State Information in Bayesian Games

Paul Cuff

Princeton University

Oct. 1, 2009

Zero-Sum Game

Payoff Matrix Π for Player A:

B

A

$p_A(a)$

	0	1
0	3	0
1	0	1

Zero-Sum Game

Payoff Matrix Π for Player A:

B

A

$p_A(a)$

$$\begin{array}{c|c|c|} & 0 & 1 \\ \hline 0 & 3 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}$$

$$\text{Value of game} = \max_{p_A} \min_{p_B} \mathbb{E}\, \Pi(A,B) = 3/4.$$

$$p_A^*(a) = [1/4, 3/4]. \qquad \square$$

General Zero-Sum Game

Payoff Matrix Π for Player A:

B

A

$p_A(a)$

$$\begin{array}{c|c|c|} & 0 & 1 \\ \hline 0 & \Pi(0,0) & \Pi(0,1) \\ \hline 1 & \Pi(1,0) & \Pi(1,1) \\ \hline \end{array}$$

General Zero-Sum Game

Payoff Matrix Π for Player A:

B

A

$p_A(a)$

	0	1
0	$\Pi(0,0)$	$\Pi(0,1)$
1	$\Pi(1,0)$	$\Pi(1,1)$

Allow payoff Π to be random, determined by a state S.

(In the literature, S is called the “type”)

Erasure Game (two states)

S is equally likely to be 0 or 1:

$S = 0$

B

A

$p_{A|S}(a|0)$

	0	1
0	3	0
e	0	1
1	$-\infty$	$-\infty$

$S = 1$

B

A

$p_{A|S}(a|1)$

	0	1
0	$-\infty$	$-\infty$
e	1	0
1	0	3

Bayesian Games

In a Bayesian game, the players each may or may not have some information about the stochastic state.

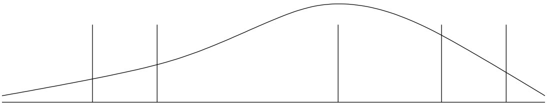
Recent related references:

- Gossner and Mertens (2001). *The value of information in zero-sum games.*
- Lehrer and Rosenberg (2004). *What restrictions do Bayesian games impose on the value of information?*
- Provan (2008). *The use of spies in strategic situations.*

Information Structure

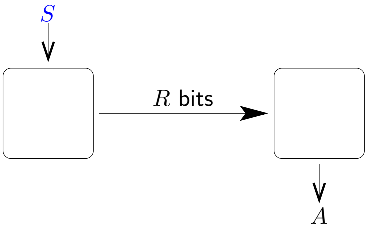
Information structure: the partition by which the state is quantized before being observed by a player of the game.

Distribution of S.



Quantization into bins.

Communication of State Information



Questions that arise:

- What is the best “information structure?” (scalar quantization)
- How about vector quantization?

Communication Tools

Tools and references from information theory:

- Han and Verdú (1993). *Approximation of output statistics.*
- Cuff (2008). *Communication requirements for generating correlated random variables.*
- Cuff, Permuter, and Cover (2009). *Coordination Capacity.*

Erasure Game (two states)

S is equally likely to be 0 or 1:

$S = 0$

		B	
		0	1
A	0	3	0
	e	0	1
	1	$-\infty$	$-\infty$

$S = 1$

		B	
		0	1
A	0	$-\infty$	$-\infty$
	e	1	0
	1	0	3

Neither know the state: Value $= 1/2$.
A knows the state: Value $= 3/2$.
B knows the state: Value $= 0$.
Both know the state: Value $= 3/4$.

Erasure Game (two states)

S is equally likely to be 0 or 1:

$S = 0$

		B	
		0	1
A	0	3	0
	e	0	1
	1	$-\infty$	$-\infty$

$S = 1$

		B	
		0	1
A	0	$-\infty$	$-\infty$
	e	1	0
	1	0	3

Neither know the state: Value $= 1/2$.
A knows the state: Value $= 3/2$.
B knows the state: Value $= 0$.
Both know the state: Value $= 3/4$.

Reminder: Concept from Rate-distortion Theory

To generate correlated actions $\sim p(a|s)$,

$R \geq I(S;A)$ is required.

Reminder: Concept from Rate-distortion Theory

To generate correlated actions $\sim p(a|s)$,

$R \geq I(S;A)$ is required.

This does not produce independent actions in the sequence.

Reminder: Concept from Rate-distortion Theory

To generate correlated actions $\sim p(a|s)$,

$R \geq I(S;A)$ is required.

This does not produce independent actions in the sequence.
Decode the action sequence after observing $k = n\frac{R}{H(A)}$ actions.

Reminder: Concept from Rate-distortion Theory

To generate correlated actions $\sim p(a|s)$,

$R \geq I(S;A)$ is required.

This does not produce independent actions in the sequence.
Decode the action sequence after observing $k = n\frac{R}{H(A)}$ actions.

What is the price of independence?

Erasure Challenge

Person A

0 1 0 0 1 1 1 1

Erasure Challenge

Person A	Person B
0 1 0 0 1 1 1 1	0 e e e e 1 e e

Erasure Challenge

Person A	Person B
0 1 0 0 1 1 1 1	0 e e e e 1 e e

How much must Person A tell Person B?

