Online Appendix

to

“Valid t-ratio Inference for IV"

For supplementary material, including updates to the original Online Appendix and a STATA package to compute tF critical values/standard error adjustments, please visit:
http://www.princeton.edu/∼davidlee/wp/SupplementarytF.html

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A.1 Notation for Appendix

This appendix collects proofs of the results claimed in the text. In the interest of being self-contained, we recapitulate our general notation. For a representative observation, the model is

$$\begin{aligned} Y &= X\beta + u \\ X &= Z\pi + v \end{aligned}$$

for an outcome Y, a single endogenous regressor X, and a single instrument Z. While suppressed, the model above allows for constants and covariates, as long as we interpret the triple (X,Y,Z) as the residuals from a regression on any covariates W and a constant.[48]

The IV estimator itself is $\widehat{\beta} = \mathbf{Z}'\mathbf{Y}/\mathbf{Z}'\mathbf{X}$, where **bold** denotes a vector, the first-stage estimator is $\widehat{\pi} = \mathbf{Z}'\mathbf{X}/\mathbf{Z}'\mathbf{Z}$, and the reduced-form estimator is $\widehat{\pi\beta} = \mathbf{Z}'\mathbf{Y}/\mathbf{Z}'\mathbf{Z}$. Note that we write the reduced-form coefficient as $\widehat{\pi\beta}$ because the reduced-form coefficient is numerically equal to the product of $\widehat{\pi}$ and $\widehat{\beta}$. The IV fitted residual is $\widehat{u} = Y - X\widehat{\beta}$, and we analogously write $\widehat{v}$ and $\widehat{\varepsilon}$ for the fitted residual from the first-stage and reduced-form regressions; we denote population analogues by v and ε, respectively.

Throughout we will be examining HAC variance estimators. Consider, for example, the first-stage estimated variance, given by

$$\widehat{V}_N(\widehat{\pi}) = (\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{v})(\mathbf{Z}'\mathbf{Z})^{-1} = \frac{\widehat{V}(Z\widehat{v})}{(\mathbf{Z}'\mathbf{Z})^2}$$

In the display above, we are using the notation $\widehat{V}_N(\cdot)$ to convey the estimated variance for a parameter. In contrast, we write $\widehat{V}(Z\widehat{v})$ (without a subscript of N) as a unifying notation for the "meat" of the sandwich variance estimator in order to cover the multitude of approaches to variance estimation encountered in applications: homoskedastic standard errors, heteroskedasticity-robust standard errors, clustered standard errors, two-way clustered standard errors, time-series approaches such as

[48] Algebra and the partitioned inverse theorem shows that ignoring covariates and constants leaves point estimates and fitted residuals (and thus variance estimators) the same, as long as we reinterpret the trio (X,Y,Z) as the residuals from a regression of each of them on W. This is a simple extension of the same point made in the regression context by Theorem 4.1 of Lovell (1963) and is an application of Theorem 6.1 of Newey and McFadden (1994).

Newey-West (1987), or yet other HAC approaches.[49] Moreover, and slightly less standardly, we will use a similar notation for covariance.

Beginning with Lemma 6, below, we will invoke three high-level assumptions that we now state.

Assumption 1 (Asymptotically Finite First-Stage)**.** $\pi_N \sqrt{\mathbf{Z}'\mathbf{Z}} \xrightarrow{p} \pi_{ZZ}$.

Note that in the main text, we wrote the true first stage parameter as π, but here we clarify that in a weak IV framework, the asymptotic sequence is one in which the parameter π shrinks towards zero. In this appendix, we clarify this with notation by writing π_N, where the parameter sequence satisfies Assumption 1.

Assumption 2 (Asymptotic Normality)**.** $\frac{1}{\sqrt{\mathbf{Z}'\mathbf{Z}}} \begin{pmatrix} \mathbf{Z}'\varepsilon \\ \mathbf{Z}'\mathbf{v} \end{pmatrix} \xrightarrow{d} N(\mathbf{0}, \Sigma)$, with $\Sigma \equiv \begin{pmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon v} \\ \sigma_{\varepsilon v} & \sigma_v^2 \end{pmatrix}$.

Assumption 2 is sufficient to imply that the first-stage and reduced-form estimators are consistent and asymptotically normal.

Assumption 3 (Consistent Variance and Covariance Estimators)**.**

$$\begin{aligned} \widehat{V}(Z\widehat{\varepsilon})/N - V\left(\mathbf{Z}'\varepsilon/\sqrt{N}\right) &\xrightarrow{p} 0 \\ \widehat{V}(Z\widehat{v})/N - V\left(\mathbf{Z}'\mathbf{v}/\sqrt{N}\right) &\xrightarrow{p} 0 \\ \widehat{C}(Z\widehat{\varepsilon}, Z\widehat{v})/N - C\left(\mathbf{Z}'\varepsilon/\sqrt{N}, \mathbf{Z}'\mathbf{v}/\sqrt{N}\right) &\xrightarrow{p} 0 \end{aligned}$$

[49] For example, if the variance matrix of the errors is taken to be spherical, we would use

$$\widehat{V}(Z\widehat{v}) = \widehat{\sigma}^2 \left(\sum_i Z_i^2 \right)$$

where $\widehat{\sigma}^2 = \frac{1}{N}\sum_i \widehat{v}_i^2$, and the sum is over the data. In contrast, if the errors were taken to be heteroskedastic, then we would use

$$\widehat{V}(Z\widehat{v}) = \sum_i Z_i^2 \widehat{v}_i^2$$

If a clustered approach is taken, with groups indexed by j and observations within group indexed by i, we would instead use

$$\widehat{V}(Z\widehat{v}) = \sum_j \mathbf{Z}_j \widehat{\mathbf{v}}_j \widehat{\mathbf{v}}_j' \mathbf{Z}_j'$$

where $\mathbf{Z}_j$ is the stack of instruments for group j, $\widehat{\mathbf{v}}_j$ is the stack of estimated residuals for group j, and the sum is over the clusters j. For two-way clustering (e.g., Cameron, Gelbach, and Miller 2011) or time-series approaches (e.g., Newey-West 1987), the results are *mutatis mutandis*.

Assumption 3 simply states that the variance estimators being employed are consistent.

A.2 Relationship between IV and reduced-form variance estimators

Lemma 1 (Relationship Between IV and Reduced-Form Variance Estimators)**.**

$$\widehat{V}_N(\hat{\beta}) \quad = \quad \frac{1}{\widehat{\pi}^2}\left\{\widehat{V}_N(\widehat{\pi\beta}) - 2\widehat{\beta}\widehat{C}_N(\widehat{\pi\beta},\widehat{\pi}\,) + \widehat{\beta}^2\widehat{V}_N(\widehat{\pi})\right\}$$

Proof: In the just-identified case with a single endogenous regressor, the standard formula for the estimated IV variance reduces so that

$$\widehat{V}_N(\hat{\beta}) \quad = \quad (\mathbf{Z}'\mathbf{X})^{-1}\widehat{V}(Z\widehat{u})(\mathbf{X}'\mathbf{Z})^{-1} = \widehat{V}(Z\widehat{u})/(\mathbf{Z}'\mathbf{X})^2$$

Similarly, the estimated variances and covariances for the reduced-form coefficient $\widehat{\pi\beta}$ and the first-stage coefficient $\widehat{\pi}$ are given by

$$\begin{aligned}
\widehat{V}_N(\widehat{\pi\beta}) &= \widehat{V}(Z\widehat{\varepsilon})/(\mathbf{Z}'\mathbf{Z})^2 \\
\widehat{V}_N(\widehat{\pi}) &= \widehat{V}(Z\widehat{v})/(\mathbf{Z}'\mathbf{Z})^2 \\
\widehat{C}_N(\widehat{\pi\beta},\widehat{\pi}) &= \widehat{C}(Z\widehat{\varepsilon},Z\widehat{v})/(\mathbf{Z}'\mathbf{Z})^2
\end{aligned}$$

where $\widehat{\varepsilon} \equiv Y - Z\widehat{\pi\beta}$ and $\widehat{v} \equiv X - Z\widehat{\pi}$ are the reduced-form and first-stage fitted residuals, respectively. For a representative observation we have $\widehat{\varepsilon} = Y - X\widehat{\beta} + X\widehat{\beta} - Z\widehat{\pi\beta} = \hat{u} + \hat{v}\hat{\beta}$, and since $\widehat{\beta}$ does not vary by observation, we have $\widehat{u}\widehat{u}' = \widehat{\varepsilon}\widehat{\varepsilon}' - 2\widehat{\beta}\widehat{\varepsilon}\widehat{v}' + \widehat{\beta}^2\widehat{v}\widehat{v}'$ which in turn implies that the middle factors of the various sandwich variance estimates are all functionally related:

$$\widehat{V}(Z\widehat{u}) \quad = \quad \widehat{V}(Z\widehat{\varepsilon}) - 2\widehat{\beta}\widehat{C}(Z\widehat{\varepsilon},Z\widehat{v}) + \widehat{\beta}^2\widehat{V}(Z\widehat{v})$$

Putting these results together, we see that

$$\begin{aligned}
\widehat{\pi}^2\widehat{V}_N(\widehat{\beta}) &= \left(\frac{\mathbf{Z}'\mathbf{X}}{\mathbf{Z}'\mathbf{Z}}\right)^2\widehat{V}_N(\widehat{\beta}) = \frac{\widehat{V}(Z\widehat{u})}{(\mathbf{Z}'\mathbf{Z})^2} = \frac{\widehat{V}(Z\widehat{\varepsilon})}{(\mathbf{Z}'\mathbf{Z})^2} - 2\widehat{\beta}\frac{\widehat{C}(Z\widehat{\varepsilon},Z\widehat{v})}{(\mathbf{Z}'\mathbf{Z})^2} + \widehat{\beta}^2\frac{\widehat{V}(Z\widehat{v})}{(\mathbf{Z}'\mathbf{Z})^2} \\
&= \widehat{V}_N(\widehat{\pi\beta}) - 2\widehat{\beta}\widehat{C}_N(\widehat{\pi\beta},\widehat{\pi}) + \widehat{\beta}^2\widehat{V}_N(\widehat{\pi})
\end{aligned}$$

and the result follows after dividing both sides of the above by $\widehat{\pi}^2$. ⊠

