Lecture 8

Wednesday, October 08, 2014
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Isometric Extension of Channel:

From Kraus operators $\{N_j\}$: $\quad \mathcal{N}^{A\to B}(\rho) = \sum_j N_j \rho N_j^\dagger$

Isometric extension: $\quad U_{\mathcal{N}}^{A\to BE} = \sum_j N_j \otimes |j\rangle^E$

arbitrary orthonormal

Can verify: 1.) $U_{\mathcal{N}}^\dagger U_{\mathcal{N}} = I$ (isometry)

2.) $$\mathrm{Tr}_E\left(U_{\mathcal{N}}(\rho^A)\right) = \mathrm{Tr}_E\left(\left(\sum_j N_j \otimes |j\rangle^E\right)\rho^A\left(\sum_k N_k \otimes |k\rangle^E\right)^\dagger\right)$$

$$= \mathrm{Tr}_E\left(\sum_{jk} N_j \rho^A N_k \otimes |j\rangle\langle k|^E\right)$$

$$= \sum_{jk} N_j \rho^A N_k^\dagger \cdot \mathrm{Tr}_E\left(\overset{\text{swap}}{|j\rangle\langle k|}^E\right)$$

$$= \sum_j N_j \rho^A N_j^\dagger$$

Again, all isometric extensions equivalent (up to isometry).
In other words, the basis is the only thing arbitrary.
environment.

Unitary:

Can interpret isometry as a unitary over a larger input state.
That is, if U is isometry $(U^\dagger U = I)$
then exists square matrix $U_2 = [U \ U']$ such that U_2 is unitary.

Interpretation: $U_{\mathcal{N}}^{A\to BE}$ defines how the channel behaves when E is initially prepared to $|0\rangle^E$

Constuct unitary $V_{\mathcal{N}}^{AE\to BE}$ such that $V_{\mathcal{N}}|\psi\rangle^A|0\rangle^E = U_{\mathcal{N}}|\psi\rangle^A$

The rest of $V_{\mathcal{N}}$ (not uniquely determined) specifies what the channel does if the environment E is prepared in $|i\rangle^E$ where $i \neq 0$.

Generalized Dephasing Channels:

Diagonals of ρ are preserved (a class of channels)

We saw that $\rho \to (1-p)\rho + pZ\rho Z$ is one of them.

Isometric extension always of this form:

$$U_{N_D}^{A\to BE} |x\rangle^{A'} \longrightarrow |x\rangle^B |\varphi_x\rangle^E$$

Don't have to be orthogonal.
Not in Kraus representation.

$$N_D(\rho) = \sum_{xx'} \langle x|\rho|x'\rangle \langle \varphi_{x'}|\varphi_x\rangle |x\rangle\langle x'|^B$$

Determine the off-diagonal coefficients.

$$N_D^c(\rho) = \sum_x \langle x|\rho|x\rangle |\varphi_x\rangle\langle\varphi_x|^E$$

Entanglement breaking

Hadamard Channels:

Def.: Complementary channel is entanglement breaking.
(Dephasing channels are special cases)

Channel performs Hadamard product:

$$\rho \xrightarrow{N} \Gamma^{\dagger} * \Sigma \quad \text{(element-wise product)}$$

where Σ depends on ρ as $\Sigma_{ij} = \langle\varphi_i|\rho|\varphi_j\rangle$

Entanglement-breaking:

Example: Classical Quantum channel.

- Measures the input $\{|k\rangle\}$
- Produces state $|\sigma_k\rangle$

Resulting ensemble: $\left\{ \left(\langle k|\rho|k\rangle, \frac{|k\rangle\langle k|\rho|k\rangle\langle k|}{\langle k|\rho|k\rangle} \otimes \sigma_k \right) \right\}$

Density operator $\sum_k \langle k|\rho|k\rangle\, |k\rangle\langle k| \otimes \sigma_k$

$\boxed{\text{A channel is entanglement breaking iff Kraus operators are rank 1.}}$

Ch. 6

Notation: $\{|i\rangle\}$ is any orthonormal basis.

Noiseless qubit channel:

$|i\rangle^A \to |i\rangle^B \quad \forall i$

By linearity $\alpha|0\rangle^A + \beta|1\rangle^A \to \alpha|0\rangle^B + \beta|1\rangle^B$

Entanglement stays intact.

