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Condorcet Methods are Less Susceptible to Strategic Voting

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Abstract Democratically choosing a single preference from three or more candidate options is not a straightforward matter. There are many competing ideas on how to aggregate rankings of candidates. However, the Gibbard-Satterthwaite theorem pessimistically concludes that no fair voting system is immune to strategic voting. In our work we analyze the likelihood of strategic voting of several popular voting systems, including Borda count, plurality and Kemeny-Young, under various vote distributions. When there are three candidates, we show that the Kemeny-Young method, and Condorcet methods in general, are categorically more resistant to strategic voting than many other common voting systems. We verify our results on voting data that we collected through an online survey on the 2012 US President Election.

Keywords Strategic Voting · Condorcet method · Kemeny Young · Borda count · Plurality

1 Introduction

The Gibbard-Satterthwaite Theorem [Gibbard(1973)], [Satterthwaite(1975)] asserts that a single-winner deterministic voting system has to satisfy one of the three undesirable properties:

1. The voting system is dictatorial.
2. There is some candidate who can never win.
3. The voting system is susceptible to strategic voting (manipulable).

In essence, any fair voting system is manipulable (susceptible to strategic voting). In other words, there always exists a voting profile where a voter would vote insincerely in order to help elect a preferred candidate. However, the Gibbard-Satterthwaite theorem focuses on the worst case rather than the “usual case”, i.e. the kind of voting profiles that are likely to occur for real world elections. Also, it does not quantify which voting systems are less manipulable than others. In order to answer questions such as “how easily is this property violated”, or “how fair is this voting system”, it is necessary to quantify degrees to which a property does or does not hold.

We investigate voting systems from a geometric perspective: Each voting profile (the collection of all votes from every voter) is represented by a point in a high-dimensional space, and a voting system is represented as a partition of the space into different regions, with each region corresponding to a winner. Strategic voting, then, becomes a property of the boundaries separating these regions. More precisely, any case of strategic voting is a case where a single vote change pushes the honest profile across some boundary.

This method is particularly helpful when analyzing anonymous voting systems: The data can be compressed into a much lower dimension – namely, from the number of voters to the number of rankings. We pursue the idea of visualizing a voting system in order to see how a voting system partitions the space of profiles. This provides insight on the cause of strategic voting, and the comparison of different voting systems. We choose to focus on the case with three candidates, both because it is the smallest interesting case (there is no strategic

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voting for a two-party election), and because visualization is easier with lower dimensions. For certain voting systems, we are able to further reduce the dimension of the space to three or fewer, making it possible to plot voting systems. Fortunately, many popular voting systems, including plurality, Borda count and Kemeny-Young belong to this category.

We exhibit a property for the Kemeny-Young method (actually, for all Condorcet methods) that distinguishes them from many other voting systems, such as plurality, Borda count and Instant Run-off voting: Some of the boundaries are completely resistant to strategic voting! For voting profiles near these boundaries, there is no strategic incentive at all. We will use the term "non-manipulable" to describe such boundaries. The Gibbard-Satterthwaite theorem only asserts that manipulable boundary exists in each voting system that does not satisfy property 1 and 2. While most voting systems seem to be entirely constructed of manipulable boundaries, Condorcet methods have many boundaries that are non-manipulable. Furthermore, these seem to be the boundaries that play the most significant role in real-world use of the voting system, as will be discussed in Section 5.

However, a boundary being "manipulable" does not mean that every voter can manipulate profiles on the boundary. Rather, a profile is manipulable as long as at least one voter can manipulate. Thus, the traditional way of measuring manipulability by counting the number of manipulable profiles, despite its simplicity, is not an ideal quantification of the degree of manipulability.

Moreover, not all voting profiles are equally significant. Some are more likely to occur in real life than others. This is addressed by introducing a probabilistic model. The motivation for using probability here is also to deviate from the traditional yet unrealistic assumption that all profiles are of equal concern to a voter; as in real elections people usually have some information on the expected vote distribution. A widely used model [Saari(1995)], [Regenwetter et al(2006)Regenwetter, Grofman, Marley, and Tsetlin] assumes that, each voter randomly draws a vote from a probability distribution p over $M!$ rankings. Thus, with M candidates, under anonymity, the preference profile is a random vector V with a multinomial distribution in $M!$ dimensions. For this reason we refer to this model as the *multinomial model*, and refer to p as the *social bias*. The *impartial culture* assumption, where each voter is completely unbiased and picks a vote with uniform distribution, is just a special case of the multinomial model (uniform social bias). But it's not a particularly inspiring one. We find it more interesting to allow for the model to represent some sort of coherence throughout the population, and this is made possible through the selection of the bias in the multinomial model. Whether or not the multinomial model is realistic, it allows us to uncover interesting properties that are missed by the uniform random votes model and that we believe would hold true under many realistic probability models. To enrich our discussion, we introduce another closely related probability model, called the sampling model. For this model, each voter holds a fixed vote (contrary to the random decision in the multinomial model), therefore the vote profile is also a fixed point in the simplex. However, the vote profile is unknown, and can only be estimated from a random sample of votes, which resembles the polls and surveys before an election. It should be emphasized that the dichotomy of manipulable and non-manipulable boundaries is independent of the probabilistic model.

Some earlier work measures the degree of manipulability by counting the number of manipulable voting profiles [Kelly(1993)]. It is shown that random voting rules (that is, winners are randomly assigned to each vote profile) have a high degree of manipulability. This approach provides a basis for detailed calculation and theory development. [Friedgut et al(2008)Friedgut, Kalai, and Nisan] define a distance metric between two voting systems as the probability that two voting systems with the same input elect different winners, with the assumption that each vote is uniformly distributed over all rankings. This gives us a tool to characterize "how dictatorial" a voting system is, by calculating the distance from the voting system to the nearest dictatorial voting system. It is shown that for any voting system with three candidates, the sum of each voter's probability of strategic voting is lower bounded by the square distance between the voting system and the set of dictatorial voting systems, multiplied by a constant. This also implies that with a fixed voting system, there always exists a voter whose probability of strategic voting is at least $\Omega(1/N)$.

The main contributions of this paper include:

1. A detailed analysis of the voting decision boundaries of three well known voting systems - plurality, Borda count and Kemeny Young: Which voting profiles near the decision boundaries are manipulable; and for each manipulable profile, which voters would want to vote strategically. For the Kemeny-Young method (and Condorcet methods in general), some decision boundaries are immune to strategic voting: No profiles near these boundaries are manipulable. This result does not hold for plurality and Borda count. All boundaries for plurality and Borda count are manipulable.
2. A new metric for manipulability of voting systems that is compatible with an arbitrary probability distribution on the space of profiles.
3. Large deviations analysis which yields the probability of strategic voting in the limit of large populations or sample sizes for a variety of social biases. In particular, the Kemeny-young method is shown to be immune to strategic voting for most vote distributions.

4. Validation using results from an online survey of the 2012 US Presidential Election. The Kemeny-Young method is largely strategy-proof under the data we collected.

Section 2 gives precise mathematical definitions of the concepts to be discussed. In Section 3 we describe the geometry of strategic voting. In section 4 we provide a probabilistic analysis of strategic voting to three widely discussed voting systems - Borda count, plurality and Kemeny-Young. In Section 5 we present our 2012 US President Election survey result. Section 6 concludes.

2 Preliminaries and Notations

2.1 Voting Systems and Strategic Voting

Let $\mathcal{C} = \{1, \ldots, M\}$ be a finite set of M candidates, where $M \geq 3$. Denote the set of permutations on $\mathcal{C}$ by S_M. Denote the number of voters by N. Each ballot x_i, $i = 1, \ldots, N$ is an element of S_M, or a strict linear ordering. A *voting system*, or a *social choice function* is a function $f : S_M^N \mapsto \mathcal{C}$. The output is called the *winner* or the *social outcome*. The input $x^N \in S_M^N$ is called a *voting profile*, or simply a *profile*. A broadly applicable voting system should be defined for any number of candidates and any number of ballots.