Lemma 2 (t-test for IV)**.**

$$\hat{t} \equiv \hat{t}(\beta_0) = \frac{\widehat{\beta} - \beta_0}{\hat{se}(\hat{\beta})} = \frac{|\widehat{\pi}|(\widehat{\beta} - \beta_0)}{\sqrt{\hat{V}_N(\widehat{\pi\beta}) - 2\hat{\beta}\hat{C}_N(\widehat{\pi\beta}, \hat{\pi}) + \widehat{\beta}^2 \hat{V}_N(\hat{\pi})}}$$

where $\hat{se}(\cdot) = \sqrt{\widehat{V}_N(\cdot)}$ *is notation for the estimated standard error of a parameter.*

PROOF: The result follows immediately from Lemma 1. ⊠

Remark (Dependence on β_0)**.** Note that while we follow standard econometric practice and write $\hat{t}$ for the estimated t-statistic, it is, of course, true that the t-statistic depends on the parameter value being tested (i.e., β_0). For statistics other than the t-statistic, we will emphasize dependence on β_0 by writing them as functions of β_0. Note that in our notation, β_0 is not necessarily the true parameter value but could also be a hypothesized—but false—parameter value (i.e., there is no reason to assume $\beta = \beta_0$).

Remark (Form of the F statistic)**.** In the just-identified context, we have

$$\widehat{F} \equiv \frac{\widehat{\pi}^2}{\widehat{V}(\widehat{\pi})} = \frac{\left((\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\right)^2}{(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{v})(\mathbf{Z}'\mathbf{Z})^{-1}} = \frac{(\mathbf{Z}'\mathbf{X})^2}{\widehat{V}(Z\widehat{v})}$$

and

$$\widehat{f} \quad = \quad \frac{\widehat{\pi}}{\sqrt{\widehat{V}(\widehat{\pi})}} = \frac{\mathbf{Z}'\mathbf{X}}{\sqrt{\widehat{V}(Z\widehat{v})}}$$

where $\widehat{f}$ is the signed t-test on the exclusion of the instrument in the first-stage regression. Note that in this context $\widehat{F}$ is the same as the "effective F statistic" described in Olea and Pflueger (2013).

A.3 t-ratio form of Anderson-Rubin statistic

Lemma 3 (Similarity of the AR-statistic and the t-statistic)**.** *The AR test statistic can be written in a form that is similar to the formula for the t-statistic for the structural parameter, but with a variance estimator that imposes the null:*

$$\hat{t}_{AR}(\beta_0) \quad = \quad \frac{\widehat{\pi}(\widehat{\beta} - \beta_0)}{\hat{\text{se}}(\widehat{\pi}(\widehat{\beta} - \beta_0))} = \frac{\widehat{\pi}(\widehat{\beta} - \beta_0)}{\sqrt{\widehat{V}_N(\widehat{\pi\beta}) - 2\beta_0\widehat{C}_N(\widehat{\pi\beta}, \widehat{\pi}) + \beta_0^2\widehat{V}_N(\widehat{\pi})}}$$

PROOF: This result is related to Proposition 2 of Van de Sijpe and Windmeijer (2021) and was noted previously in Theorem 5 of Feir, Lemieux and Marmer (2016). For the former, the result is a special case; for the latter the proof is omitted. In our context, the result follows straightforwardly because for any given approach to variance estimation, the *AR* test of the null hypothesis $\beta = \beta_0$ can be obtained by: (1) forming the residual $u_0 = Y - X\beta_0$, (2) regressing u_0 on Z, and (3) using an F test to test the null hypothesis that the coefficient on Z in that regression is zero, where the F test adopts the desired approach to inference for the original IV model.[50]

This gives rise to concepts of the *AR* coefficient and the *AR* standard error, by which we mean simply the coefficient and standard error from the regression in that third step, respectively. Consider each in turn. The *AR* coefficient is

$$\frac{\mathbf{Z}'(\mathbf{Y}-\mathbf{X}\beta_0)}{\mathbf{Z}'\mathbf{Z}} \quad = \quad \widehat{\pi\beta} - \widehat{\pi}\beta_0 = \widehat{\pi}(\widehat{\beta} - \beta_0)$$

where the last result follows since the reduced-form coefficient $\widehat{\pi\beta}$ is the product of the first-stage coefficient $\widehat{\pi}$ and the estimated structural parameter $\widehat{\beta}$. The *AR* standard error can be thought of in two ways. First, and more standardly, let $\widehat{u}_0$ denote the fitted *AR* regression residual. Then the estimated *AR* standard error is the square root of

$$(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{u}_0)(\mathbf{Z}'\mathbf{Z})^{-1} = \frac{\widehat{V}(Z\widehat{u}_0)}{(\mathbf{Z}'\mathbf{Z})^2}$$

Second, since the *AR* coefficient is a linear combination of the reduced-form and first-stage coefficients, as shown above, it is the square root of

$$\widehat{V}_N(\widehat{\pi\beta}) - 2\beta_0\widehat{C}_N(\widehat{\pi\beta}, \widehat{\pi}) + \beta_0^2\widehat{V}_N(\widehat{\pi})$$

The lemma follows from the second result. We will use the first characterization of the *AR* standard error in Lemma 4 below. ⊠

In light of Lemmas 2 and 3, it is not surprising that there is a numerical equivalence allowing one to obtain $\hat{t}$ from $\hat{t}_{AR}(\beta_0)$ and other quantities, the subject to which we turn next.

[50] If covariates are part of the model, then the degrees of freedom for the F test should be adjusted to reflect the dimension of the covariates W that were partialled out in the first step.

A.4 From $\hat{t}_{AR}(\beta_0)$ to $\hat{t}$

Lemma 4 (Dependence of $\hat{t}$ on $\hat{t}_{AR}(\beta_0)$, $\widehat{\rho}(\beta_0)$ and $\widehat{f}$)**.**

$$\hat{t}^2 = \frac{\hat{t}^2_{AR}(\beta_0)}{1 - 2\widehat{\rho}(\beta_0)\frac{\hat{t}_{AR}(\beta_0)}{\widehat{f}} + \frac{\hat{t}^2_{AR}(\beta_0)}{\widehat{f}^2}}$$

where

$$\widehat{\rho}(\beta_0) \equiv \frac{\widehat{C}(Z\widehat{u}_0, Z\widehat{v})}{\sqrt{\widehat{V}(Z\widehat{u}_0)}\sqrt{\widehat{V}(Z\widehat{v})}}$$

and as emphasized in the second remark after Lemma 1, $\widehat{f}$ is the t-ratio test on the exclusion of the instrument in the first-stage regression, i.e., $\widehat{F} = \widehat{f}^2$.

Proof: We first note that the *IV* or structural residual combines the *AR* regression residual $\hat{u}_0$ with the first-stage residual $\hat{v}$. To see this, recall that the outcome for the *AR* regression is $u_0 = Y - X\beta_0$ and observe that the *AR* regression's predicted value is $Z\widehat{\pi}(\widehat{\beta} - \beta_0)$. But then

$$\widehat{u}_0 = Y - X\beta_0 - Z\widehat{\pi}(\widehat{\beta} - \beta_0)$$

Then add and subtract $X\beta_0$ and $Z\widehat{\pi}(\widehat{\beta} - \beta_0)$ from the *IV* residual $\widehat{u}$ to obtain

$$\begin{aligned} \widehat{u} &= Y - X\widehat{\beta} = Y - X\widehat{\beta} + X\beta_0 - X\beta_0 + Z\widehat{\pi}(\widehat{\beta} - \beta_0) - Z\widehat{\pi}(\widehat{\beta} - \beta_0) \\ &= \widehat{u}_0 - \widehat{v}(\widehat{\beta} - \beta_0) \end{aligned}$$

As in the proof of Lemma 1, and for the same reasons, we can use the result above to re-write the meat of the IV variance estimate:

$$\widehat{V}(Z\widehat{u}) = \widehat{V}(Z\widehat{u}_0) - 2(\widehat{\beta} - \beta_0)\widehat{C}(Z\widehat{u}_0, Z\widehat{v}) + (\widehat{\beta} - \beta_0)^2\widehat{V}(Z\widehat{u}_0)$$

Next, note that $\hat{t}^2$ and $\hat{t}^2_{AR}(\beta_0)$ differ only to the extent $\widehat{V}(Z\widehat{u})$ and $\widehat{V}(Z\widehat{u}_0)$ differ:

$$\begin{aligned} \hat{t}^2 &= \frac{(\widehat{\beta} - \beta_0)^2}{\widehat{V}(Z\widehat{u})/(\mathbf{Z}'\mathbf{X})^2} \\ \hat{t}^2_{AR}(\beta_0) &= \frac{\widehat{\pi}^2(\widehat{\beta} - \beta_0)^2}{\widehat{V}(Z\widehat{u}_0)/(\mathbf{Z}'\mathbf{Z})^2} = \frac{(\widehat{\beta} - \beta_0)^2}{\widehat{V}(Z\widehat{u}_0)/(\mathbf{Z}'\mathbf{X})^2} \end{aligned}$$

