

$\mathcal{F}^{B^n \to \hat{B}}$

Resulting State: $\omega^{\hat{A}\hat{B}} \triangleq \left(\mathcal{E}^{A^n \to \hat{A}} \otimes \mathcal{F}^{B^n \to \hat{B}}\right)\varphi^{A^n B^n}$

Error Metric: $\left\| \omega^{\hat{A}\hat{B}} - \Phi^{\hat{A}\hat{B}} \right\|_1 \leq \varepsilon$

Bell state.

Rate: $E = \frac{1}{n} \log \dim(\hat{A})$

Achievability: Rate E is achievable if $\forall \varepsilon > 0$
$\exists n, \mathcal{E}, \mathcal{F}$ at rate E s.t. error $< \varepsilon$.

Theorem: $\sup\{E : \text{achievable}\} = H(A)_\rho$

Proof: **Achievability:**

Measure the type. Map to $\hat{A}$ $(\hat{B})$ with cyclic overlap.

Consider Schmidt decomposition:

$$|\varphi\rangle^{AB} = \sum_{x\in\mathcal{X}} \sqrt{p(x)}\, |x\rangle^A |x\rangle^B$$

$$\Rightarrow |\varphi\rangle^{A^nB^n} = \sum_{x^n\in\mathcal{X}^n} \sqrt{p(x^n)}\, |x^n\rangle^{A^n} |x^n\rangle^{B^n}$$

tensor product states.

Product distribution

$$= \sum_t \sum_{x^n\in T_t} \sqrt{p(x^n)}\, |x^n\rangle|x^n\rangle$$

$$= \sum_t \sqrt{p(x^t)} \sum_{x^n\in T_t} |x^n\rangle|x^n\rangle$$

$$= \sum_t \sqrt{p(t)|T_t|}\, |\Phi_t\rangle^{A^nB^n}$$

Maximally entangled state in each type class.

Encoder construction:

1.) Measure if strongly typical:

$$\mathcal{E}_1^{A^n\to YA^n}(\sigma^{A^n}) = |0\rangle\langle 0|^Y \otimes \overbrace{(I-\Pi_\delta^{A^n})\,\sigma^{A^n}\,(I-\Pi_\delta^{A^n})}^{\sigma_0^{A^n}}$$
$$+\, |1\rangle\langle 1|^Y \otimes \underbrace{\Pi_\delta^{A^n}\,\sigma^{A^n}\,\Pi_\delta^{A^n}}_{\sigma_1^{A^n}}$$

2.) Measure the type.

$$\mathcal{E}_2^{YA^n\to YTA^n}(\sigma^{YA^n}) = |0\rangle\langle 0|^Y \otimes |e\rangle\langle e|^T \otimes \sigma_0^{A^n}$$
$$+ \sum_{t\in T_\delta^{(n)}} |1\rangle\langle 1| \otimes |t\rangle\langle t|^T \otimes \Pi_t\, \sigma_1^{A^n}\, \Pi_t$$

3.) Divide the typical set in chunks of size 2^{nR}:

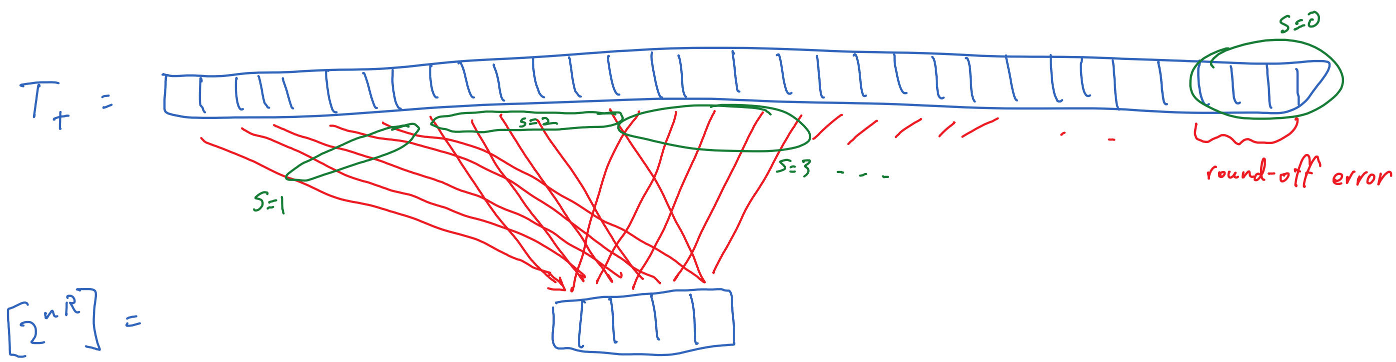
$$|T_t| \ge 2^{n H(t) - d\log(n+1)} \ge 2^{n(H(A)_\rho - \eta)}$$

for n large enough where $\eta \to 0$ as $\delta \to 0$.

If t is in the typical set.

Since $R < H(A)$ then choose δ small enough s.t. $R < H(A) - \eta$

$\Rightarrow |T_t|$ is exponentially larger than 2^{nR}



$$\mathcal{E}_3^{YTA^n \to YSTA^n}\left(\sigma^{YTA^n}\right) = |0\rangle\langle 0|^Y \otimes |0\rangle\langle 0|^S \otimes |e\rangle\langle e|^T \otimes \sigma_0^{A^n}$$
$$+ \sum_{t,s} |1\rangle\langle 1|^Y \otimes |s\rangle\langle s|^S \otimes |t\rangle\langle t|^T \otimes \Pi_{s,t}\, \sigma_1^{A^n}\, \Pi_{s,t}$$

Isometrically map the subspace projected onto by $\Pi_{s,t}$ to $\hat{A}$.

$$\mathcal{E}_4^{YSTA^n \to YST\hat{A}}$$

$\mathcal{E}_5$ drops Y, S, and T.