A voting system f is *neutral* if it commutes with permutations on $\mathcal{C}$ [Kalai(2002)]. Intuitively, a neutral voting system is not biased in favor of or against any candidate. This implies, in particular, the second condition of the Gibbard-Satterthwaite Theorem, that there is a possibility for each candidate to win.

A voting system f is *anonymous* or *symmetric* if the "names" of the voters do not matter [Kalai(2002)], i.e.

$$f(x_1, \ldots, x_N) = f(\sigma(x_1, \ldots, x_N)) \tag{1}$$

for any permutation σ on $1, \ldots, N$.

A voting system f is *dictatorial* if the output only depends on the input of a single voter. Formally,

$$f(x_1, \ldots, x_N) = f(x'_1, \ldots, x'_{i-1}, x_i, x'_{i+1}, \ldots, x'_N) \tag{2}$$

for some voter i (the dictator). Clearly, an anonymous voting system is not dictatorial. In fact, it is the "least dictatorial" in the sense that each voter has the same impact on the outcome.

It is debatable whether a single-winner voting system should allow ties. While allowing ties results in more complexity, it is logically possible that two candidates are equally preferred with every possible evaluation. For example, in a plurality election, half of the voters choose a and the other half choose b. Another disadvantage of forbidding ties is that a voting system cannot be completely neutral and anonymous if the number of candidates is equal to a sum of non-trivial divisors of the number of voters [Moulin(1983)]. To maintain both fairness and the single-winner property, we allow a little bit of randomness at the boundary. If two candidates a and b are tied according to the voting rule, we randomly pick a winner between them. A voter whose sincere preference is abc considers $f(x^N) = a$ a better outcome than $f(x^N) = \{a, b\}$, and considers $f(x^N) = b$ as worse than it.

For an anonymous voting system, the input to the voting system $(x_1, \ldots, x_N)$ can be mapped to a vector $v \in \mathbb{N}^{M!}$. The ith entry is the number of type-i votes, which is the ith permutation in S_M (in any fixed order). The social choice function can therefore be rewritten as $f : \mathbb{N}^{M!} \mapsto [M]$. When M is small, anonymous voting systems require a much smaller input dimension, that does not depend on N.

A further consolidation is achieved by normalizing the vote profile by dividing by the number of voters, N. The resulting vector is the empirical distribution of votes $v \in [0, 1]^{M!}$. Thus, any anonymous voting system is a function of the number of voters, N, and the empirical distribution of votes. We say that a voting system is *scale invariant* if it is only a function of the empirical distribution and not N. All such empirical distributions are contained in a probability simplex, called a *vote simplex* $\mathcal{V} = \{v \in \mathbb{R}^{M!} : \sum_{i=1}^{M!} v_i = 1\}$.

Scale invariant voting systems as defined above are anonymous, but not vice versa. A counterexample is a voting system that elects a if an odd number of voters list a as their top choice, and elect b otherwise. It is anonymous but not scale invariant. We assume scale invariance in our paper and treat a voting system as a mapping from $\mathcal{V}$ to the set of candidates $\mathcal{C}$:

$$f : \mathcal{V} \mapsto \mathcal{C} \tag{3}$$

For a normalized voting profile $v \in \mathcal{V}$, define the *vote change* $\Delta_{i,j}$ as an $M!$-dimensional vector whose ith entry is $-\frac{1}{N}$, jth entry is $\frac{1}{N}$ and all other entries are 0. This definition comes in handy when we analyze the possible strategic choices for voters. Type-i voters are called *pivotal voters* if there exists $j \in \mathcal{C}$ such that

$$f(v + \Delta_{i,j}) \neq f(v) \tag{4}$$

That is, a type-i voter is able to change the voting outcome by changing his vote alone. A pivotal voter has the incentive for *strategic voting* if $f(v + \Delta_{ij}) >_i f(v)$, where $a >_i b$ means the type-i voters prefer a to b. In other words, if type-i is a voter's sincere preference, then she could benefit from the voting system by voting differently from sincere preference. A voting profile in which there exists at least one strategic voter is called *manipulable*. The set of pivotal profiles, denoted by T, is called the *boundary set*, or simply *boundary*. When the number of voters approaches infinity, the boundary set degenerates to a space with dimension $M!-2$, which is one dimension lower than the profile space. We use the term "boundary" to refer to voting systems with both finitely many and infinitely many voters, as long as there is no ambiguity.

3 Geometry of Strategic Voting

3.1 From 6 Dimensions to 3 Dimensions

Borda count and Kemeny-Young methods are examples of pairwise comparison-based voting systems. For these voting systems, the input can be written as a matrix $P \in \mathbb{R}^{M\times M}$, where P_{ij}, the entry at the ith row and jth column, is the percentage of voters who prefer i to j, $i \neq j$. Since $P_{ij}+P_{ji}=1$, only the upper or lower triangle of P (excluding the diagonal) is all the information we need. Thus the data dimension is reduced dramatically from $M!$ to $M(M-1)/2$. If there exists a candidate i such that $P_{ij} > \frac{1}{2}$ for all j, then i is called the *Condorcet winner*. Any voting system that elects the Condorcet winner, whenever one exists, is known as a *Condorcet method*. However, a Condorcet winner may not always exist as the pairwise comparison result is not guaranteed to be transitive. For example, if there are only three voters, whose votes are *abc*, *bca* and *cba* respectively, then a, b and c form a "majority cycle". This phenomenon is famously known as the *Condorcet paradox*, and has been extensively studied, such as in [Saari(1995)], [Regenwetter et al(2006)Regenwetter, Grofman, Marley, and Tsetlin]. For the case with three candidates, we list the pairwise comparison as a vector

$$u = (P_{ab}, P_{bc}, P_{ca}) \tag{5}$$

Denote the set of all pairwise comparison vectors by $\mathcal{U}$. Note that two vertices of the cube, $[0,0,0]$ and $[1,1,1]$ represent cyclic pairwise comparisons and do not correspond to any ranking, and $\mathcal{U}$ is the convex hull of the remaining 6 vertices. Similar to [Saari(1995)] , we list $3! = 6$ permutations in the following order, and we will stick to this order for the rest of this paper:

$$abc, acb, cab, cba, bca, bac \tag{6}$$

The transformation from $\mathcal{V}$ to $\mathcal{U}$ can be described by a matrix A:

$$\mathbf{A} = \begin{bmatrix} 1\,1\,1\,0\,0\,0 \\ 1\,0\,0\,0\,1\,1 \\ 0\,0\,1\,1\,1\,0 \end{bmatrix} \tag{7}$$

We list several advantages of this representation:

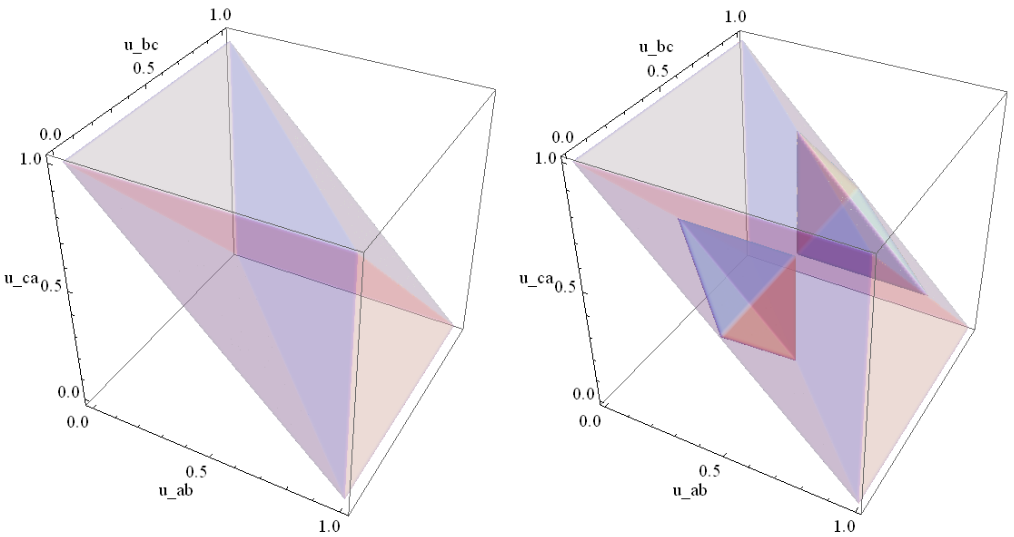
1. The Condorcet paradox (see Figure 1) is intuitively explained: The cube $[0,1]^3$ can be divided into eight sub-cubes with side length 1/2 each, two of which corresponds to conflicting pairwise preferences. That is, the majority vote between each pair of candidates is $a > b, b > c, c > a$ or $a < b, b < c, c < a$.
2. Sen's value restriction theorem [Sen(1966)] is intuitively explained: If we get rid of any two spanning vertices on the same edge of the cube (equivalent to the value-restricted condition), the convex hull of the remaining four vertices does not intersect the two non-transitive sub-cubes.
3. Any vote change Δ_{ij} always results in a shift of the pairwise comparison profile, as $A\Delta_{ij} \neq 0$. This means that every single vote change can be observed in the space of pairwise comparison.