Then, using the above result on $\widehat{V}(Z\widehat{u})$, we obtain

$$\begin{aligned}\frac{\hat{t}^2}{\hat{t}_{AR}^2(\beta_0)} &= \frac{\widehat{V}(Z\widehat{u}_0)}{\widehat{V}(Z\widehat{u})} = \frac{\widehat{V}(Z\widehat{u}_0)}{\widehat{V}(Z\widehat{u}_0) - 2(\widehat{\beta}-\beta_0)\widehat{C}(Z\widehat{u}_0, Z\widehat{v}) + (\widehat{\beta}-\beta_0)^2\widehat{V}(Z\widehat{v})} \\ &= \frac{1}{1 - 2\widehat{\rho}(\beta_0)(\widehat{\beta}-\beta_0)\sqrt{\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u}_0)}} + (\widehat{\beta}-\beta_0)^2\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u}_0)}}\end{aligned}$$

Finally, note that

$$\frac{\hat{t}_{AR}(\beta_0)}{\widehat{f}} = \frac{\widehat{\pi}(\widehat{\beta}-\beta_0)}{\sqrt{\widehat{V}(Z\widehat{u}_0)}\Big/(\mathbf{Z}'\mathbf{Z})}\frac{\sqrt{\widehat{V}(Z\widehat{v})}\Big/(\mathbf{Z}'\mathbf{Z})}{\widehat{\pi}} = (\widehat{\beta}-\beta_0)\sqrt{\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u}_0)}}$$

and the result follows. ⊠

A.5 From $\hat{t}$ to $\hat{t}_{AR}^2$

Lemma 5 (Dependence of $\hat{t}_{AR}$ on $\hat{t}$, $\widehat{\widehat{\rho}}$, and $\widehat{F}$)**.**

$$\hat{t}_{AR}^2 = \frac{\hat{t}^2}{1 + 2\widehat{\widehat{\rho}}\frac{\hat{t}}{\sqrt{\widehat{F}}} + \frac{\hat{t}^2}{\widehat{F}}}$$

where

$$\widehat{\widehat{\rho}} = \frac{\widehat{C}(Z\widehat{u}, Z\widehat{v})}{\sqrt{\widehat{V}(Z\widehat{u})}\sqrt{\widehat{V}(Z\widehat{v})}}$$

Proof: The proof is similar to that of Lemma 4, but with some subtle differences. First, as in Lemma 4 we have $\widehat{u}_0 = \widehat{u} + (\widehat{\beta}-\beta_0)\widehat{v}$, which allows us to decompose $\widehat{V}(Z\widehat{u}_0)$, yielding

$$\frac{\hat{t}_{AR}^2(\beta_0)}{\hat{t}^2} = \frac{\widehat{V}(Z\widehat{u})}{\widehat{V}(Z\widehat{u}_0)} = \frac{\widehat{V}(Z\widehat{u})}{\widehat{V}(Z\widehat{u}) + 2(\widehat{\beta}-\beta_0)\widehat{C}(Z\widehat{u}, Z\widehat{v}) + (\widehat{\beta}-\beta_0)^2\widehat{V}(Z\widehat{v})}$$

The result follows after dividing top and bottom by $\widehat{V}(Z\widehat{u})$ and recognizing that

$$\frac{\hat{t}}{\sqrt{\widehat{F}}} \quad = \quad \left(\widehat{\beta}-\beta_0\right)\sqrt{\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u})}}$$

Note that unlike Lemma 4, the dependence is 1) on $\widehat{F}$ as opposed to $\widehat{f}$ and 2) on the (generalized) correlation between the IV residual and the first-stage residual, as opposed to the (generalized) correlation between the AR residual and the first-stage residual. ⊠

A.6 Rejection probabilities for tests based on t-ratio

We next derive an asymptotic version of Lemma 3.

Lemma 6 (Limiting Distribution of $\hat{t}^2$ Under Weak IV Asymptotics)**.** *Under Assumptions 1, 2, and 3, we have*

$$\hat{t}^2 \quad \overset{d}{\to} \quad \frac{t_{AR}^2(\beta_0)}{1-2\rho(\beta_0)\frac{t_{AR}(\beta_0)}{f}+\frac{t_{AR}^2(\beta_0)}{f^2}} \equiv t^2(\beta_0)$$

where

$$\rho(\beta_0) \quad = \quad \lim_{N\to\infty}\frac{C\left(\frac{1}{N}\mathbf{Z}'\mathbf{u_0},\frac{1}{N}\mathbf{Z}'\mathbf{v}\right)}{\sqrt{V\left(\frac{1}{N}\mathbf{Z}'\mathbf{u_0}\right)}\sqrt{V\left(\frac{1}{N}\mathbf{Z}'\mathbf{v}\right)}}$$

and $t_{AR}(\beta_0)$ and f are distributed jointly normal with unit variances, correlation $\rho(\beta_0)$, and means that are given below.

Proof: We will show that regardless of whether β_0 is the true parameter or not,

$$\begin{pmatrix}\hat{t}_{AR}(\beta_0)\\ \widehat{f}\end{pmatrix} \quad \overset{d}{\to} \quad \begin{pmatrix}t_{AR}(\beta_0)\\ f\end{pmatrix} \sim N\left(f_0\begin{pmatrix}\frac{\Delta(\beta_0)}{\sqrt{1+2\rho\cancel{(\beta_0)}\Delta(\beta_0)+\Delta^2(\beta_0)}}\\ 1\end{pmatrix},\begin{pmatrix}1 & \rho(\beta_0)\\ \rho(\beta_0) & 1\end{pmatrix}\right)$$

where

$$\begin{aligned}\Delta(\beta_0) \quad &= \quad (\beta-\beta_0)\frac{\sqrt{\sigma_v^2}}{\sqrt{\sigma_\varepsilon^2-2\beta\sigma_{\varepsilon v}+\beta^2\sigma_v^2}}\\ f_0 \quad &= \quad \frac{\pi_{ZZ}}{\sqrt{\sigma_v^2}}\end{aligned}$$

* Corrected September 20, 2023

from which the result follows.

Since $\varepsilon = u + v\beta$, the *AR* outcome $u_0 = Y - X\beta_0$ can be written as

$$
\begin{aligned}
u_0 &= X\beta + u - X\beta_0 = (Z\pi + v)(\beta - \beta_0) + u = Z\pi(\beta - \beta_0) + v(\beta - \beta_0) + \varepsilon - v\beta \\
&= Z\pi(\beta - \beta_0) + \varepsilon - v\beta_0
\end{aligned}
$$

which means the *AR* coefficient is given by

$$
\begin{aligned}
(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{u}_0 &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left[\mathbf{Z}'\pi_N(\beta - \beta_0) + \varepsilon - \mathbf{v}\beta_0\right] \\
&= \pi(\beta - \beta_0) + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left(\varepsilon - \beta_0\mathbf{v}\right)
\end{aligned}
$$

The *AR* standard error is the square root of the estimated variance of the above, i.e.:

$$
(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}\left(\left(\mathbf{Z}'\mathbf{Z}\right)^{-1/2}\mathbf{Z}'(\varepsilon - \beta_0 v)\right) = (\mathbf{Z}'\mathbf{Z})^{-1}\left(\widehat{\sigma}_\varepsilon^2 - 2\beta_0\widehat{\sigma}_{\varepsilon v} + \beta_0^2\widehat{\sigma}_v^2\right)
$$

and therefore from Assumption 1 the *AR* statistic is given by

$$
\begin{aligned}
\hat{t}_{AR}(\beta_0) &= \frac{\pi_N(\beta - \beta_0) + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left(\varepsilon - \beta_0 v\right)}{\sqrt{(\mathbf{Z}'\mathbf{Z})^{-1}\left(\widehat{\sigma}_\varepsilon^2 - 2\beta_0\widehat{\sigma}_{\varepsilon v} + \beta_0^2\widehat{\sigma}_v^2\right)}} \\
&= \frac{\pi_{ZZ}(\beta - \beta_0)}{\sqrt{\sigma_\varepsilon^2 - 2\beta_0\sigma_{\varepsilon v} + \beta_0^2\sigma_v^2}} + \frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'\left(\varepsilon - \beta_0 v\right)}{\sqrt{\widehat{\sigma}_\varepsilon^2 - 2\beta_0\widehat{\sigma}_{\varepsilon v} + \beta_0^2\widehat{\sigma}_v^2}} + o_p(1)
\end{aligned}
$$

Similarly,

$$
\widehat{f} = \frac{(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}}{\sqrt{(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{v})(\mathbf{Z}'\mathbf{Z})^{-1}}} = \frac{(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}'\pi_N + v)}{\sqrt{(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}((\mathbf{Z}'\mathbf{Z})^{-1/2}Z\widehat{v})}} = \frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'(\mathbf{Z}'\pi_N + v)}{\sqrt{\widehat{\sigma}_v^2}}
$$

Some algebra shows that

$$
C\left(\hat{t}_{AR}(\beta_0), \widehat{f}\right) = \frac{\sigma_{\varepsilon v} - \beta_0\sigma_v^2}{\sqrt{\sigma_v^2}\sqrt{\sigma_\varepsilon^2 - 2\beta_0\sigma_{\varepsilon v} + \beta_0^2\sigma_v^2}} + o_p(1)
$$

