Lecture 9
Tuesday, October 14, 2014
10:23 AM

Follow-up with Ch. 5-6 first

Hadamard Channel:

Complementary Ch. is entanglement-breaking.

$\Rightarrow$ Isometric extension: $\sum_i c_i \underbrace{|\varphi_i\rangle^E \langle\psi_i|^A}_{\text{arbitrary rank 1 Kraus operators}} \otimes |i\rangle^B = U^{A\to BE}$

$$\Rightarrow \mathrm{Tr}_E\left(U^{A\to BE}\rho^A U^\dagger\right) = \mathrm{Tr}_E\left(\sum_{ij} c_i c_j^* \langle\psi_i|\rho|\psi_j\rangle\, |\varphi_i\rangle\langle\varphi_j|^E \otimes |i\rangle\langle j|^B\right)$$

$$= \sum_{ij} \underbrace{c_i c_j^* \langle\psi_i|\rho|\psi_j\rangle}_{\Sigma} \underbrace{\langle\varphi_j|\varphi_i\rangle}_{\Gamma^\dagger} |i\rangle\langle j|^B$$

$$\rho \to C^\dagger\left(\Sigma * \Gamma^\dagger\right)C$$

$\Sigma = B^\dagger_\rho B$, $\quad CC^\dagger = I$

$\Gamma = A^\dagger A$, $\quad BB^\dagger = I$

$\Gamma_{ii} = 1$

Constraints

$A = [\,|\varphi_1\rangle\ |\varphi_2\rangle \dots]$, $\quad B = [\,c_1|\psi_1\rangle\ c_2|\psi_2\rangle \dots]$

Generalized Dephasing Channel:

$C = B^{-1} \Rightarrow$ B and C are square
$\Updownarrow$
B and C are unitary

Degradable:

Bob can synthesize the channel to the environment.

That is, $\exists\, \mathcal{N}^{B\to E}$ such that $\mathcal{N}^{B\to E}\circ\mathcal{N}^{A\to B} = \mathcal{N}^{A\to E}$

For $\mathcal{N}^{B\to E}$ use Kraus operators $\{|\varphi_i\rangle^E\langle i|^B\}$.

That is the classical-quantum channel:
1.) Measures B with $\{|i\rangle\}$
2.) Produces $|\varphi_i\rangle$ accordingly.

This channel is entanglement-breaking (Kraus operators are rank 1)

Proof that entanglement-breaking $\Leftrightarrow$ Kraus operators rank 1.

$\Rightarrow$ First consider $\mathcal{N}^{A\to B'}$ operating on $|\phi\rangle^{AB}$ $\leftarrow$ Maximally entangled.

$$\left(\underline{I}^{B} \otimes \mathcal{N}_{EB}^{A\to B'}\right)\left(\Phi^{AB}\right) = \underbrace{\sum_z p_z(z)\, |\varphi_z\rangle\langle\varphi_z|^{B} \otimes |\psi_z\rangle\langle\psi_z|^{B'}}$$

Generic seperable state.

Use the following Kraus operators:

$$\{A_z\} \quad \text{where} \quad A_z = \sqrt{d\, p_z(z)}\, |\psi_z\rangle\langle\varphi_z^*|$$

1.) Check that these are valid Kraus operators.

$$\sum_z A_z^\dagger A_z = \sum_z d\, p_z(z)\, |\varphi_z^*\rangle\langle\varphi_z^*|$$

Notice that $\mathrm{Tr}_{B'}\left(\mathcal{N}^{A\to B'}(\Phi)\right)$ must be $\mathrm{Tr}_A\left(\Phi^{BA}\right) = \pi = \frac{1}{d}\underline{I}$

And
$$= \mathrm{Tr}_{B'}\left(\sum_z p_z(z)\, |\varphi_z\rangle\langle\varphi_z|^{B} \otimes |\psi_z\rangle\langle\psi_z|^{B'}\right)$$

$$= \sum_z p_z(z)\, |\varphi_z\rangle\langle\varphi_z|^{B}$$

$$\sum_z A_z^\dagger A_z = d\left(\sum_z p_z(z)\, |\varphi_z\rangle\langle\varphi_z|^{B}\right)^*$$

$$= d\left(\frac{1}{d}\underline{I}\right)^* = \underline{I}$$

2.) Check what these Kraus operators do to Φ^{BA}

$$\rho_\Phi = \frac{1}{d}\sum_{ij} |i\rangle\langle j|^{B} \otimes |i\rangle\langle j|^{A}$$

Apply Kraus:
$$\cancel{\frac{1}{d}}\sum_{ij} |i\rangle\langle j|^{B} \otimes \sum_z \underbrace{\cancel{d}\, p_z(z)}_{\text{sqrt}}\, \underbrace{|\psi_z\rangle \langle\varphi_z^*|i\rangle}\langle j|\varphi_z^*\rangle\langle\psi_z|^{B'}$$

scalar

$$= \sum_{ijz} p_z(z)\, \underbrace{|i\rangle\langle j|\varphi_z^*\rangle\langle\varphi_z^*|i\rangle\langle j|^{B}}_{\text{transpose}} \otimes |\psi_z\rangle\langle\psi_z|^{B'}$$

$$= \sum_z p_z(z) \left(|\varphi_z^*\rangle\langle\varphi_z^*| \right)^{T\, B} \otimes |\psi_z\rangle\langle\psi_z|^{B'}$$

$\Leftarrow$ Take arbitrary $\{ c_z |\varphi_z\rangle^{B'}\langle\psi_z|^A \}_z$ Kraus operators.