In terms of densities:

$\rho \to \rho$

Kraus operators: $\{I\}$

Isometric extension: $\sum_{i=0}^{1} |i\rangle^B\langle i|^A \otimes |0\rangle^E$

$[q \to q]$

Noiseless Classical Channel:

$|i\rangle\langle i|^A \to |i\rangle\langle i|^B$

$|i\rangle\langle j|^A \to 0$

Only these rank 1 matrices are preserved, and probabilistic combinations.

i.e. $\rho \to \sum_{i=0}^{1} |i\rangle^B\langle i|^A \rho |i\rangle^A\langle i|^B$ Kraus operators $\{|i\rangle^B\langle i|^A\}$

rank 1

$\Rightarrow$ Entanglement-breaking

If $\{|i\rangle^A\}$ and $\{|i\rangle^B\}$ are the computational basis,

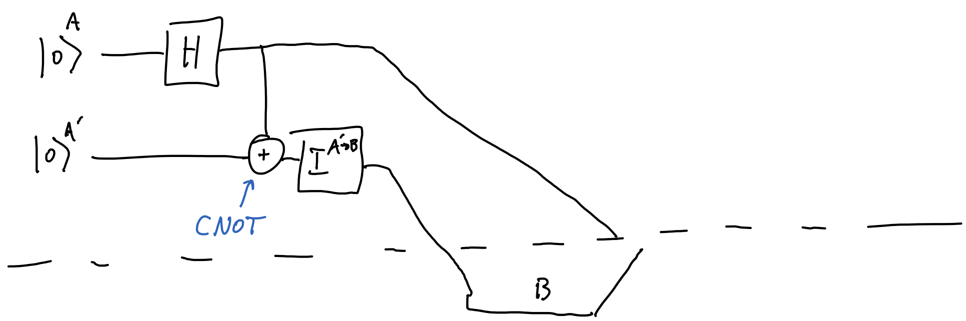
$\rho \to D$ where D is diagonal and $D_{ii} = \rho_{ii}$

$[c \to c]$

Elementary Coding: $[q \to q] \geq [c \to c]$

- Intuitively:
  - Encode classical info. into orthogonal states of ρ $\{|i\rangle^A\}$
  - Classical info. $X \sim p(x)$. Prepare $|X\rangle^A$
  - Send using quantum channel $|X\rangle^B$
  - Measure using $\{|i\rangle^B\}$
- Formally: Use quantum channel to synthesis classical channel.
  - Bob measures channel output with $\{|i\rangle\langle i|^B\}$ measurement.

Entanglement distribution: $[q \to q] \geq [qq]$



- Alice:
  - Prepare $|0\rangle^A |0\rangle^{A'}$
  - Hadamard on A system: $\left(\frac{|0\rangle^A + |1\rangle^A}{\sqrt{2}}\right)|0\rangle^{A'}$
  - CNOT results in Bell state: $|\Phi^+\rangle^{AA'} = \frac{|00\rangle^{AA'} + |11\rangle^{AA'}}{\sqrt{2}}$
  - Send A' to Bob: $|\Phi^+\rangle^{AB}$

The classical analogy is $[c \to c] \geq [cc]$

Super-dense coding: $[q \to q] + [qq] \geq 2[c \to c]$

Protocol: · Alice has a two bit message to sent: $M \in \{0,1,2,3\}$

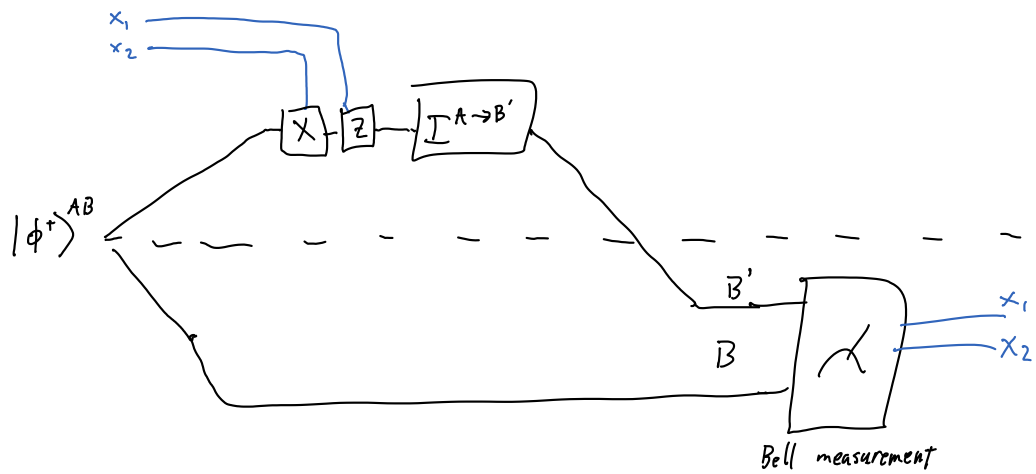
Alice and Bob share an ebit $|\phi^+\rangle^{AB}$

Alice: Perform I, X, Z, or XZ on her side of the ebit depending on if message is 0, 1, 2, or 3 respectively.