Borda count determines the winner by giving each candidate a certain number of points corresponding to the position in which she is ranked by each voter. For three candidates, the first place receives 2 points, the second receives 1 point and the bottom receives none. Once all votes have been counted the candidate with the most points is the winner. Alternatively, the score can be interpreted as the total number of "pairwise wins". In fact, Borda count is the only voting system that is both a positional rule and pairwise comparison-based [Saari(1995)]. The partitioning of $\mathcal{U}$ by Borda count is shown in Figure 2. The mapping from both the voting profile v and the pairwise comparison profile u to a *score profile* $S^B = (S^B_a, S^B_b, S^B_c) \in \mathbb{R}^3$ is detailed in equation (8)

$$\mathbf{S^B} = \mathbf{B}v = \left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{A} + \mathbf{1}_{3,6} \right) v = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} u + \mathbf{1}_{3,1} \tag{8}$$

where $\mathbf{1}_{m,n}$ is the m by n matrix of all 1's, and

$$\mathbf{B} = \begin{bmatrix} B_a \\ B_b \\ B_c \end{bmatrix} = \begin{bmatrix} 2\,2\,1\,0\,0\,1 \\ 1\,0\,0\,1\,2\,2 \\ 0\,1\,2\,2\,1\,0 \end{bmatrix} \tag{9}$$



(a) Set $\mathcal{U}$ (b) Condorcet paradox

Fig. 1: Set $\mathcal{U}$ of pairwise comparison profiles and the set of Condorcet paradox

Plurality is another positional rule where the candidate with the most first-place vote wins. Unlike Borda count, it cannot be represented in the space of pairwise comparisons. However, the conversion to 3 dimensions is very intuitive by simply combining two rankings sharing the same top choice. We omit the formal matrix description here.

The Kemeny-Young method assigns a score to each *ranking* based on the "total distance" of each ranking to the entire profile. A voting profile is mapped to a score profile of 6 rankings: $S^K \in \mathbb{R}^6$. The score of each ranking, known as the *Kemeny score*, is equal to the sum of the Kendall tau distance between the ranking and each vote. The top choice of the ranking with the lowest Kemeny score is chosen as the winner. Due to this observation, and the following factorization, the Kemeny-Young method can be represented in the space of pairwise comparisons.

$$\mathbf{S^K} = \mathbf{K}v = (\mathbf{1}_{6,3}A + A^T\mathbf{1}_{3,6} - 2A^TA)v = (\mathbf{1}_{3,6} - 2A)^T u + A^T\mathbf{1}_{3,1} \tag{10}$$

where

$$\mathbf{K} = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \end{bmatrix} \tag{11}$$

3.2 Strategic Voting in Different Voting Systems

3.2.1 Borda count

Based on the score profile, we can conclude which types of voters have strategic incentive. Consider an example:

Example 1. In an election using Borda count, the scores of three candidates are $S_a^B = 100, S_b^B = 100$ and $S_c^B = 70$. A voter with preference *cba*, learning that her top choice c has no chance of winning, is willing to *compromise* and vote *bca* instead of *cba* in order to help her second choice win. Another voter with preference *abc*, knowing that b threatens her top choice a, will *bury* b by voting *acb*. In both cases, voters change the result to their benefit by not voting honestly.

A voter is pivotal if by changing her vote alone, some losing candidate can outscore the winner or become tied with it. However, if the runner up is lagging behind by 2 points or more, then none of the pivotal voters has the incentive to change his vote. For example, if a leads b by 2 points, then some voter with honest preference *bac*, *bca* or *cba* would like to help b defeat a. But the best they can do is to shrink the difference to 1 point, which is not enough to change the result. Table 1 enumerates all cases of strategic voting. If the leading candidate and the runner-up are both *ab*, it means a and b are tied as first. Otherwise, we assume the runner-up is only 1 point behind the leading candidate. The 6 columns on the right indicates how a strategic voter would vote given their sincere preferences. Cases of strategic voting are shaded in gray.

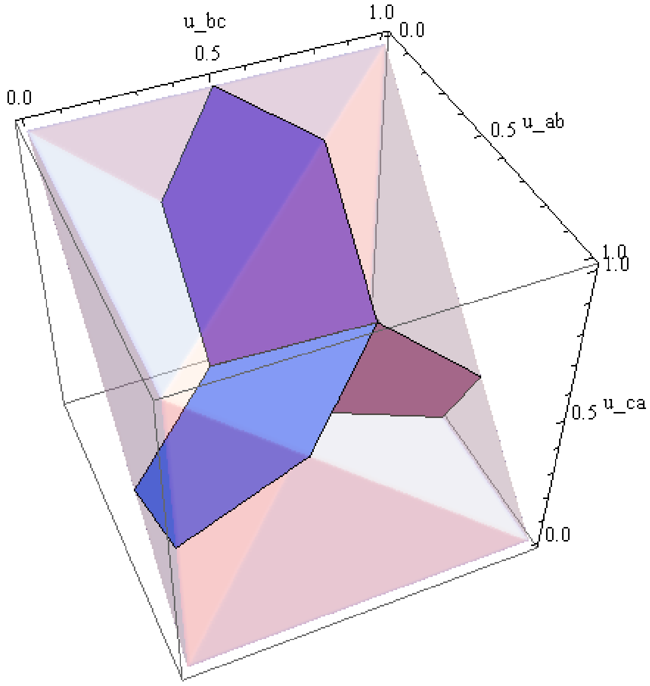


Fig. 2: How Borda count partitions $\mathcal{U}$

Leading	Runner-up	*abc*	*acb*	*cab*	*cba*	*bca*	*bac*
ab	*ab*	*acb*	*acb*	*acb*	*bca*	*bca*	*bca*
ac	*ac*	*abc*	*abc*	*cba*	*cba*	*cba*	*abc*
bc	*bc*	*bac*	*cab*	*cab*	*cab*	*bac*	*bac*
a	*b*	*abc*	*acb*	*cab*	*bca*	*bca*	*bca*
a	*c*	*abc*	*acb*	*cba*	*cba*	*cba*	*bac*
b	*c*	*abc*	*cab*	*cab*	*cab*	*bca*	*bac*
c	*a*	*abc*	*abc*	*cab*	*cba*	*bca*	*abc*
b	*a*	*acb*	*acb*	*acb*	*cba*	*bca*	*bac*
c	*b*	*bac*	*acb*	*cab*	*cba*	*bac*	*bac*

Table 1: Strategic voting table for Borda count. Grey cells are the strategic voters

3.2.2 plurality

The analysis of strategic voting for plurality is similar to Borda count. For plurality, the first place gets all the credit and no score is assigned to the second place. The candidate with the most first place votes is declared the winner. In Table 2 we list all cases of strategic voting for plurality where a and b are either tied as the winner or differ by 1 point. Other cases are completely symmetric and hence omitted. We denote the case where a and b are tied at the top by putting ab in both the “Leading” and the “Runner-up” column. It can be seen that or every winner/runner-up combination, there is chance for strategic voting for at least one type of voter.