Apply Channel to A: $I^B \otimes N^{A\to B'}$

$$\sum_z |c_z|^2 \left(I^B \otimes |\varphi_z\rangle^{B'}\langle\psi_z|^A \right) \rho^{BA} \left(I^B \otimes |\psi_z\rangle^A\langle\varphi_z|^{B'} \right)$$

$$= \sum_z |c_z|^2 \left(I \otimes \langle\psi_z|^A \right) \rho^{BA} \left(I \otimes |\psi_z\rangle^A \right) \otimes |\varphi_z\rangle\langle\varphi_z|^{B'}$$

Seperable.

Ch.5.

Coherent Measurement

Perform arbitrary measurement $J \sim \{M_j^S\}$ using von Neumann measurements (simple to interpret)

Consider the isometry $\sum_j M_j^S \otimes |j\rangle^{S'} = U^{S\to SS'}$

Measure S' in the basis $\{|j\rangle^{S'}\}$ (von Neumann)

If $\{M_j\}$ were Kraus operators for a channel, then $U^{S\to SS'}$ is isometric extension.

where S' is environment.

Isometry may result in entanglement between S and S'.

$$U\rho U^\dagger = \sum_{jj'} M_j \rho M_{j'}^\dagger \otimes |j\rangle\langle j'|^{S'}$$

Upon measuring S', the entanglement breaks (Kraus $\{ |j\rangle\langle j|^{S'} \}$)

Measurement channel

Result: $\sum_j M_j \rho M_j^\dagger \otimes |j\rangle\langle j|^{S'}$ Classic-Quantum state.

S' holds the measurement outcome.

- Consider the isometric extension of the measurement channel: $V = \sum_j |j\rangle^{S'} |j\rangle^E \langle j|^S$

$$\Rightarrow \left(I^{S} \otimes V^{S' \to S'E}\right) U^{S \to SS'} \rho\, U^{\dagger} (I \otimes V)^{\dagger} = \sum_{jj'} M_j \rho M_{j'}^{\dagger} \otimes |j\rangle\langle j'|^{S'} \otimes |j\rangle\langle j'|^{E}$$

Quantum Instrument:

Generization of above ("most general")

Results in Classical-quantum state.

$$\rho \longrightarrow \sum_j \mathcal{E}_j^{A\to B}(\rho) \otimes |j\rangle\langle j|^J$$

CPTR (like a noisy channel)

Measurement outcome

Can be represented with $\{M_{jk}\}$ where $\mathcal{E}_j^{A\to B}(\rho) = \sum_k M_{jk} \rho M_{jk}^{\dagger}$

$$\sum_{jk} M_{jk}^{\dagger} M_{jk} = I \qquad \left(\sum_k M_{jk}^{\dagger} M_{jk} \le I \right)$$

Interpret as two measurements:

Isometry: $U^{A \to BJE_JE_K} = M_{jk} \otimes |j\rangle^J \otimes |j\rangle^{E_J} \otimes |k\rangle^{E_K}$

Ch. 6

Comments on Optimality:

$[q\to q] \ge [qq]$ (Entang.-dist.)

Suppose $[q\to q] \ge C[qq]$ for $C>1$.

If we use classical comm. and teleportation:

$$C[qq] + 2C[c\to c] \ge C[q\to q]$$

$$\Rightarrow [q\to q] + 2C[c\to c] \ge C[q\to q]$$

Repeat: $[q\to q] + \infty[c\to c] \ge \infty[q\to q]$

Must argue that classical comm. cannot produce quantum comm.

/ comm.

Super-dense coding or Teleportation:

Suppose that either $[q \to q] + [qq] \geq 2C\,[c \to c]$ or $2\,[c \to c] + [qq] \geq C\,[q \to q]$
for $C > 1$.

Use entanglement and cycle between the two:

$$[q \to q] + \infty[qq] \overset{\text{1 round}}{\geq} C\,[q \to q] + \infty[qq] \;\; \ldots \;\; \geq \infty[q \to q]$$

Must argue that you cannot use entanglement to produce comm.

Protocols generalize $(d > 2)$:

We label the qudit resources: 1 qudit channel $= \log_2 d\,[q \to q]$
1 edit $= \log_2 d\,[qq]$
$\vdots$

Example of resource inequality:

Qudit super-dense coding: $\log d\,[q \to q] + \log d\,[qq] \geq 2\log d\,[c \to c]$

Same inequalities (divide by $\log d$)

Ch. 7

Equivalence with abundance of resource:

Notice the duality: If $[qq]$ is abundant:

$$2\,[c \to c] + \cancel{[qq]} \geq [q \to q]$$

$$[q \to q] + \cancel{[qq]} \geq 2\,[c \to c]$$

$$[q \to q] = 2\,[c \to c]$$

$\Rightarrow$ Relationship between entanglement-assisted classical capacity
and " " quantum capacity.

If communication is abundant $([c \to c])$:

$$[q \to q] \geq [qq]$$

$$[qq] + \cancel{2[c \to c]} \geq [q \to q]$$

$$[qq] = [q \to q]$$

$\Rightarrow$ Relationship between entanglement generation capacity and quantum capacity if classical comm. is free.

Next Time: Coherent Bit Channel: $[q \to qq]$

Isometry: $|i\rangle^A \longrightarrow |i\rangle^A |i\rangle^B$