Resulting state: $|\phi^+\rangle^{\overset{B'}{\cancel{A}}B}$, $|\phi^-\rangle^{\overset{B'}{\cancel{A}}B}$, $|\Psi^+\rangle^{\overset{B'}{\cancel{A}}B}$, or $|\Psi^-\rangle^{\overset{B'}{\cancel{A}}B}$ (see Ch. 3.5.6)

· Send A to Bob (B')

Bob: Measure in basis defined by Bell states.

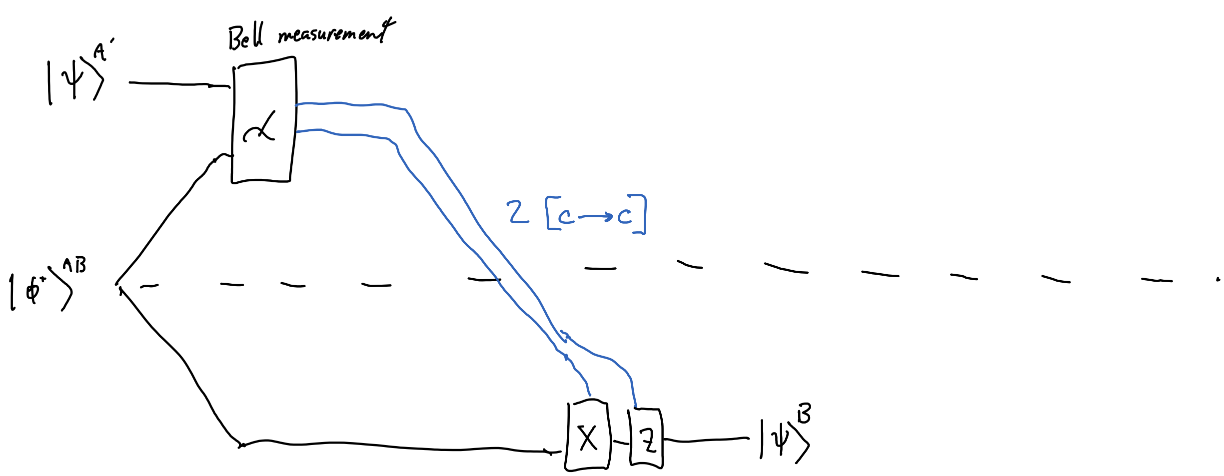


Quantum Teleportation:

$$2[c\to c] + [qq] \geq [q\to q]$$

· Dual to super-dense coding. Switch encoder and decoder.

· Both consume an ebit (the process is not reversible)



Protocol: Original State $|\psi\rangle^{A'} |\Phi^+\rangle^{AB} \overset{(1)}{=} \frac{1}{2}\left(|\Phi^+\rangle^{A'A} |\psi\rangle^{B} + |\Phi^-\rangle^{A'A} Z|\psi\rangle^{B} + |\Psi^+\rangle^{A'A} X|\psi\rangle^{B} + |\Psi^-\rangle^{A'A} XZ|\psi\rangle^{B} \right)$

Alice measure A' and A in Bell basis:

Result: One of these 4 with equal prob.

Notice: · Bob's state is no longer entangled

· The measurement is not informative about $|\psi\rangle^{A'}$

Send measurement to Bob.

Bob: Undoes appropriate Pauli operator.

Informative Theory:

Noisy resource inequalities:

Entenglement Generation (noisy entanglement dist.)

$$\langle N \rangle \geq E\,[qq]$$

Quantum Capacity $\langle N \rangle \geq Q\,[q \to q]$

"Noisy Super-dense coding" (Part 1)

$$\langle \rho^{AB} \rangle + Q\,[q \to q] \geq C\,[c \to c]$$

Entanglement-assisted classical capacity: (Part 1)

$$\langle N \rangle + E[qq] \geq C\,[c \to c]$$

"Noisy Teleportation": $\langle \rho^{AB} \rangle + C\,[c \to c] \geq Q\,[q \to q]$