Leading	Runner-up	*abc*	*acb*	*cab*	*cba*	*bca*	*bac*
ab	*ab*	*a*	*a*	*a*	*b*	*b*	*b*
a	*b*	*a*	*a*	*c*	*b*	*b*	*b*
b	*a*	*a*	*a*	*a*	*c*	*b*	*b*

Table 2: Strategic voting table for plurality voting

3.2.3 The Kemeny-Young Method

We start with a lemma that states that the boundaries of the Kemeny-Young method are always between two rankings with Kendall tau distance 1 or 2, but not 3.

Lemma 1 *For a voting profile with 3 candidates, the Kemeny distance between the optimal ranking and the runner-up is 1 or 2 if the optimal ranking is unique.*

Proof: Without a loss of generality, assume abc is the optimal ranking. We have $S^K_{abc} + S^K_{cba} = S^K_{bca} + S^K_{acb}$. (This holds even if the votes are partial rankings.) Therefore, either bca or acb has a lower Kemeny score than cba. So a Kemeny distance of 3 is impossible. However, the runner-up might have Kemeny distance 2 to the winner. An example is shown in Table 3:

Ranking	Votes Received	Kemeny Score
abc	3	8
acb	0	11
cab	2	10
cba	0	13
bca	2	10
bac	0	11

Table 3: An example of Kemeny-Young

Here the winner is abc, with cab and bca being tied as runner-ups. They both have Kemeny distance of 2 to the optimal ranking. Table 4 lists all cases where a pivotal voter exists, assuming the leading ranking is abc. Note that

1. A voter who can change the optimal ranking from abc to acb is not considered a pivotal voter, since she does not change the overall result.
2. The runner-up is either tied with, or 1 or 2 points higher than the leading ranking. Otherwise, there is no pivotal voter. This follows from Lemma 1.
3. We do not consider the cases where three rankings are tied or almost tied as these do not come up in the multinomial analysis.

Observe that Kemeny-Young is categorically different from Borda count and plurality voting in the following way: Strategic voting happens only when the boundary separating two "winning" rankings with Kemeny distance 2. For those with Kemeny distance 1, none of the six types of voters can benefit from modifying their vote. For example, suppose $S^K_{abc} = S^K_{bac} - 1$. Three types of voters – bac, bca and cba would like to help b to win. However, no matter how they change their vote, there is simply no way to fill in the gap! We plot the strategic and non-manipulable boundaries in the space of pairwise comparisons in Figure 3. The transparent tetrahedrons are the regions of Condorcet paradox. Note that all manipulable boundaries reside within

We end this section by summarizing the above conclusions with a theorem:

Theorem 1 *The following propositions hold for a voting system with three candidates:*

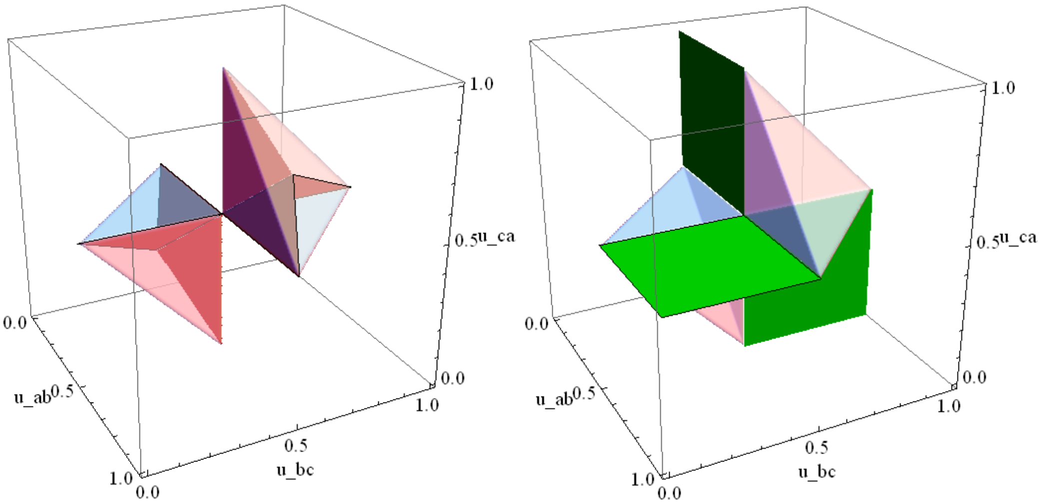
1. *For Borda count and plurality, strategic voting is possible at every boundary: Each pivotal profile is either manipulable, or adjacent to a manipulable profile.*
2. *For the Kemeny-Young method, strategic voting is possible at a boundary if and only if it separates two rankings with Kemeny distance 2.*

4 Probabilistic Analysis for Strategic Voting

In this section, we introduce a new metric for measuring the manipulability of voting systems, the *conditional incentive*. It is defined as the probability that a random voter has incentive for strategic voting *given* that she is a pivotal voter. We believe that the conditional incentive more accurately measures the likelihood of strategic voting than simply counting the number of manipulable profiles. We then apply it to two probabilistic models and provide a simple algorithm to calculate the conditional incentive.

Optimal	Runner-up	*abc*	*acb*	*cab*	*cba*	*bca*	*bac*
abc	*acb*	*abc*	*acb*	*cab*	*cba*	*bca*	*bac*
abc	*bac*	*abc*	*acb*	*cab*	*cba*	*bca*	*bac*
abc	*cab*	*abc*	*acb*	*cab*	*cba*	*cab* or *cba*	*bac*
abc	*bca*	*abc*	*acb*	*cab*	*cba*	*bca*	*bca* or *cba*

Table 4: Pivotal and Strategic voters for Kemeny-Young



(a) manipulable boundaries (b) Non-manipulable boundaries

Fig. 3: Strategic and non-manipulable boundaries of the Kemeny-Young method

4.1 Conditional Incentive

We reserve the capital P for profile distributions and use lower case letters for vote profiles and the probability distribution of votes. Let P_N be the profile distribution with N voters. For example, under the *impartial culture* assumption, each voter uniformly randomly picks a vote. In this case P_N is a multinomial distribution, and the probability that we get a particular profile $(v_1, \ldots, v_6) \in \mathcal{V}$ is:

$$P_N((v_1, \ldots, v_6)) = \frac{N!}{6^N (Nv_1)! \cdots (Nv_6)!} \tag{12}$$

Definition 1 *Define the mean incentive $I(P_N)$ as the probability that a random voter has the incentive to manipulate:*

$$I(P_N) = \sum_{v \in V} P_N(v) \sum_{i=1}^{M!} v_i \mathbf{1}\{\textit{type-i voter is strategic}\} \tag{13}$$

For infinitely many voters, let

$$I(P) = \lim_{N \to \infty} I(P_N) \tag{14}$$

if the limit exists.

Definition 2 *Define the mean effect $E_m(P_N)$ as the probability that a random voter is a pivotal voter. It is obvious that $I(P_N) \leq E_m(P_N)$.*

$$E_m(P_N) = \sum_{v \in V} P_N(v) \sum_{i=1}^{M!} v_i \mathbf{1}\{\textit{type-i voter is pivotal}\} \tag{15}$$

For infinitely many voters, let

$$E_m(P) = \lim_{N \to \infty} E_m(P_N) \tag{16}$$

if the limit exists.

Definition 3 *The conditional incentive $I_c(P_N)$ is defined as the probability that a random voter is a strategic voter, given that she is pivotal.*

$$I_c(P_N) = \frac{I(P_N)}{E_m(P_N)} \tag{17}$$

For infinitely many voters, let

$$I_c(P) = \lim_{N \to \infty} I_c(P_N) \tag{18}$$

if the limit exists.

For a scale invariant voting system, we expect $E_m(P) = 0$. We do not show a formal proof, but intuitively the boundary should grow "thinner" as the number of voters grows. In other words, the voting result stabilizes after more and more people have voted. However, the conditional incentive I_c does not necessarily converge to 0, as will be analyzed in the following section. A voting system may have a very low mean incentive simply because there is a small fraction of pivotal profiles. But what we are really after is a voting system with a low conditional incentive. After all, in the settings of Gibbard-Satterthwaite Theorem, everyone votes *as if* their vote makes the difference.