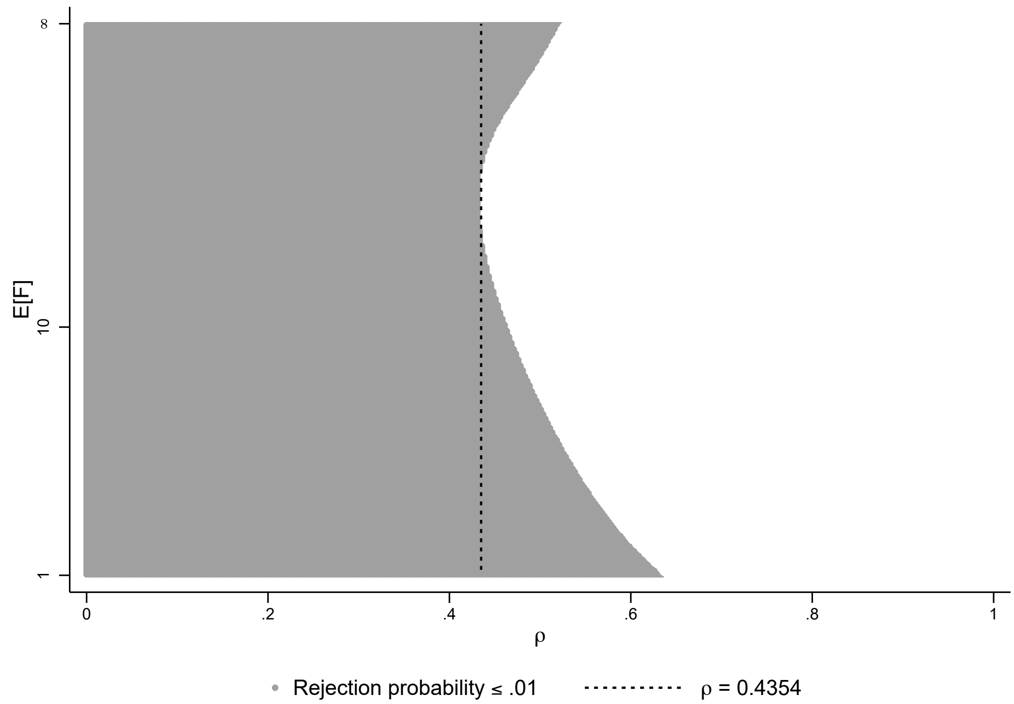
and that the first term in the above is equal to $\rho(\beta_0)$.

Putting these results together, we have

$$
\begin{pmatrix} \hat{t}_{AR}(\beta_0) - f_0 \frac{\Delta(\beta_0)}{\sqrt{1 + 2\rho\cancel{(\beta_0)}\Delta(\beta_0) + \Delta^2(\beta_0)}} \\ \widehat{f} - f_0 \end{pmatrix} = \begin{pmatrix} \frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'(\varepsilon - \beta_0 v)}{\sqrt{\widehat{\sigma}_\varepsilon^2 - 2\beta_0\widehat{\sigma}_{\varepsilon v} + \beta_0^2\widehat{\sigma}_v^2}} \\ \frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'\mathbf{v}}{\sqrt{\widehat{\sigma}_v^2}} \end{pmatrix} + o_p(1)
$$

*Corrected September 20, 2023

Figure A1: Combinations of $E[F]$, ρ for $Pr[t^2 > 2.576^2] \leq 0.01$



Vertical axis scale uses the transformation $\frac{E[F]/10}{1+E[F]/10}$. Shaded region represents all combinations of $E[F], \rho$ such that the rejection probability is less than or equal to 0.01. Dashed line is the maximum ρ such that the region to the left is shaded.

and joint asymptotic normality of $(\hat{t}_{AR}(\beta_0), \widehat{f})$ of the stated form follows from Assumptions 1, 2, and 3 and the continuous mapping theorem. ⊠

A.7 Some numerical findings and other results derived from the rejection probabilities

Result 1a. *In addition to the IV model in (1), consider the restriction that* $E[F] \geq \bar{F}$. *The smallest value of* $\bar{F}$ *such that* $\Pr\left[t^2 > 1.96^2\right] \leq 0.05$ *is 142.6 .*

Result 1b. *In addition to the IV model in (1), consider the restriction that* $|\rho| < \bar{\rho}$. *The largest value of* $\bar{\rho}$ *such that* $\Pr\left[t^2 > 1.96^2\right] \leq 0.05$ *is 0.565.*

Result 1c. *For the 1 percent level of significance, there exists no* $\bar{F}$ *such that* $\Pr\left[t^2 > 2.576^2\right] \leq 0.01$ *for all* $E[F] \geq \bar{F}$, *and the largest* $\bar{\rho}$ *such that* $\Pr\left[t^2 > 2.576^2\right] \leq$

0.01 *for all* $|\rho| \leq \bar{\rho}$ *is 0.43. The full set of values of* $|\rho|, E[F]$ *for which* $\Pr\left[t^2 > 2.576^2\right] \leq 0.01$ *is illustrated in Figure A1.*

Result 2a. $\Pr\left[\left\{t^2 > 1.96^2\right\} \cap \{F > 10\}\right] \leq 0.113$ *for all values of* $\rho, E[F]$. *This implies that confidence intervals are* $\hat{\beta}_{IV} \pm 1.96 \cdot \hat{SE}\left(\hat{\beta}_{IV}\right)$ *when* $F \geq 10$ *and* $(-\infty, \infty)$ *when* $F < 10$*, and should be interpreted as 88.7 percent confidence intervals.*

Result 2b. $\Pr\left[\left\{t^2 > 1.96^2\right\} \cap \{F \geq 104.7\}\right] \leq 0.05$ *for all values of* $\rho, E[F]$.

Result 2c. $\Pr\left[\left\{t^2 > 3.43^2\right\} \cap \{F > 10\}\right] \leq 0.05$ *for all values of* $\rho, E[F]$.

Result 2d. *Let AR be the statistic of There exists no finite threshold* $\bar{F}$ *such that* $\Pr\left[\left\{t^2 > 1.96^2\right\} \cap \{F \geq \bar{F}\}\right] + \Pr\left[\left\{AR > 1.96^2\right\} \cap \{F < \bar{F}\}\right] \leq 0.05$ *for all values of* $\rho, E[F]$.

Derivation of Results 1a-b-c, 2a-b-c-d:

Recall

$$t^2(f, t_{AR}) = \frac{f^2 t_{AR}^2}{f^2 - 2\rho_0 f t_{AR} + t_{AR}^2}$$

Lemma 7. *For* $\rho_0 = \pm 1$*, suppose* $f = f_0^* + \rho_0 t_{AR}$*. Then, for* $q > 0$*,*

$$\{t_{AR} : t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) \geq q\} = \begin{cases} (-\infty, \underline{f}_A^*] \cup [\bar{f}_A^*, \infty) & \text{if } |f_0^*| < 4\sqrt{q} \\ (-\infty, \underline{f}_A^*] \cup \{-\frac{\rho_0 f_0^*}{2}\} \cup [\bar{f}_A^*, \infty) & \text{if } |f_0^*| = 4\sqrt{q} \\ (-\infty, \underline{f}_A^*] \cup [\underline{f}_B^*, \bar{f}_B^*] \cup [\bar{f}_A^*, \infty) & \text{if } |f_0^*| > 4\sqrt{q} \end{cases}$$

where

$$\underline{f}_A^* = \frac{-\rho_0 f_0^* - \sqrt{f_0^{*2} + 4|f_0^*|\sqrt{q}}}{2}; \qquad \bar{f}_A^* = \frac{-\rho_0 f_0^* + \sqrt{f_0^{*2} + 4|f_0^*|\sqrt{q}}}{2}$$

$$\underline{f}_B^* = \frac{-\rho_0 f_0^* - \sqrt{f_0^{*2} - 4|f_0^*|\sqrt{q}}}{2}; \qquad \bar{f}_B^* = \frac{-\rho_0 f_0^* + \sqrt{f_0^{*2} - 4|f_0^*|\sqrt{q}}}{2}$$

PROOF:

$$t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) = \frac{1}{f_0^{*2}}(\rho_0 f_0^* + t_{AR})^2 t_{AR}^2$$

Let $\underline{\tau} = \min\{-\rho_0 f_0^*, 0\}$ and $\bar{\tau} = \max\{-\rho_0 f_0^*, 0\}$. Note $t^2(f_0^* + \rho_0 t_{AR}, t_{AR})$ is a quartic polynomial, monotonically decreasing on $(-\infty, \underline{\tau})$ and $(-\frac{\rho_0 f_0^*}{2}, \bar{\tau})$ and monotonically increasing on $(\underline{\tau}, -\frac{f_0}{2})$ and $(\bar{\tau}, \infty)$. So the solutions to $t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) = q$ are as follows:

$$\{t_{AR} : t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) = q\} = \begin{cases} \{\underline{f}_A^*, \bar{f}_A^*\} & \text{if } |f_0^*| < 4\sqrt{q} \\ \{\underline{f}_A^*, \bar{f}_A^*, -\frac{\rho_0 f_0^*}{2}\} & \text{if } |f_0^*| = 4\sqrt{q} \\ \{\underline{f}_A^*, \bar{f}_A^*, \underline{f}_B^*, \bar{f}_B^*\} & \text{if } |f_0^*| > 4\sqrt{q} \end{cases}$$

The result follows. ⊠

Remarks:

1. This result characterizes the rejection region for Wald when $\rho_0 = \pm 1$ under the null and alternative.

2. Our asymptotic approximation is based on: $\begin{pmatrix} t_{AR} \\ f \end{pmatrix} \sim N\left(\begin{pmatrix} t_1 \\ f_0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix}\right)$ When $\rho_0 = \pm 1$, $f = f_0 - \rho_0 t_1 + \rho_0 t_{AR}$. So, Lemma 7 can be used to characterize the corresponding Wald rejection region with $f_0^* = f_0 - \rho_0 t_1$. Note that under the null, $t_1 = 0$ and $f_0^* = f_0$.

