Appendix C. Proofs

Condition A1. Random draw from population. Let μ be a probability measure on $(\Omega, \mathcal{F})$. Each $\omega \in \Omega$ represents an individual. $(\Omega, \mathcal{F}, \mu)$ describes the probabilities of drawing individuals from a (possibly infinite) population.

Condition A2. Stochastic treatment assignment. For each $\omega \in \Omega$, let υ_ω be a probability measure on $(\Delta,\mathcal{D})$. $(\Delta,\mathcal{D}, \upsilon_\omega)$ describes the probabilities associated with receiving the treatment (or, in the RDD, the score V), for each individual ω. Assume that for any $B \in\mathcal{D}$, $\upsilon_\omega(B)$ as a function of ω is measurable $\mathcal{F}$. Let $\mathcal{G}$ be the σ-field consisting of all sets $\Omega \times A$, where $A \in\mathcal{D}$.

Condition A3. Probabilities for the overall experiment. Define P as follows: $\forall E \in\mathcal{F}\times\mathcal{D}$, $P(E) = \int_\Omega \upsilon_\omega [\delta : (\omega, \delta) \in E] \mu(d\omega)$. It can be shown that P is a probability measure on $(\Omega \times \{0, 1\}, \mathcal{F} \times \mathcal{D})$.

Condition A4. Pre-determined characteristics. Let $X = x(\omega)$ be a real-valued function that is measurable $\mathcal{F}\times\mathcal{D}$. It follows that it is also measurable $\mathcal{F}$.

Condition A5. Finite first moments. E_P and E_μ denote expectations with respect to probability measures P and μ, respectively. Where appropriate, Y, Y_1, Y_0, $\frac{f_\omega(0)}{f(0)}Y$, $\frac{f_\omega(0)}{f(0)}Y_1$, and $\frac{f_\omega(0)}{f(0)}Y_0$ are each assumed to be integrable P and integrable μ.

Condition B1. Binary treatment assignment model. Let $\Delta = \{0, 1\}$ and $\mathcal{D}= \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$. Define the random variable D as $D = \delta$, $\delta \in \Delta$, which is measurable $\mathcal{F}\times\mathcal{D}$.

Condition B2. Regression discontinuity design. Let $\Delta = \mathbb{R}$, and $\mathcal{D}=\mathcal{R}^1$ be the class of linear Borel sets. Define the random variable V – measurable on $\mathcal{F}\times\mathcal{D}$ – as $V(\delta) = \delta$, $\delta \in \Delta$, and let $D = 1[V \geq 0]$.

Condition C1. Potential outcomes. Let $Y_1 = y_1(\omega)$, $Y_0 = y_0(\omega)$, be real-valued functions that are measurable $\mathcal{F}\times\mathcal{D}$ (and hence measurable $\mathcal{F}$). Let $Y = DY_1 + (1 - D) Y_0$.

Condition C2. Potential outcome function. Let $Y = y(\omega, \delta)$ be a real-valued function that is measurable $\mathcal{F}\times\mathcal{D}$. Let $y(\cdot, \cdot)$ be continuous in the second argument except at $\delta = 0$, where the function is only continuous from the right. Define the function $Y^+ = y(\omega, 0)$ and $Y^- = \lim_{\varepsilon\to 0^+} y(\omega, -\varepsilon)$.

Condition D1. Treatment randomization. v_ω is identical for all $\omega \in \Omega$

Condition D2. Continuous density of score. Let $F_\omega(\delta) = v_\omega(-\infty, \delta]$, and $f_\omega(\delta)$ its derivative with respect to δ. Let $f(\delta) = \int_\Omega f_\omega(\delta)\,\mu(d\omega)$. Assume that $0 < f_\omega(\delta)$, and $f_\omega(\delta)$ is continuous in δ on $\mathbb{R}$. (Note that if v_ω is measurable $\mathcal{F}$, one can show that in this set-up, so too are F_ω and f_ω).

Proposition 1. If Conditions A1-A5, B1, C1, and D1 hold, then:

a) $\forall F \in \mathcal{F}$, $P[F \times \Delta | D = 1] = P[F \times \Delta | D = 0] = P[F \times \Delta] = \mu[F]$

b) $E_P[Y|D = 1] - E_P[Y|D = 0] = E_\mu[Y_1 - Y_0] \equiv ATE$

c) $\forall x_0 \in \mathbb{R}$, $P[X \le x_0 | D = 1] = P[X \le x_0 | D = 0] = P[X \le x_0] = \mu[\omega : X \le x_0]$