We illustrate the concept using the example of Borda count. We show the score profiles instead of the vote profiles because 1) the pattern of strategic voting for Borda count can be directly read from the score profile, and 2) score profiles are easily shown on a two-dimensional plot. Figure 4 depicts a microscopic view of the score profiles of Borda count near the boundary $S_a^B = S_b^B$. Here each dot represents a score profile (S_a^B, S_b^B, S_c^B). All score profiles lie on a plane $S_a^B + S_b^B + S_c^B = 3N$.The vertical line in the middle represents the boundary $S_a^B = S_b^B$. To the left of the boundary are the score profiles such that $S_a^B > S_b^B > S_c^B$ and to the right are the profiles satisfying $S_b^B > S_a^B > S_c^B$. S_c^B is the same for all profiles on the same horizontal level, and increases when moving upwards. The set of manipulable score profiles is surrounded by the rectangle. The fractions over each dot denotes the number of types of strategic voter (numerator) and types of pivotal voter (denominator). For example, 2/5 means that if a voting profile (among many) maps to this score profile, then 5 out of 6 types of voter are pivotal, but only two types are strategic voters. The procedure of counting follows from Table 1. Figure 5 shows how a vote change translates to the change of the score profile. In the left figure, a *bac*-type voter benefits from strategic voting by changing her vote to *bca* (arrow number 3). In the right figure, an *abc*-type voter have three options (arrow 2,3 and 5) to make b the winner by changing her vote, but she has no incentive to do so.

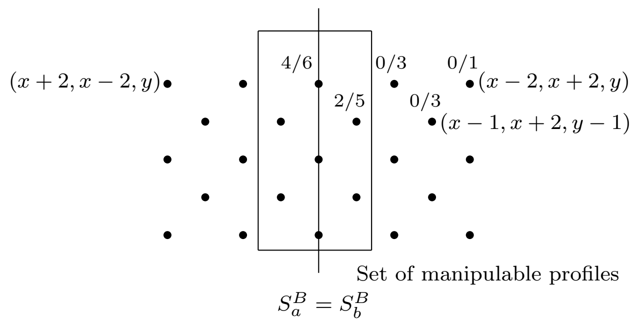
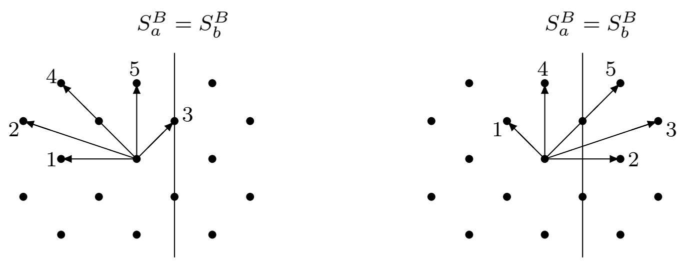


Fig. 4: Score profiles with pivotal and strategic voters for Borda count



(a) Possible vote changes for voter *bac*. The arrows 1-5 denotes the resulting score profile if the vote is changed to *abc*, *acb*, *bca*, *cab* or *cba*, respectively.

(b) Possible vote changes for voter *abc*. The arrows 1-5 denotes the resulting score profile if the vote is changed to *acb*, *bac*, *bca*, *cab* or *cba*, respectively.

Fig. 5: The outcome of vote changes by two voter types, one with strategic incentive (*bac*) and one without (*abc*), in Borda count.

It is easy to count the number of pivotal and manipulable score profiles. Unfortunately, not all score profiles are equiprobable. Even if they were, the ratio of voters supporting each ranking given the score profile is non-uniform. Calculating the conditional incentive $I_c(P_N)$ with large N seems to be a formidable task due to the very large number of pivotal profiles. However, given the specific profile distribution $\{P_N\}$, the calculation of conditional incentive is much simplified using large deviation analysis.

4.2 Multinomial Model

In real elections, people usually have some prior knowledge on the possible vote distribution, which they could get from polls, surveys, social networks, etc. However, there is still quite a lot of randomness remain due to sampling error, faulty or biased sampling method, mishandling of data among others. Both models we are about to introduce simulate the reality with a unimodal distribution that represents some sort of coherence throughout the population. The first probabilistic model, known as the *multinomial model*, assumes that the behavior of each individual voter is intrinsically random: Each voter draws a random vote the (normalized) voting profile v is subject to a multinomial distribution: $P_N \sim Multi(N, p)$, where p_0, called the social bias, is a probability distribution over 6 rankings. For example, $p = [0.3, 0.2, 0.5, 0, 0, 0]$ means that the probability that any individual voter votes for *abc* is 0.3.

However, the problem can be much simplified by using large deviation analysis. To find the conditional incentive as $N \to \infty$, we first find the information projection of p onto the set of all pivotal profiles T:

$$q^* = \text{argmin}_{q \in T} D(q||p) \tag{19}$$

Here $D(q||p) = \sum_i q_i \log(q_i/p_i)$, the Kullback-Leibler divergence or relative entropy from q to p, is a metric of the information lost when p is used to approximate q [Cover et al(1991)Cover, Thomas, Wiley et al]. Let v be the empirical vote distribution drawn with bias p. For any $\epsilon > 0$, denote by q^ϵ the neighborhood of q^* with radius ϵ (we do not specify a distance metric). By Sanov's Theorem [Sanov(1958)],

$$\lim_{N \to \infty} \mathbb{P}[v \in q^\epsilon | v \in T] = 1 \tag{20}$$

In other words, if strategic voting happens at all, with high probability the profile is contained in a small neighborhood of q^* as the number of voters grow. Therefore, $I_c(P_N)$ can be calculated by just looking at a few pivotal profiles near q^*. $D(q||p)$ is a convex function of q, but T may not be a convex set: The convex combination of two pivotal profiles is not necessarily pivotal. However, if T consists of "linear pieces", i.e. T can be written as a finite union of subsets, each subset being a hyperplane bounded by linear inequalities, then convex optimization can be applied to minimize $D(q||p)$ over each linear piece and take the minimum. Most voting systems we encounter in practice have piecewise linear boundaries. To focus on the central idea, assume the optimal solution q^* is located on the boundary T defined by $L \cdot q = 0$.

$$\text{Minimize } D(q||p) = \sum_{i=1}^{6} q_i \log \frac{q_i}{p_i} \tag{21}$$

$$\text{subject to } Lq = 0 \tag{22}$$

$$0 \le q_i \le 1 \text{ for } i = 1 \dots 6 \tag{23}$$

$$\sum_{i=1}^{6} q_i = 1 \tag{24}$$

The solution can be found using the KKT conditions:

$$q_i^* = \frac{p_i e^{\lambda L_i}}{\sum_i p_i e^{\lambda L_i}} \qquad \text{for } i = 1, \dots, 6 \tag{25}$$

where λ is the Lagrange multiplier for the constraint (22). λ can be solved by plugging (25) into (22).

The following result further simplifies the calculation of conditional incentive.

Theorem 2 *Suppose the social bias p satisfies $p_i > 0$ for all i and $p \notin T$. Let $q^* = argmin_{q \in T} D(q||p)$. For a profile v satisfying $||v - q^*||_1 < \epsilon$ for some $\epsilon > 0$ a vote change Δ_{ij} satisfying $L \cdot \Delta_{ij} = 0$,*

$$\left| \frac{P_N(v + \Delta_{ij})}{P_N(v)} - 1 \right| \le \epsilon \cdot \frac{\max_i p_i}{\min_i p_i} \tag{26}$$

This theorem implies the following. Consider any profile v near q^*, and draw a hyperplane orthogonal to L. Other profiles on this hyperplane are almost equiprobable to v, and the difference in probability vanishes as N goes to infinity. Therefore, a single profile can be used to represent the entire hyperplane in (17). Of course, as v moves further and further away from q^* the probability would decrease, but with a fairly large N the approximation can be good enough.