3. Under the null, $f_0^* = f_0$, so define

$$\underline{f}_A = \frac{-\rho_0 f_0 - \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{2}; \qquad \bar{f}_A = \frac{-\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{2}$$

$$\underline{f}_B = \frac{-\rho_0 f_0 - \sqrt{f_0^2 - 4|f_0|\sqrt{q}}}{2}; \qquad \bar{f}_B = \frac{-\rho_0 f_0 + \sqrt{f_0^2 - 4|f_0|\sqrt{q}}}{2}$$

   Then,

$$\Pr_{f_0, \rho_0 = \pm 1}(t^2 \geq q) = \begin{cases} \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) & \text{if } |f_0| \leq 4\sqrt{q} \\ \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B) & \text{if } |f_0| > 4\sqrt{q} \end{cases}$$

   where Φ denotes the standard normal c.d.f.

4. This result can also be used to characterize $\{t_{AR} : t^2 \geq q, f^2 \geq \bar{F}\}$ by intersecting the set given with $(-\infty, -\sqrt{\bar{F}} - \rho_0 f_0^*] \cup [\sqrt{\bar{F}} - \rho_0 f_0^*, \infty)$.

Corollary 1. *Under the null,*

(a) $\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = \Pr_{-f_0,\rho_0=-1}(t^2 \geq q, f^2 \geq \bar{F}) = \Pr_{-f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$

(b) $\lim_{f_0 \downarrow 0} \Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1 - [\Phi(\sqrt{\bar{F}}) - \Phi(-\sqrt{\bar{F}})]$

(c) $\lim_{f_0 \longrightarrow \infty} \Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$

PROOF:

(a) Subscripting t with ρ_0 to denote its direct dependence on ρ_0, note that $t^2_{\rho_0}(f_0^* + \rho_0 t_{AR}, t_{AR}) = t^2_{-\rho_0}(-(f_0^* + \rho_0 t_{AR}), t_{AR}) = t^2_{-\rho_0}(-f_0^* + (-\rho_0)t_{AR}), t_{AR})$ and $f^2 = (f_0^* + \rho_0 t_{AR})^2 = (-f_0^* + (-\rho_0)t_{AR})^2$. The first equality follows.
Next, $t^2_{\rho_0}(f_0^* + \rho_0 t_{AR}, t_{AR}) = t^2_{\rho_0}(-f_0^* + \rho_0(-t_{AR}), (-t_{AR}))$ and $f^2 = (f_0^* + \rho_0 t_{AR})^2 = (-f_0^* + \rho_0(-t_{AR}))^2$. Under the null, $t_1 = 0$ and $t_{AR} \sim N(0,1)$ is symmetrically distributed about zero. The second equality follows.

(b) Note that $\underline{f}_A$, $\bar{f}_A \longrightarrow 0$ as $f_0 \longrightarrow 0$. The result follows.

(c)

$$\bar{f}_A = \frac{-\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{2} \left(\frac{\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}} \right) = \frac{2\sqrt{q}}{\rho_0 \frac{f_0}{|f_0|} + \sqrt{1 + \frac{4\sqrt{q}}{|f_0|}}}$$

Hence, $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \bar{f}_A = \sqrt{q}$. Similarly, $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \underline{f}_A = -\infty$; $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \bar{f}_B = -\sqrt{q}$; $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \underline{f}_B = -\infty$. When $\rho_0 = 1$, as $f_0 \longrightarrow \infty$, $\sqrt{\bar{F}} - f_0 \longrightarrow -\infty$, so that the rejection probability is determined by $\bar{f}_A$ and $\bar{f}_B$ asymptotically. Result (c) follows. ⊠

Remarks:

1. Note that results on rejection probabilities for Wald follow setting $\bar{F} = 0$, $\Pr_{f_0,\rho_0}(t^2 \geq q) = \Pr_{f_0,\rho_0}(t^2 \geq q, f^2 \geq 0)$.

2. By part (a), under the null, to characterize $\Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q, f^2 \geq \bar{F})$, it suffices to focus on the case where $\rho_0 = 1$ and $f_0 \geq 0$.

3. From (b), by choosing $\bar{F}$ close to zero, the worst case rejection probability for $\{t^2 \geq q, f^2 \geq \bar{F}\}$ is arbitrarily close to one.

4. By parts (a) and (b), $\lim_{f_0 \longrightarrow 0} \Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q) = 1$

Corollary 2. *Under the null, there exists $\bar{f}_0 > 0$ large enough that for any $f_0 > \bar{f}_0$,*

(a) if $q < 4$, then
$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) > 0;$$

(b) if $q > 4$, then
$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0.$$

PROOF:

Set $\rho_0 = 1$. As $f_0 \longrightarrow \infty$, $\Pr_{f_0,\rho_0=1}(t^2 \geq q) = \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)$. Define $v = \frac{1}{f_0}$. So $v \downarrow 0$ as $f_0 \longrightarrow \infty$. From the Proof of Lemma 1, for $f_0 > 0$, we have $\bar{f}_A = \frac{2\sqrt{q}}{1+\sqrt{1+4v\sqrt{q}}}$. Similarly, $\bar{f}_B = \frac{-2\sqrt{q}}{1+\sqrt{1-4v\sqrt{q}}}$.

$$\frac{\partial \bar{f}_A}{\partial v} = \frac{-4q}{(1+\sqrt{1+4v\sqrt{q}})^2\sqrt{1+4v\sqrt{q}}} \; ; \quad \frac{\partial \bar{f}_B}{\partial v} = \frac{-4q}{(1+\sqrt{1-4v\sqrt{q}})^2\sqrt{1-4v\sqrt{q}}}$$

Let $w = 4v\sqrt{q}$.

$$\begin{aligned}
\frac{\partial}{\partial v}\left[1-\Phi(\bar{f}_A)+\Phi(\bar{f}_B)\right] &= \phi(\bar{f}_B)\frac{\partial \bar{f}_B}{\partial v} - \phi(\bar{f}_A)\frac{\partial \bar{f}_A}{\partial v} \\
&= \phi(\bar{f}_B)\frac{\partial \bar{f}_B}{\partial v}\left[1 - \frac{(1+\sqrt{1-4v\sqrt{q}})^2\sqrt{1-4v\sqrt{q}}}{(1+\sqrt{1+4v\sqrt{q}})^2\sqrt{1+4v\sqrt{q}}}\frac{\phi(\bar{f}_A)}{\phi(\bar{f}_B)}\right] \\
&= -\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right|\left[1 - \frac{(1+\sqrt{1-w})^2\sqrt{1-w}}{(1+\sqrt{1+w})^2\sqrt{1+w}}\exp\left(2q\left[\frac{-1}{(1+\sqrt{1+w})^2}+\frac{1}{(1+\sqrt{1-w})^2}\right]\right)\right]
\end{aligned}$$

Using a first-order expansion of the bracketed term in the final expression above, we find that as $w \downarrow 0$,

$$\begin{aligned}
\frac{\partial}{\partial v}\left[1-\Phi(\bar{f}_A)+\Phi(\bar{f}_B)\right] &= -\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right|\left[(4-q)2v\sqrt{q}+o(v)\right] \\
&= \left[(q-4)2v\sqrt{q}+o(v)\right]\cdot\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right|
\end{aligned}$$

Notice from the Proof of Lemma 1, $\lim_{\rho_0=1,f_0\longrightarrow\infty}\underline{f}_A = \lim_{\rho_0=1,f_0\longrightarrow\infty}\underline{f}_B = -\infty$. Correspondingly, it is straightforward to show that the terms $\Phi(\underline{f}_A)$ and $\Phi(\underline{f}_B)$ do not have a first-order effect on the derivative above (for cases $\rho_0 = 1$ and $f_0 \longrightarrow \infty$, or $\rho_0 = -1$ and $f_0 \longrightarrow -\infty$).

So, under the null, for $q \neq 4$, as $f_0 \longrightarrow \infty$,

$$\begin{aligned}
\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q) &= \frac{\partial}{\partial f_0}\left[\Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)\right] \\
&= -\frac{1}{f_0^2}\frac{\partial}{\partial v}\left[\Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)\right] \\
&= -v^2\frac{\partial}{\partial v}\left[\Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)\right] \\
&= (4-q)\cdot\Bigg[\underbrace{2\sqrt{q}\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right| v^3}_{>0}\Bigg] + o(v^3)
\end{aligned}$$

This established the result for the Wald rejection region. The generalization to $\{t^2 \geq q, f^2 \geq \bar{F}\}$ is straightforward and follows the argument above, as $\Phi(\bar{f}_A)$ and $\Phi(\bar{f}_B)$ are still the dominant terms in the derivative. ⊠

Remarks:

1. Putting Corollary 1(c) and Corollary 2 together, we see that the rejection probability for Wald with $\rho_0 = 1$ asymptotes to $1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$ as $f_0 \longrightarrow \infty$. When $q < 4$, the Wald rejection probability approaches its asymptote from below. This means that for large enough f_0, $\Pr_{f_0,\rho_0=1}(t^2 \geq q) < 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$. Given Corollary 1(a) and continuity of the Wald rejection probability, there exists a value f_0 such that $\Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q) = 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$.

2. When $q > 4$, the rejection probability for Wald with $\rho_0 = \pm 1$ is decreasing as it asymptotes to $1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$. Generally, there will *not* be a value of f_0 such that $\Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q) = 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$.

3. $q = 4$ corresponds to test size 4.55%. So, $q < 4$ corresponds to test size > 4.55%, and $q > 4$ corresponds to test size < 4.55%.