Proof. a) $P[F \times \Delta | D = 1] = P[(F \times \Delta) \cap (\Omega \times \{1\})] / P[\Omega \times \{1\}]$. Numerator is $\int_{F\times\{1\}} P(d(\omega, \delta))$. This is equal to $\int_F \left[\int_{\{1\}} v_\omega(d\delta)\right] \mu(d\omega) = v_\omega(\{1\}) \cdot \mu[F]$ by 18.20.c of Billingsley (1995) and by D1. Similarly, denominator is $v_\omega(\{1\})$. Similar argument holds for $P[F \times \Delta | D = 0]$. b) Need to show that conditional expectation of Y_1 given $\mathcal{G}$, evaluated at $D = 1$ is equal to $E_\mu[Y_1]$. It can be shown that the conditional expectation of Y given $\mathcal{G}$ can be written as $\alpha(\delta_0) \equiv \frac{1}{P[\Omega\times\{\delta_0\}]} \int_{\Omega\times\{\delta_0\}} Y_{\delta_0} P(d(\omega, \delta))$, for $\delta_0 = 0$ and 1. Consider the case when $\delta_0 = 1$. We then have $\frac{1}{P[\Omega\times\{1\}]} \int_{\Omega\times\{1\}} Y_1 P(d(\omega, \delta)) = \frac{1}{P[\Omega\times\{1\}]} \int_\Omega \left[\int_{\{1\}} Y_1 v_\omega(d\delta)\right] \mu(d\omega)$ by 18.20.c of Billingsley (1995). Because Y_1 is only a function of ω, and by D1, this becomes $\frac{v_\omega(\{1\})}{P[\Omega\times\{1\}]} \int_\Omega Y_1 \mu(d\omega)$ which is equal to $\int_\Omega Y_1 \mu(d\omega) = E_\mu[Y_1]$; a similar argument shows that $\alpha(0) = E_\mu[Y_0]$. c) By A4, for every $x_0 \in \mathbb{R}$, $F \equiv [\omega : X(\omega) \le x_0]$ is in $\mathcal{F}$, and thus c) follows from a).

Proposition 2 If Conditions A1-A5, B2, C1, and D2 hold, then:

a) $\forall F \in \mathcal{F}$, $P[F \times \Delta | V = v]$ is continuous in v at $v = 0$

b) $E_P[Y|V = 0] - \lim_{\varepsilon\to 0^+} E_P[Y|V = -\varepsilon] = E_P[Y_1 - Y_0 | V = 0] = E_\mu\left[\frac{f_\omega(0)}{f(0)}(Y_1 - Y_0)\right] \equiv ATE^*$

c) $\forall x_0 \in \mathbb{R}$, $P[X \le x_0 | V = v]$ is continuous in v at $v = 0$

Proof. a) Fix $F \in \mathcal{F}$, and consider the function $\alpha : \Omega \times \Delta \to \mathbb{R}$, $\alpha(z, \delta) \equiv \frac{\int_F f_\omega(\delta)\mu(d\omega)}{f(\delta)}$. It suffices to show 1) that $\alpha(z, \delta)$ is a version of the conditional probability of $F \times \Delta$ given $\mathcal{G}$, and 2) that $\alpha(z, \delta)$ is continuous in δ on $\mathbb{R}$. First, for each $\Omega \times A$ we have – by 18.20.c and 18.20.d of Billingsley

(1995) – $\int_{\Omega\times A}\alpha(z,\delta)\,P(d(z,\delta)) = \int_A \frac{\int_F f_\omega(\delta)\mu(d\omega)}{f(\delta)} v(d\delta)$, where v is a probability measure defined by $v(B) = \int_\Omega v_\omega(B)\,\mu(d\omega)$, for all $B\in\mathcal{D}$. v has density f with respect to Lebesgue measure because for all $B\in\mathcal{D}$, $\int_B f(\delta)\,d\delta = \int_B[\int_\Omega f_\omega(\delta)\,\mu(d\omega)]d\delta = \int_\Omega[\int_B f_\omega(\delta)\,d\delta]\mu(d\omega) = \int_\Omega v_\omega(B)\,\mu(d\omega)$, by Fubini's theorem, and because $f_\omega(\delta)$ is a density of v_ω. Thus, by theorem 16.11 of Billingsley (1995), $\int_A \frac{\int_F f_\omega(\delta)\mu(d\omega)}{f(\delta)} v(d\delta) = \int_A[\int_F f_\omega(\delta)\,\mu(d\omega)]d\delta$, which equals $\int_F[\int_A f_\omega(\delta)\,d\delta]\mu(d\omega)$, by Fubini's theorem. This equals $\int_F v_\omega(A)\,\mu(d\omega) = P[F\times A]$, because f_ω is a density and by 18.20.c of Billingsley (1995).

Second, to show continuity of $\alpha(z,\delta)$, it suffices to show that for any $F\in\mathcal{F}$ and any sequence $\delta_n\to 0$, $\int_F f_\omega(\delta_n)\,\mu(d\omega)\to\int_F f_\omega(0)\,\mu(d\omega)$. This follows from dominated convergence, noting that $f_\omega(\delta_n)\le g_\omega$, if $g_\omega\equiv\sup_n f_\omega(\delta_n)$, which is finite for each ω, because $f_\omega(\delta_n)$ converges to $f_\omega(0)$, by D2.