Theorem 3 *Assume $p \notin T$, $p_i > 0$ for $i = 1, \dots, 6$, and the information projection q^* is on the boundary defined by $Lq = C$. The conditional incentive $I_c(p)$ is given by*

$$I(p) = \sum_{d=-D}^{D} \sum_{i=1}^{6} e^{\lambda k} q_i \mathbb{1}\{\textit{type-i voters are strategic on the hyperplane } Lq = C + \frac{d}{N}\} \tag{27}$$

$$EF(p) = \sum_{k=-1}^{1} \sum_{i=1}^{6} e^{\lambda k} q_i \mathbb{1}\{\textit{type-i voters are pivotal on the hyperplane } Lq = C + \frac{d}{N}\} \tag{28}$$

$$I_c(p) = \frac{I(p)}{EF(p)} \tag{29}$$

where D is chosen appropriately such that all pivotal profiles are included. Whether type-i voters are pivotal or strategic can be verified by enumerating all possible vote changes.

4.2.1 Application to Borda count

Without a loss of generality, let us assume the social bias p is closest to the boundary between a and b. Lemma 2 says that all pivotal profiles are such that the score difference between a and b is at most 4:

Lemma 2 *For Borda count, suppose $v \in \{v : |S_a^B - S_b^B| \le 4, \min(S_a^B, S_b^B) > S_c\}$, and $v_i > 0$ for $i = 1, \dots, 6$, then there exists at least one pivotal voter in v.*

Proof Since a voter contributes 2 points to her top choice and 0 point to her bottom choice, by reversing the order she could increase or decrease the point difference of two candidates by at most 4. Therefore, if a and b are 5 points or more apart no single voter can change the outcome. Q.E.D.

We explicitly write the convex program for Borda count. B_a and B_b are the vectors defined in (8).

$$\text{Minimize } D(q||p) = \sum_{i=1}^{6} q_i \log \frac{q_i}{p_i} \tag{30}$$

$$\text{subject to } (B_a - B_b)q = 0 \tag{31}$$

$$0 \le q_i \le 1 \text{ for } i = 1 \dots 6 \tag{32}$$

$$\sum_{i=1}^{6} q_i = 1 \tag{33}$$

Figure 6 plots what happens near q^*. These hyperplanes are actually 4-dimensional, but are plotted as 2-dimensional for the sake of visualization. $I_c(p)$ is given by Theorem 3. The sign of λ does not matter. If $\lambda > 0$, then the hyperplanes with $k > 0$ are closer to p, and vice versa. Choosing $\lambda > 0$ or $\lambda < 0$ results in the same value of $I_c(p)$.

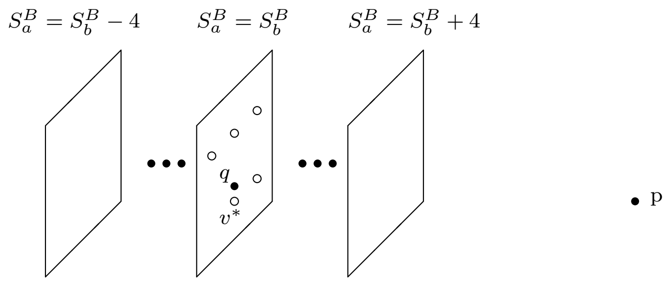


Fig. 6: Hyperplanes parallel to $S_a^B = S_b^B$

For illustration, we pick 8 different social biases, and set the number of voters to $N = 10000$. The conditional incentives computed analytically using Theorem 3 are very close to the simulation results (Table 5), except for the uniform distribution. This is expected since the uniform distribution is at the intersection of 3 boundaries.

Are there any profiles with zero conditional incentive? The answer is yes, but such profiles require $p_i = 0$ for some i. One example is $p = [.01, .49, 0, 0, .49, .01]$. Though all 4 types of voters can be pivotal voters, only *abc* and *bac* can potentially vote strategically (refer to Table 1). But nobody holds that preference anyway. In real elections with three candidates, it is very unlikely for a ranking to have no support at all, as suggested by historical voting data in [Regenwetter et al(2006)Regenwetter, Grofman, Marley, and Tsetlin].

p	$I_c(P_N)$ (simulation)	$I_c(p)$ (analytical limit)
uniform	0.2706	0.1577
[0.2 0.2 0.18 0.18 0.12 0.12]	0.2474	0.2453
[0.2 0.15 0.15 0.15 0.2 0.15]	0.2597	0.2605
[0.18 0.18 0.17 0.17 0.15 0.15]	0.2636	0.2595
[0.2 0.2 0.1 0.1 0.2 0.2]	0.2371	0.2399
[0.19 0.18 0.17 0.16 0.15 0.15]	0.2628	0.2599
[0.2 0.18 0.17 0.16 0.15 0.14]	0.2585	0.2556
[0.2 0.15 0.18 0.17 0.15 0.15]	0.2618	0.2521

Table 5: Conditional incentive for Borda count with $N = 10,000$ voters

4.2.2 Application to plurality

Calculation of $I_c(p)$ for plurality involves the same procedure as Borda count. We solve for the information projection of p on the boundary set T^P, find the pivotal and strategic voters on hyperplanes parallel to the boundaries, and calculate the mean incentive and mean effect separately. Table 6 compares the analytical and simulation results of the conditional incentive with $N = 10000$ voters. The analytical and simulation results are very close except when p is uniform.

p	$I_c(P_N)$ (simulation)	$I_c(p)$ (analytical limit)
uniform	0.2177	0.1829
[0.2 0.2 0.18 0.18 0.12 0.12]	0.1598	0.1631
[0.2 0.15 0.15 0.15 0.2 0.15]	0.2010	0.1983
[0.18 0.18 0.17 0.17 0.15 0.15]	0.2005	0.1988
[0.2 0.2 0.1 0.1 0.2 0.2]	0.1335	0.1342
[0.19 0.18 0.17 0.16 0.15 0.15]	0.2006	0.1993
[0.2 0.18 0.17 0.16 0.15 0.14]	0.1941	0.1933
[0.2 0.15 0.18 0.17 0.15 0.15]	0.2007	0.2000

Table 6: Conditional incentive for plurality voting with $N = 10000$ voters

4.2.3 Application to Kemeny-Young

Finding the information projection on the boundary set for Kemeny-Young is trickier than Borda count. This is because the Kemeny-Young method partitions the probability simplex $\mathcal{V}$ into non-convex cells. Therefore the information projection q^* may not always fall on the interior of a boundary, but instead on the intersection of two linear pieces. Figure 7 illustrates an example, where q^* is at the intersection of two boundaries $S^K_{abc} = S^K_{bac}$ and $S^K_{acb} = S^K_{bac}$. If the social bias favors a, then the information projection may fall on the intersection of the two boundaries. Once that happens, we need to look at the microscopic structure near q^* to determine which pivotal profiles are manipulable and which are not. Again, for visualization purpose we are showing the space of profiles in a lower dimension. The two boundaries are actually 4-dimensional subspaces, and the intersecting "line" is actually a 3-dimensional subspace.

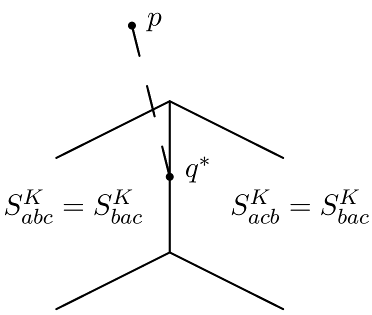


Fig. 7: The information projection falls on the intersection of $S^K_{abc} = S^K_{bac}$ and $S^K_{acb} = S^K_{bac}$

Again, we compare the simulation and analytical results of the conditional incentive for the Kemeny-Young method in Table 7. The last distribution is adjusted slightly because we want to show an example where the social bias is not on the boundary and is closer to a manipulable boundary.

p	$I_c(P_N)$ (simulation)	$I_c(p)$ (analytical limit)
uniform	0.0830	0
[0.2 0.2 0.18 0.18 0.12 0.12]	0	0
[0.2 0.15 0.15 0.15 0.2 0.15]	0.0939	0
[0.18 0.18 0.17 0.17 0.15 0.15]	0.0	0
[0.2 0.2 0.1 0.1 0.2 0.2]	0.0	0
[0.19 0.18 0.17 0.16 0.15 0.15]	0.0	0
[0.2 0.18 0.17 0.16 0.15 0.14]	0.0	0
[0.205 0.15 0.18 0.17 0.15 0.145]	0.162	0.196

Table 7: Conditional incentive for Kemeny-Young with $N = 10000$ voters

4.3 Sampling Model

The sampling model assumes a finite number of voters N, and each individual voter holds a fixed preference. Therefore, unlike the multinomial model, the vote distribution is exactly the social bias $p \in \mathcal{P}_N$. However, voters do not know p completely. Instead, they observe p^s, the profile of a small group of randomly chosen voters. Suppose the group size is N', $N' \ll N$. The sample follows a multinomial distribution $Multi(p, N_0)$. A rational voter would then estimate which boundary p most probably falls on and would then determine her strategy of manipulation accordingly.