Derivation of Result 1a: We use numerical evidence to verify that for a given $f_0 > 0$, the largest null rejection probability occurs when $\rho_0 = 1$. As discussed in Remark 1 above, taking $q = 1.96^2 < 4$, Corollary 2(a) and Corollary 1(c) then tell us that there exists f_0 such that $\Pr_{f_0,\rho_0=1}(t^2 \geq q) < 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})] = 0.05$. From Lemma 7, we have $\Pr_{f_0,\rho_0=1}(t^2 \geq q) = \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + [\Phi(\bar{f}_B) - \Phi(\underline{f}_B)]\mathbf{1}\{|f_0| > 4\sqrt{q}\}$. Given the formulas for $\underline{f}_A$, $\bar{f}_A$, $\underline{f}_B$, and $\bar{f}_B$ above, it is straightforward to solve for the smallest f_0 such that $\Pr_{f_0,\rho_0=1}(t^2 \geq q) = 0.05$ and verify that

$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q) > 0$ for any larger f_0 (so that $\Pr_{f_0,\rho_0=1}(t^2 \geq q)$ must be smaller than its asymptotic value of 0.05 for all larger f_0). The solution is $f_0 = 11.9$. Hence $E(F) = E(f^2) = Var(f) + [E(f)]^2 = 1 + (11.9)^2 = 142.6$. ⊠

Derivation of Results 1b and 1c: Taking $q = 2.576^2 > 4$, Corollary 2(b) says that for large enough f_0, $\Pr_{f_0,\rho_0=1}(t^2 \geq q) > 0.01$. We verify that the derivative $\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q) < 0$ for large enough f_0 and then verify the inequality numerically for any smaller values of f_0. The findings in Results 1b and 1c for $\rho_0 < 1$ are obtained numerically. An analogous figure to Figure A1 for the 5 percent level is given in Lee et al. (2020).

⊠

Define $\phi^* = \frac{\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}$, and $\bar{\phi} = \begin{cases} 4\sqrt{q} & \text{if } \bar{F} \leq 4\sqrt{q} \\ \frac{\bar{F}}{\sqrt{\bar{F}}-\sqrt{q}} & \text{if } \bar{F} > 4q \end{cases}$

Lemma 8. *Under the null, for* $\bar{F} > 0$*,*

if $0 < f_0 < \phi^*$*,*

$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) > 0\,;$$

and if $\phi^* < f_0 < \bar{\phi}$*,*

$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0.$$

Proof: For $0 < f_0 < \phi^*$, $-f_0 + \sqrt{\bar{F}} > \bar{f}_A(f_0)$, and for $f_0 > \phi^*$, $-f_0 + \sqrt{\bar{F}} < \bar{f}_A(f_0)$. Let $\underline{\phi} = \begin{cases} \frac{\bar{F}}{\sqrt{q}-\sqrt{\bar{F}}} & \bar{F} < q \\ \infty & \bar{F} \geq q \end{cases}$. If $0 < f_0 < \underline{\phi}$, then $-f_0 - \sqrt{\bar{F}} < \underline{f}_A(f_0)$. Moreover, $\frac{\partial}{\partial f_0}[-f_0 - \sqrt{\bar{F}}] < 0$ and $\frac{\partial}{\partial f_0}\underline{f}_A(f_0) < 0$ for $f_0 > 0$. For $0 < f_0 < \bar{\phi}$, we can show that $[\underline{f}_B(f_0), \bar{f}_B(f_0)] \cap \left((-\infty, -f_0 - \sqrt{\bar{F}}] \cup [-f_0 + \sqrt{\bar{F}}, \infty)\right) = \emptyset$. Hence,

$$\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = \begin{cases} 1 - \Phi(-f_0 + \sqrt{\bar{F}}) + \Phi(-f_0 - \sqrt{\bar{F}}) & \text{if } 0 < f_0 < \phi^* \\ 1 - \Phi(\bar{f}_A(f_0)) + \Phi(-f_0 - \sqrt{\bar{F}}) & \text{if } \phi^* < f_0 < \underline{\phi} \end{cases}.$$

For $0 < f_0 < \phi^*$,

$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = \phi(-f_0 + \sqrt{\bar{F}}) - \phi(-f_0 - \sqrt{\bar{F}}) > 0$$

since $|-f_0+\sqrt{\bar{F}}| < |-f_0-\sqrt{\bar{F}}|$. And, for $\phi^* < f_0 < \bar{\phi}$, $\frac{\partial}{\partial f_0}[1-\Phi(\bar{f}_A(f_0))+\Phi(-f_0-\sqrt{\bar{F}})] < 0$ and $\frac{\partial}{\partial f_0}[1-\Phi(\bar{f}_A(f_0))+\Phi(\underline{f}_A(f_0))] < 0$. The result follows. ⊠

Remarks:

1. Lemma 8 characterizes a local maximum in $\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$. The maximum occurs at $f_0 = \phi^*$, which is the smallest maximizing point for $f_0 > 0$.

2. Importantly, note that the derivative of $\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$ is discontinuous at $f_0 = \phi^*$, so this maximizer is well separated, which is useful for our numerical analysis.

3. We know the asymptotic value of this rejection probability by Corollary 1(c). In addition, numerical experimentation shows another bounded local maximum can sometimes be the global maximizer when $q > 4$, as might be expected given Corollary 2.

4.
   (9)
   $$\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right)$$

   - $\frac{\partial}{\partial \bar{F}}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0$ and $\frac{\partial}{\partial q}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0$
   - $\lim_{\bar{F}\downarrow 0}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1$
   - $\lim_{\bar{F}\longrightarrow\infty}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1-\Phi(\sqrt{q})$.
   - Clearly, $\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$ cannot be a global maximizer over $f_0 > 0$ if $\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 1-[\Phi(\sqrt{q})-\Phi(-\sqrt{q})]$

Size Calculations

Equation (9) is a key step in our size calculation results. We use Lemma 8 and numerical evidence to verify that $\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$ is a global maximizer over f_0, ρ_0. To achieve a size γ test, we solve

$$\gamma = 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right)$$

Note that the expression on the right-hand side is monotonic decreasing in both $\bar{F}$ and q, so that solving this equation for $\bar{F}$ or q is straightforward.

Derivation of Result 2a: Set $\bar{F} = 10$ and $q = 1.96^2$. Then, $\gamma = 0.113$.[51] $\boxtimes$

Derivation of Result 2b: Set $q = 1.96^2$ and $\gamma = 0.05$. Then, solve for $\bar{F}$ yielding $\bar{F} = 104.7$. $\boxtimes$

Derivation of Result 2c: Set $\bar{F} = 10$ and $\gamma = 0.05$. Then, solve for q yielding $q = 3.4$. $\boxtimes$

Derivation of Result 2d: Let $f_0 = \phi^*$. Then,

$$\Pr_{f_0=\phi^*,\rho_0=1}(\{t^2 \geq q, f^2 \geq \bar{F}\} \cup \{t_{AR}^2 \geq q, f^2 < \bar{F}\}) = \begin{cases} 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right) \\ \text{if } \bar{F} \leq \frac{q}{2} \\ 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi(-\sqrt{q}) \\ \text{if } \bar{F} > \frac{q}{2} \end{cases}$$

Note that $1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi(-\sqrt{q}) > 1-\Phi(\sqrt{q})+\Phi(-\sqrt{q})$. When $\bar{F} = \frac{q}{2}$, the expressions in the bracket above are equal. Since we already know $1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right)$ is decreasing in $\bar{F}$, we can conclude that for all $\bar{F}$,

$$\Pr_{f_0=\phi^*,\rho_0=1}(\{t^2 \geq q, f^2 \geq \bar{F}\} \cup \{t_{AR}^2 \geq q, f^2 < \bar{F}\}) > 1-\Phi(\sqrt{q})+\Phi(-\sqrt{q}).$$

Plugging in $q = 1.96^2$ yields the stated result. $\boxtimes$

A.8 Imposing Restrictions on ρ and Inference on ρ

A.8.1 Interpretation and Adjustment of the Usual Critical Values under Different Assumptions about ρ

For a range of assumptions about ρ, the table below reports 1) the correct significance levels for the tests $t^2 > 1.96^2$ and $t^2 > 2.576^2$ and 2) the necessary minimal adjustments to the usual critical values to restore the 5 percent and 1 percent significance levels. The $|\rho| \leq 0.565$ and $|\rho| \leq 0.435$ necessary restrictions were reported

[51] To be precise, we set q to the 95% quantile of the χ_1^2 distribution.

in Lee et al. (2020), and both the $|\rho| \leq 0.565$ and $|\rho| \leq 0.760$ restrictions were reported in Angrist and Kolesár (2021).

Table 5: Additional Assumptions about $|\rho|$: Actual Significance Levels for $|t| > 1.960$, $|t| > 2.576$ and Necessary Critical Values, Standard Error Adjustments for Correct Significance Levels

Assumed upper bound on $\|\rho\|$	Actual significance level, critical value for $\|t\|$ is 1.960	Critical Value for $\|t\|$ (Std. Error Adj. Factor), 5% level	Actual significance level, critical value for $\|t\|$ is 2.576	Critical Value for $\|t\|$ (Std. Error Adj. Factor), 1% level
1.000	1.000	∞ (∞)	1.000	∞ (∞)
0.950	0.396	5.656 (2.886)	0.302	7.663 (2.975)
0.850	0.186	3.158 (1.611)	0.099	4.315 (1.675)
0.760	**0.100**	**2.451 (1.251)**	**0.042**	**3.500 (1.359)**
0.650	0.063	2.108 (1.076)	0.023	3.017 (1.171)
0.565	**0.050**	**1.960 (1.000)**	**0.016**	**2.794 (1.083)**
0.500	0.050	1.960 (1.000)	0.012	2.671 (1.035)
0.435	**0.050**	**1.960 (1.000)**	**0.010**	**2.576 (1.000)**

Note: Numbers in parentheses are the correct critical values divided by 1.96 (third column) and 2.576 (fifth column).