b) Consider the function $\beta:\Omega\times\Delta\to\mathbb{R}$, $\beta(z,\delta) = \int_\Omega Y\frac{f_\omega(\delta)}{f(\delta)}\mu(d\omega)$. It suffices to show that 1) $\beta(z,\delta)$ is a version of the conditional expectation of Y given $\mathcal{G}$, and 2) $\beta(z,0) = E_P[Y_1|V=0] = E_\mu\left[\frac{f_\omega(\delta)}{f(\delta)}Y_1\right]$ and $\lim_{\varepsilon\to 0^+}\beta(z,-\varepsilon) = E_P[Y_0|V=0] = E_\mu\left[\frac{f_\omega(\delta)}{f(\delta)}Y_0\right]$. First, for all $\Omega\times A\in\mathcal{G}$, we have $\int_{\Omega\times A}\beta(z,\delta)\,P(d(z,\delta)) = \int_A[\int_\Omega Y\frac{f_\omega(\delta)}{f(\delta)}\mu(d\omega)]v(d\delta)$ by 18.20.c and 18.20.d of Billingsley (1995). This is equal to $\int_\Omega[\int_A Y\frac{f_\omega(\delta)}{f(\delta)}v(d\delta)]\mu(d\omega) = \int_\Omega[\int_A Yf_\omega(\delta)\,d\delta]\mu(d\omega)$ because v has density f (see above). This is equal to $\int_\Omega[\int_A Yv_\omega(d\delta)]\mu(d\omega) = \int_{\Omega\times A} YP(d(\omega,\delta))$, because v_ω has density f_ω, and by 18.20.c of Billingsley (1995). Second, let $\delta=0$. $\int_\Omega Y\frac{f_\omega(0)}{f(0)}\mu(d\omega) = \int_\Omega Y_1\frac{f_\omega(0)}{f(0)}\mu(d\omega) = E_P[Y_1|V=0]$, by the definition of Y, and the same argument above. Also, $\int_\Omega Y_1\frac{f_\omega(0)}{f(0)}\mu(d\omega) = E_\mu\left[\frac{f_\omega(0)}{f(0)}Y_1\right]$. Finally, let $\delta_n<0$, $\delta_n\to 0$. $\frac{f_\omega(\delta_n)}{f(\delta_n)}\to\frac{f_\omega(0)}{f(0)}$, by D2. Need to show $\lim_n\int_\Omega Y_0\frac{f_\omega(\delta_n)}{f(\delta_n)}\mu(d\omega) = \int_\Omega Y_0\frac{f_\omega(0)}{f(0)}\mu(d\omega)$. This follows from dominated convergence with $|Y_0\frac{f_\omega(\delta_n)}{f(\delta_n)}|$ dominated by $|Y_0\frac{g_\omega}{\inf_n f(\delta_n)}|$ (same g_ω as above).By the same argument as above, $\int_\Omega Y_0\frac{f_\omega(0)}{f(0)}\mu(d\omega) = E_P[Y_0|V=0] = E_\mu\left[\frac{f_\omega(0)}{f(0)}Y_0\right]$.

c) By A4, for every $x_0\in\mathbb{R}$, $F\equiv[\omega : X(\omega)\le x_0]$ is in $\mathcal{F}$, and thus c) follows from a).

Proposition 3

If Conditions A1-A5, B2, C2, and D2 hold, then:

a) and c) of Proposition 2 are true, and

b) $E_P[Y|V=0] - \lim_{\varepsilon \to 0^+} E_P[Y|V=-\varepsilon] = E_\mu\left[\frac{f_\omega(0)}{f(0)}(Y^+ - Y^-)\right] \equiv ATE^{**}$

Proof. For a) and c), see the proof to Proposition 2. b) First, following the argument the proof to Proposition 2, $\beta(z,\delta)$ is a version of the conditional expectation of Y given $\mathcal{G}$. Second, let $\delta = 0$. $\int_\Omega Y \frac{f_\omega(0)}{f(0)} \mu(d\omega) = \int_\Omega Y^+ \frac{f_\omega(0)}{f(0)} \mu(d\omega) = E_\mu\left[\frac{f_\omega(0)}{f(0)} Y^+\right]$. Finally, let $\delta_n < 0$, $\delta_n \to 0$. $\frac{f_\omega(\delta_n)}{f(\delta_n)} \to \frac{f_\omega(0)}{f(0)}$, by D2. Need to show $\lim_n \int_\Omega Y \frac{f_\omega(\delta_n)}{f(\delta_n)} \mu(d\omega) = \int_\Omega Y^- \frac{f_\omega(0)}{f(0)} \mu(d\omega)$. This follows from dominated convergence with $|Y \frac{f_\omega(\delta_n)}{f(\delta_n)}|$ dominated by $|h_\omega \frac{g_\omega}{\inf_n f(\delta_n)}|$ (same g_ω as above) where $h_\omega \equiv \sup_n |y(\omega,\delta_n)|$, which is finite for each ω, because $y(\omega,\delta_n) \to Y^-$, by C2.It follows that $\int_\Omega Y^- \frac{f_\omega(0)}{f(0)} \mu(d\omega) = E_\mu\left[\frac{f_\omega(0)}{f(0)} Y^-\right]$.