To find the closest boundary to p^s, we are faced with the problem of choosing a prior distribution for p. A prior that give more weight to the center (equiprobable for all votes) might be preferred (in real elections, people do often see close matches rather than landslide victories), but for now we assume uniform prior for generality and lack of good model for non-uniform priors. Thus maximizing $P(p|p^s)$ is equivalent to maximizing $P(p^s|p) \doteq 2^{-N'D(p^s||p)}$. We would like to find a distribution $\bar{p}$ that maximizes the KL-Divergence $D(p^s||p)$ among all $p \in T$:

$$\bar{p} = \text{argmax}_{p\in T} D(p^s||p) = \text{argmin}_{p\in T} - \sum_{i=1}^{6} p_i^s \log p_i \tag{34}$$

We first consider the general case. The boundary T is defined by an equation $h(p) = 0$. For now, put the inequality constraints aside and assume that the information projection is on the interior of a boundary piece. Since log is a concave function, we are solving the constrained convex program for p:

$$\text{minimize } -\sum_{i=1}^{6} q_i \log p_i \tag{35}$$

$$\text{subject to } h(p) = 0 \tag{36}$$

$$\sum_{i=1}^{6} p_i = 1 \tag{37}$$

We can use the KKT condition to get a necessary condition for the optimal solution $\bar{p}$. Let λ_b and λ_p be the Lagrange multipliers corresponding to the boundary restriction (36) and the probability simplex restriction (37). To get rid of the minus sign, we flip the sign of the Lagrange multipliers.

$$\frac{q_i}{p_i} - \frac{\partial h}{\partial p_i} \cdot \lambda_b - \lambda_p = 0, i = 1, \ldots, 6 \tag{38}$$

or, after rearrangement,

$$p_i = \frac{q_i}{h_i' \lambda_b + \lambda_p} \tag{39}$$

where $h_i' = \partial h / \partial p_i$. The 6 equations of (38) together with (36) and (37) form a system of 8 equations and 8 unknowns. However, with a general h it is difficult to find the analytical solution. The values of λ_b and λ_p could be found by appllying numerical methods. Using (39), we will show that a small shift of the social bias parallel to the boundary defined by $h(p)$ does not change the KL-Divergence $D(p||q)$.

$$\begin{aligned}\frac{P(p+\Delta_{ij}|p^s)}{P(p|p^s)} &= \frac{P(p^s|p+\Delta_{ij})}{P(p^s|p)} \\ &= \left(\frac{p_i+1/N}{p_i}\right)^{p_i^s K}\left(\frac{p_j-1/N}{p_j}\right)^{p_j^s K} \\ &\approx 1 \end{aligned} \tag{40}$$

For the sampling model, the change of probability is much less sensitive to perturbations on the social bias. Therefore, the conditional incentive can be calculated by sampling at $\bar{p}$ alone.

5 2012 Presidential Election Survey

In our 2012 US Presidential Election Survey, we asked each respondent to evaluate 11 Republican, Democrat and independent candidates through an online survey form. Although Hilary Clinton was not a candidate for the 2012 election, she was included for her popularity. The respondents were presented the 11 candidates in randomized order, and were asked to assign a numerical score (on a scale from 0 to 100) to each candidate. A 100 indicates the strongest support for that candidate to be president, and 0 indicates the strongest opposition. The respondents had the choice to not rate candidates they did not know or did not want to evaluate. The respondents also optionally provided basic demographic information (age, gender, level of education, state of residence, party affiliation and so on) for better analysis of the data. We recruited respondents through three channels:

1. Google Adwords and FiveThirtyEight.com. Our link was shown to random Google users and NYTimes (FiveThirtyEight blog) readers in the US. Users who clicked on our ads were taken to our online survey.
2. Mercer County Panel. The respondents were residents of Mercer County, New Jersey.
3. Amazon Mechanical Turk. The respondents were Amazon Mechanical Turk users living in the US.

We collected 1,650 valid responses in total (responses that rated at least one candidate were considered valid). Among the respondents, 359 identified themselves as Republicans, 391 as Democrats, and 537 as independent.

Method	Valid Responses
Google/FiveThirtyeight	887
Mercer County Panel	291
Mechanical Turk	472
Total	1650

Based on the collected data, we ranked the candidates using five different ranking methods: Kemeny-Young, Borda count, range voting, instant run-off voting, and plurality voting. Except for range voting, all scores are first converted to rankings. Kemeny-Young and single run-off voting are compatible with partial rankings, whereas for Borda count, we assign all unranked candidates the “average score”. For example, if someone rates Obama 80, Romney 75, Ron Paul 72 and leaves all other candidates unrated, then a score of 3 is assigned to Obama, 1 is assigned to Ron Paul and 2 is assigned to everyone else (including Romney). Thus a voter rating more candidates contributes more information, which hopefully gives voters a reason to rate as many candidates as possible. The outcome with different methods are quite similar, but not exactly the same. Only Borda count puts Mitt Romney at the top, while all other methods decide on Barack Obama as the winner.

As in previous sections, we plot the results in the pairwise comparison space. In Figure 8, each dot represents the vote profile for one subset of three candidates. There are 11 candidates, hence $\binom{11}{3} = 165$ different subsets. The two dark tetrahedrons denote the regions of Condorcet paradox. We reordered the candidates so that all dots fall in the same sub-cube ($u_{ab} > 0.5, u_{bc} > 0.5, u_{ca} < 0.5$). Therefore, the same candidate, e.g. Mitt Romney, may be attached to the name a in one dot but attached to the name b in another dot. This, however, does not affect the cycle-free property of the voting profile. If we had not done so, the dots would scatter around the space, but still would not be inside the Condorcet paradox region. By sorting it is easier to see that no dots falls inside the “Condorcet paradox” region.

Some historical voting data [Regenwetter et al(2006)Regenwetter, Grofman, Marley, and Tsetlin] report the same finding that real elections are free of Condorcet paradoxes.

What can we read from these data pertaining to strategic voting? In the last section we showed that for the Kemeny-Young method, all manipulable boundaries are inside the region of Condorcet paradox. It is reasonable

Rank	Kemeny-Young	plurality	Borda count	Run-off	Range Voting
1	Barack Obama	Barack Obama	Mitt Romney	Barack Obama	Barack Obama
2	Hillary Clinton	Mitt Romney	Barack Obama	Mitt Romney	Hillary Clinton
3	Mitt Romney	Ron Paul	Hillary Clinton	Ron Paul	Mitt Romney
4	Ron Paul	Hillary Clinton	Ron Paul	Hillary Clinton	Ron Paul
5	Jon Huntsman	Newt Gingrich	Jon Huntsman	Newt Gingrich	Newt Gingrich
6	Rick Santorum	Rick Santorum	Rick Santorum	Rick Santorum	Jon Huntsman
7	Newt Gingrich	Jon Huntsman	Newt Gingrich	Jon Huntsman	Rick Santorum
8	Rick Perry	Rick Perry	Rick Perry	Rick Perry	Rick Perry
9	Michele Bachmann	Michele Bachmann	Gary Johnson	Michele Bachmann	Gary Johnson
10	Gary Johnson	Gary Johnson	R. Lee Wrights	Gary Johnson	Michele Bachmann
11	R. Lee Wrights	R. Lee Wrights	Michele Bachmann	R. Lee Wrights	R. Lee Wrights