Erasure Challenge

Person A	Person B
0 1 0 0 1 1 1 1	0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits

Erasure Challenge

Person A	Person B
0 1 0 0 1 1 1 1	0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits

Erasure Challenge

Person A	Person B
0 1 0 0 1 1 1 1	0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits

Erasure Challenge

Person A

0 1	0 0	1 1	1 1

Person B

0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization

Erasure Challenge

Person A

0 1	0 0	1 1	1 1

Person B

0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization

Erasure Challenge

Person A

0 1	0 0	1 1	1 1

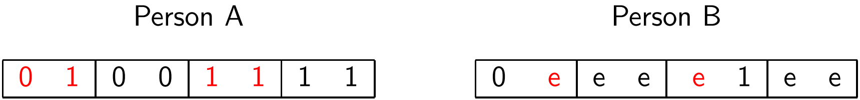
Person B

0 1	e e	1 1	e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization

Erasure Challenge



Person A

0	1	0	0	1	1	1	1

Person B

0	e	e	e	e	1	e	e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization

Erasure Challenge

Person A

0 1	0 0	1 1	1 1

Person B

0 e	e e	e 1	e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization
  $\log_2 \binom{4}{2} + 4$ bits

Erasure Challenge

Person A

0 1	0 0	1 1	1 1

Person B

0 e	e e	e 1	e e

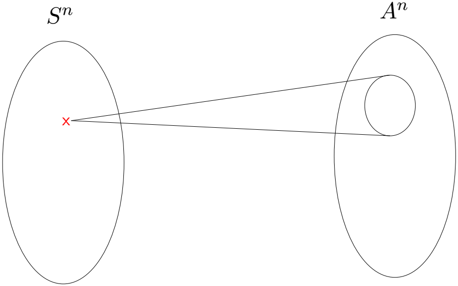
How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization
  $\log_2 \binom{4}{2} + 4$ bits $= \log_2 96 = 6.58$ bits

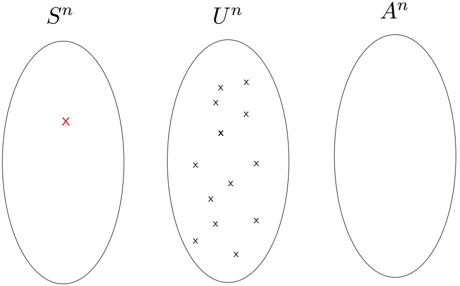
Generating Correlated Sequences

Generating Correlated Sequences

Generating Correlated Sequences

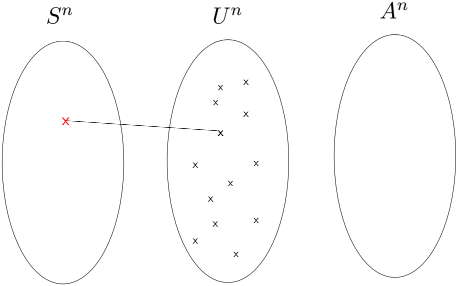


Generating Correlated Sequences



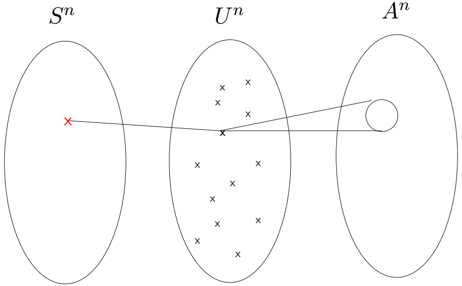
- Generate a codebook with extra u^n sequences $\sim \prod_{i=1}^{n} p(u_i)$.

Generating Correlated Sequences



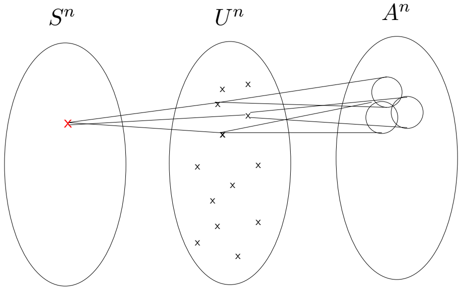
- Generate a codebook with extra u^n sequences $\sim \prod_{i=1}^{n} p(u_i)$.
- Encoder randomly chooses among u^n sequence that are correlated with x^n and sends the index i.

Generating Correlated Sequences



- Generate a codebook with extra u^n sequences $\sim \prod_{i=1}^{n} p(u_i)$.
- Encoder randomly chooses among u^n sequence that are correlated with x^n and sends the index i.
- Decoder generates y^n randomly conditioned on $u^n(i)$.

Generating Correlated Sequences



$$R \geq I(X;U) + I(U;Y|X).$$

Resolvability: [Wyner 75] [Han, Verdú 93]

Degenerate Game (counter-example)

$S = 0$

B

		0
A	0	0
	1	-1

$S = 1$

B

		0
A	0	-1
	1	0

The expected payoff is simply the negative Hamming distortion.

No need for randomizing.

Bayesian State Communication

Simple idea first:

Choose U such that $S - U - A$ form a Markov chain and $R > I(S;U)$.

B doesn't know S: Payoff $\geq \frac{R}{I(S,A;U)} \underline{\Pi}_{p_{A|S}} + \frac{I(S,A;U) - R}{I(S,A;U)} \underline{\Pi}^{(U)}_{p_{A|U}}$.

B knows S: Payoff $\geq \frac{R - I(S;U)}{I(A;U|S)} \underline{\Pi}^{(S)}_{p_{A|S}} + \frac{I(S,A;U) - R}{I(A;U|S)} \underline{\Pi}^{(S,U)}_{p_{A|U}}$

Bayesian State Communication

More complexity:

Choose U_1 and U_2 such that $S - (U_1, U_2) - A$ form a Markov chain and $R > I(S; U_1, U_2)$.

Generate a U_1 codebook and a U_2 codebook for each U_1 sequence.

The opponent learns U_1 early and U_2 late.

Bottomline

Summary:

- Rate-distortion type coding is not suited for games.
- Generating i.i.d. sequences plays a partial role.
- Causality of decisions creates a time-varying result even with i.i.d. codebooks.