A.8.2 Assumptions on ρ imply Assumptions on β

As noted by Van de Sijpe and Windmeijer (2021), inspecting the definition of ρ is instructive because it reveals a tight connection between ρ and β. Specifically, we have

$$
\begin{aligned}
(10) \qquad \rho = \rho(\beta) &\equiv \frac{C(Zu, Zv)}{\sqrt{V(Zu)\,V(Zv)}} \\
&= \frac{C(Z\varepsilon, Zv) - \beta V(Zv)}{\sqrt{\left(V(Z\varepsilon) + \beta^2 V(Zv) - 2\beta C(Z\varepsilon, Zv)\right) V(Zv)}} \\
&= \frac{\rho_{RF} - \beta \frac{\sqrt{V(Zv)}}{\sqrt{V(Z\varepsilon)}}}{\sqrt{\left(1 + \left(\beta \frac{\sqrt{V(Zv)}}{\sqrt{V(Z\varepsilon)}}\right)^2 - 2\rho_{RF}\beta \frac{\sqrt{V(Zv)}}{\sqrt{V(Z\varepsilon)}}\right)}}
\end{aligned}
$$

where the reduced-form population residual is $\varepsilon \equiv v\beta + u$, and the correlation between the reduced form and first-stage errors (multiplied by Z) is $\rho_{RF} \equiv Corr(Zv, Z\varepsilon)$. Since the variance and ρ_{RF} terms can be consistently estimated (irrespective of the first-stage strength) from the residuals of the first-stage and reduced-form regressions, there is a one-to-one relationship between the unknown parameters β and ρ.[52]

Imposing the restriction $|\rho| \leq 0.565$ is therefore equivalent to

$$\frac{V(Zv)}{V(Z\varepsilon)}\beta^2 - 2\rho_{RF}\frac{\sqrt{V(Zv)}}{\sqrt{V(Z\varepsilon)}}\beta + \frac{\rho_{RF}^2 - 0.565^2}{1-0.565^2} \leq 0,$$

giving endpoints for the interval for β,

$$\frac{\sqrt{V(Z\varepsilon)}}{\sqrt{V(Zv)}}\left(\rho_{RF} \pm \sqrt{1-\rho_{RF}^2}\frac{0.565}{\sqrt{1-0.565^2}}\right).$$

This interval omits information on π and the reduced-form coefficient $\pi\beta$. Indeed, the constraint on β from from any assumption about $\rho(\beta)$ exists even when the instrument is irrelevant ($\pi = 0$). These resultant *a priori* bounds on β need not include 0; they exclude 0 if and only if $|\rho_{RF}| > 0.565$ regardless of the *IV* regression. This may concern researchers wishing information from the IV strategy as to whether $\beta = 0$, rather than the imposition of *a priori* assumptions about ρ.[53]

Of the 66 specifications (drawn from 10 separate studies) for which the calculation is possible (see details below), 30 percent (weighted by the reciprocal of specifications within a study to give each study implicitly equal weight) of the time the *a priori* restriction $|\rho| \leq 0.565$ excludes $\beta = 0$. 42 percent (weighted) of the time the *a priori* restriction $|\rho| \leq 0.435$ rules out $\beta = 0$.

A.8.3 Confidence Intervals for ρ

The one-to-one mapping between ρ and β in Equation (10) makes clear that it is possible to make statistical inferences about the unknown parameter ρ, even in the presence of the same issues that affect inference on β (e.g., an unknown strength of the instrument). A simple and direct valid confidence interval for ρ consists of the endpoints $\rho(\hat{\beta}^L)$ and $\rho(\hat{\beta}^U)$ using (10), where $\left[\hat{\beta}^L, \hat{\beta}^U\right]$ is the *tF* confidence interval for β.[54]

[52]This point is also emphasized by Van de Sijpe and Windmeijer (2021), who note an alternative way to state our Equation (10), namely $\rho_{RF} = (\rho + \beta r)/(1 + 2\beta\rho r + \beta^2 r^2)$ where $r = \sqrt{V(Zv)/V(Zu)}$. See their Equation (7) and their Section 4.

[53]This is especially true if the specification is used as a "placebo" test.

[54]Alternatively the *AR* confidence set could be used as the input to Equation (10) to construct a valid confidence set for ρ, since *AR* inference also accommodates $|\rho|$ as large as 1.

To shed light on what these intervals might look like in practice, we compute 95-percent confidence intervals for ρ for the same subsample of 66 specifications (as described below). We find that 41 percent of the confidence intervals for ρ are contained within the ± 0.565 interval, of which 55 percent include $\rho = 0$, which is consistent with no endogeneity; 34 percent are contained within ± 0.435. Furthermore, 57 percent, 41 percent, 36 percent, and 24 percent of the intervals contain the values $|\rho|$ of 0, 0.7, 0.8, and 0.9, respectively. The data are consistent with a broad range of values for ρ (including $|\rho|$ much larger than 0.565 or 0.435) that could not be statistically rejected at conventional levels of significance. Finally, we find that 18 percent of the confidence intervals for ρ do not intersect with the $|\rho| \leq .565$ region, therefore rejecting that that hypothesis at the 5 percent level of significance.[55],[56]

A separate, more involved question is how one could incorporate information on these confidence sets from into any particular study. Different approaches are possible, including empirical Bayes and hierarchical Bayes. Precisely how to incorporate prior information on $|\rho|$ is beyond the scope of our paper, but could be an interesting avenue for future research.

A.8.4 Recovery of the Reduced-Form Covariance Matrix from the *AER* Sample of Studies

$V(Z\varepsilon)$, $V(Zv)$, and $C(Z\varepsilon, Zv)$ (and therefore ρ_{RF}) can be consistently estimated, under both weak-IV and strong-IV asymptotics since each of can also be consistently estimated by sample analogues $\hat{V}(Z\hat{v}_{RF}), \hat{V}(Z\hat{v})$, and $\hat{V}(Z\hat{\varepsilon})$ This can be done directly from the microdata using the residuals from the reduced form and first-stage regressions, with no need to compute $\hat{\beta}$.

Without the microdata in hand, we recover the covariance and variance estimates above from the five reported statistics: $\hat{\beta}, \hat{\pi}, \hat{V}_N\left(\widehat{\pi\beta}\right)$, $\hat{V}_N(\hat{\pi})$, and $\hat{V}_N\left(\hat{\beta}\right)$.

[55] Note that tF confidence intervals for ρ are, by construction, more conservative than (and entirely contain) confidence intervals on ρ that could be derived from the usual ± 1.96 confidence intervals. Using the usual *IV* confidence intervals, we find the same 18 percent that do not intersect with the $|\rho| \leq .565$ region. An equivalent way to state this is that 18 percent of the time the usual ± 1.96 confidence intervals for β do not intersect with the bounds on β implied by the $|\rho| \leq .565$ assumption.

[56] All considerations so far refer to size and confidence level. When the null is not true, the parameter $\rho(\beta)$ can differ from the structural correlation ρ.

By using the relation that $\hat{\varepsilon} = \hat{v}\hat{\beta} + \hat{u}$, one can express the desired estimators as

$$\hat{V}(Z\hat{\varepsilon}) = N\hat{V}(Z)^2 \hat{V}_N\left(\widehat{\pi\beta}\right)$$
$$\hat{C}(Z\hat{\varepsilon}, Z\hat{v}) = \frac{N\hat{V}(Z)^2 \hat{V}_N\left(\widehat{\pi\beta}\right) + \hat{\beta}^2 N\hat{V}(Z)^2 \hat{V}_N(\widehat{\pi}) - \hat{V}(Z)^2 N\hat{\pi}^2 \hat{V}_N\left(\hat{\beta}\right)}{2\hat{\beta}}$$
$$\hat{V}(Z\hat{v}) = N\hat{V}(Z)^2 \hat{V}_N(\widehat{\pi})$$

Information about $N\hat{V}(Z)^2$ is not needed since they cancel out in using these quantities in the sample analogues of $\frac{V(Zv)}{V(Zv_{RF})}$ and ρ_{RF}.

Starting with the sample of 255 specifications from Table 1, we make the following restrictions: 1) all five statistics are nonmissing, 2) identical sample sizes are reported for the 2SLS, first-stage, and reduced form regressions, 3) the ratio of the reduced-form and first stage coefficients is within 5 percent of the reported 2SLS point estimate, 4) the variance estimates result in $|\hat{\rho}_{RF}| \leq 1$.[57] The resulting sample consists of 66 specifications drawn from 10 separate studies.

A.9 Power curves: AR, tF, and step functions (c^*, F^*)

Figure A2 contains the power curves for the eight remaining scenarios as described in the text. A black diamond represents the rejection probability from 250,000 Monte Carlo simulations, each with a sample size of 1,000.