Table 8: Voting outcome of different voting systems based on survey data based

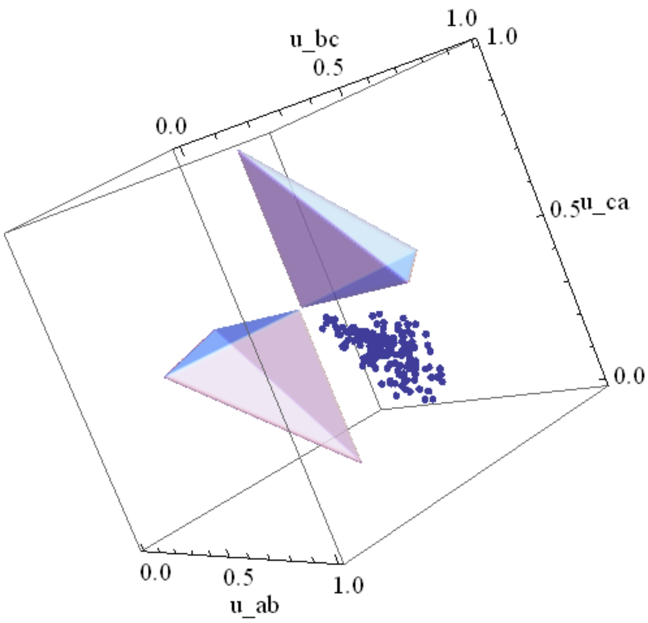


Fig. 8: Condorcet paradox never occurs for our collected voting data

to suspect that most data points are closer to non-manipulable boundaries than manipulable boundaries. In Figure 9, we plotted the log-distance (using the KL-divergence as the distance metric) of each sample to its maximum likelihood prior on the manipulable and non-manipulable boundaries, respectively. The log-scale is used to prevent the data points from clustering near the origin. The figure shows that all but a few profiles are closer to a non-manipulable boundary than a manipulable boundary. This suggests that for a real world election using the Kemeny-Young method, the vote count is more likely to be on a non-manipulable boundary than on a manipulable boundary.

For comparison, we also show the data points with the Borda count and plurality in Figure 10(a) and 10(b), respectively. We plot against the boundary set of Borda count T^B in Figure 10(a). Most data points locate in the winning region of a, but there are also some data points located along the boundary between a and b. Therefore, for Borda count not only is the conditional incentive high, but the absolute probability (measured by the mean effect) is also likely to be high in real elections. The data points are plotted in a different space for plurality in Figure 10(b). The three axes denotes the portion of the plurality votes that go to a, b or c, respectively. The polytope inside the cube is the vote simplex, and all data points reside on the simplex. Different colors (or shades of gray) denote different winning regions.

6 Concluding Remarks

It is difficult to verify whether in real elections votes are distributed multinomially. However, for Condorcet methods, the categorical difference between “manipulable boundary” and “non-manipulable boundary” applies universally. Given the theoretical calculation and simulation results under the multinomial model, we believe that the Kemeny-Young method, and Condorcet methods in general are less manipulable than competing methods.

The concept of strategic voting is used in different contexts. First, ballots may not always be in the form of a (partial or complete) rank-ordering. For example, Range voting and Majority judgement requires a voter to assign an “evaluation”, in the form of a numerical value, to each candidate. Balinski and Laraki

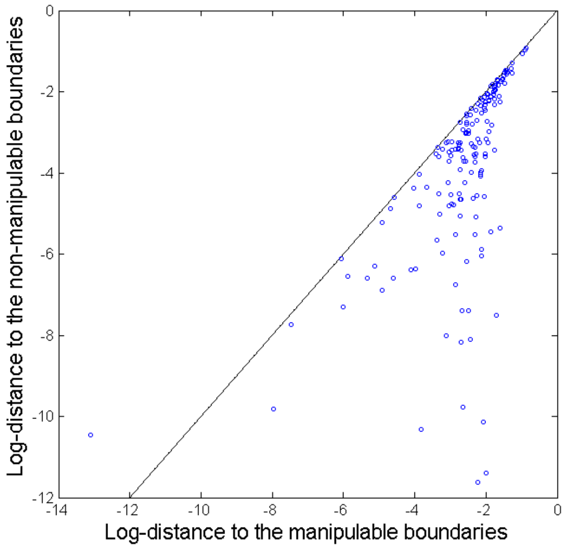
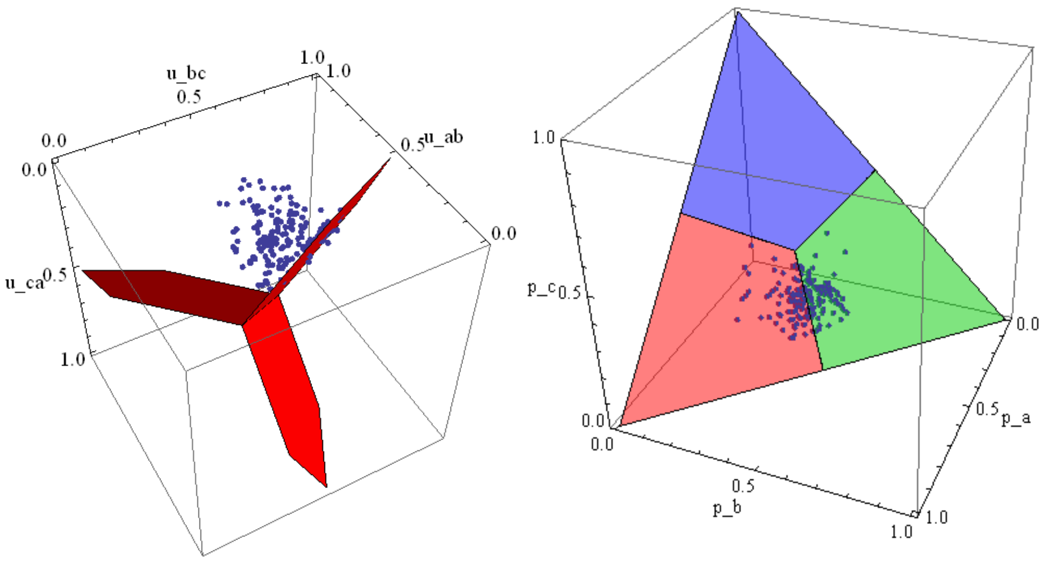


Fig. 9: Kemeny-Young is mostly immune to strategic voting for our collected voting data



(a) Borda count (b) Plurality

Fig. 10: Election survey data for Borda count and plurality

[Balinski and Laraki(2007)] argue that Majority judgement is "strategy-proof" in the sense that each voter cannot improve the status of a candidate if that candidate's score (median of every voter's evaluation) is lower than her sincere evaluation. However, if we convert the scores of each candidate into a ranking (or pick the top scorer), then a voter can still help her favourite by lowering the score of other candidates. Range voting, in contrast is very susceptible to strategic voting. Voters will exaggerate the difference between their top choice and the bottom choice, to the point that it virtually becomes approval voting. For approval voting, each voter converts a preferential order into a binary opinion for each candidate: Approve or disapprove. The choice of the approval cut-off can be tactical. But it is debatable whether this should be considered as strategic voting. In sum, the exact meaning of strategic voting is highly dependent on the voting rules.

Strategic voting becomes more clearly defined when we focus on ranked voting systems. Both Arrow's Impossiblity Theorem and the Gibbard-Satterthwaite Theorem apply to ranked voting systems. The likelihood for strategic voting changes across voting systems, and within a voting system with different social biases. By modelling the voting process as a multinomial distribution, we can accurately estimate the conditional incentive, or the probability of strategic voting given that a pivotal voter exists. For plurality voting and Borda count, manipulable profiles appear everywhere near the boundary between "winning regions". For Condorcet methods, the existence of non-manipulable boundaries implies that for certain biases, the conditional incentive approaches zero as the number of voters increase.

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