B Detailed Discussion of the tF Critical Value Function and Proofs

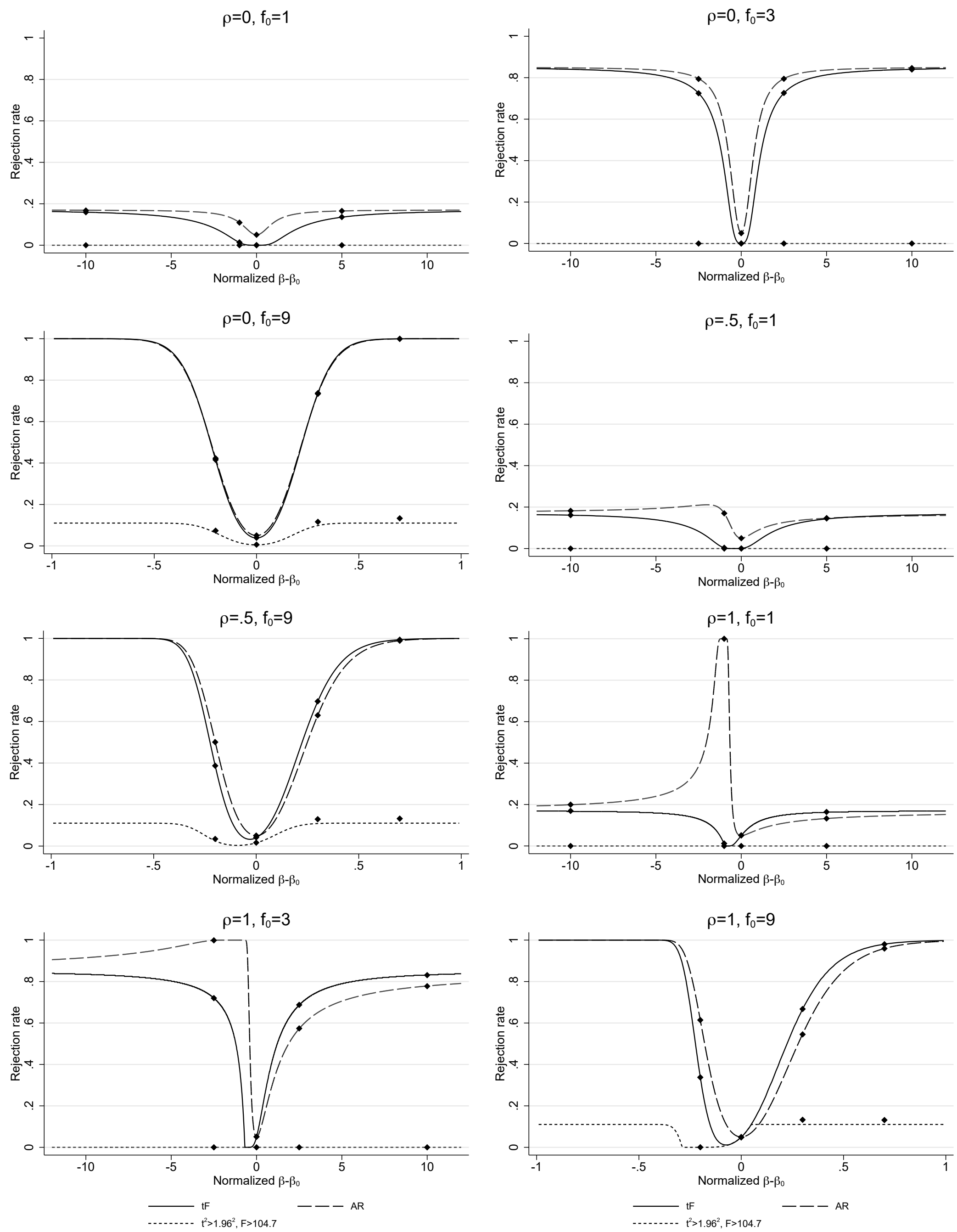
Section III.B described how to construct a critical value function $c_\alpha(F)$ that smoothly adjusts according to the first-stage F-statistic and that also controls size,

$$\text{Pr}_{\Delta(\beta_0)=0,\rho,f_0}\left[t^2 > c_\alpha(F)\right] \leq \alpha. \tag{11}$$

In this Appendix Section, we introduce four properties that define a class $\mathbb{C}$ of

[57]For 3) although the ratio of the reduced-form and first-stage coefficients should equal the 2SLS point estimate, we make an allowance for rounding and other small discrepencies in the authors' reporting. Given the micro-data all estimates of $\hat{\rho}_{RF}$ will be, by construction, less than 1 in absolute value (restriction 4). The fact that this is not always true suggests some inconsistency in reporting by the authors of the studies.

Figure A2: Power Curves



possible critical value functions.[58] We show that the constructed tF critical value function, described in section III.B and in more detail in section B.3, exists and is an element of the class $\mathbb{C}$. We also show that the construction process described in III.B leads to the *only* possible critical value functions in $\mathbb{C}$. Finally, in sections B.4 and B.5, we prove additional properties that further motivate the tF critical value function.

B.1 Critical Value Function Properties

In this section, we define a class $\mathbb{C}$ of candidate critical value functions. We will only consider critical value functions of F that are positive and continuous, and have a plateau structure.[59] The required plateau structure is that for sufficiently large values of F, the critical value function is constant. This structure simplifies the description of the critical value function for practitioners (and also is aligned with the notion that the critical value should never fall below the standard chi-square quantile $q_{1-\alpha}$ for strong instruments).[60] Since all candidate critical value functions will have the plateau structure it will be convenient to adopt a notation that includes this structure. Suppose a candidate critical value function is constant for all $F \geq \tilde{F}$. Then we will denote the critical value function by $\tilde{c}(F;\tilde{F})$. Let $\underline{c} \equiv \tilde{c}\left(\tilde{F};\tilde{F}\right)$, so that $\tilde{c}(F;\tilde{F}) = \underline{c}$ for $F \geq \tilde{F}$.

Next we provide an overview of the properties underlying the class of candidate critical value functions $\mathbb{C}$, before introducing the formal notation and statements.

First, based on the intuition that increasing F signals a stronger instrument and less distortion to the t-statistic, we require the critical value function $\tilde{c}(F;\tilde{F})$ to be decreasing in F for $F < \tilde{F}$. Hence, we are considering a class of continuous functions that are decreasing for a range $F \in (\underset{\sim}{F},\tilde{F})$ and that plateau at a constant level for $F \geq \tilde{F}$, where $0 \leq \underset{\sim}{F} < \tilde{F}$.[61]

Second, we require that, for small values of f_0 and $|\rho| = 1$, the null rejection probability is exactly equal to α. This is motivated by the conjecture of Stock and

[58] In this Appendix Section, we use $c_\alpha(\cdot)$ for the tF critical value function, $\tilde{c}_\alpha(\cdot)$ for the decreasing segments of the critical value function, as described in Section III.B, and $\tilde{c}(\cdot;\cdot)$ as a critical value function with a plateau structure, as introduced below.

[59] Following the approach of Stock and Yogo (2005), our study exclusively focuses on critical value functions of the first-stage F-statistic. Critical value functions that depend on more information from the data (for example, the sign of f) might also be possible and could lead to some efficiency gains.

[60] It is possible that other forms of the critical value function could lead to efficiency gains, but we impose the plateau structure here.

[61] Allowing the domain of candidate critical value functions to be $(\underset{\sim}{F},\infty)$, where possibly $\underset{\sim}{F} > 0$, allows the inclusion of critical value functions where no rejections occur for a range of small values of $F \leq \underset{\sim}{F}$, or equivalently the critical value function is set to infinity for this range of F values.

Yogo (2005) that for a fixed value of f_0, the "worst case" null rejection probability occurs when $\rho = \pm 1$. We are able to theoretically verify this conjecture for small values of f_0 and values of $|\rho|$ arbitrarily close to 1. We show this in Appendix Section B.5.

It is possible to show that critical value functions that satisfy the above restrictions will 1) have $\underset{\sim}{F} = q_{1-\alpha}$, which means that the domain of the critical value function is $(q_{1-\alpha}, \infty)$, and 2) asymptote to infinity as $F \downarrow \underset{\sim}{F} = q_{1-\alpha}$. Deriving the critical value function along this asymptote is a key technical challenge to obtaining the tF critical value function.

We add a third requirement that puts an upper bound on the magnitude of $\tilde{F}$ (where the plateau begins) in a way that we make precise below. In general, the bound is increasing, as the decreasing part of the critical value function uniformly increases. This requirement does not stem from an *a priori* motivation; instead, it is a technical restriction that allows us to ultimately establish existence of elements of the class $\mathbb{C}$. That said, the third requirement alone captures a wide range of potential critical value functions. It will turn out that this bound is irrelevant for $\alpha = .05$ but relevant for other α such as .01.

The fourth requirement is a technical condition that characterizes the critical value function as it asymptotes. This property will be used to prove a certain form of uniqueness of the candidate critical value functions.

To state the above required properties more formally, it is useful to adopt some notation for candidate critical value functions.

- As noted above, without loss of generality, we can assume that the domain of the candidate critical value functions is $F \in (q_{1-\alpha}, \infty)$. So, we let $\tilde{c}(F;\tilde{F})$ denote a positive function continuous in F with domain $F \in (q_{1-\alpha}, \infty)$ that is constant on $F \in [\tilde{F}, \infty)$ for $\tilde{F} > q_{1-\alpha}$.

- Let $\bar{f}(f_0)$ denote the maximum value of f among all intersections between the graphs $\tilde{c}(F;\tilde{F})$ and $t^2 = \frac{f^2(f-f_0)^2}{f_0^2}$ for a given f_0; let $\underline{f}(f_0)$ denote the minimum value among all intersections.

- Let $\bar{f}_0$ be the value of f_0 such that $[\bar{f}(f_0)]^2 = \tilde{F}$; specifically

$$\bar{f}_0 = \frac{\tilde{F}}{\sqrt{\tilde{F}} + \sqrt{\tilde{c}(\tilde{F};\tilde{F})}}. \tag{12}$$

For every critical value function with the plateau structure, there is a point where the decreasing segment meets the plateau. This point is at $f^2 = \tilde{F}$ with critical value

$\tilde{c}(\tilde{F};\tilde{F})$. Then, $\bar{f}_0$ is the f_0 associated with the "W" curve whose right "arm" passes through this point. The horizontal axis in Figure 7 denotes values of f, so a candidate critical value function that plateaus at A^{**} (with coordinates $(\sqrt{\tilde{F}}, \tilde{c}(\tilde{F};\tilde{F}))$) would have a corresponding value $\bar{f}_0 = f_0'$.

We are now in a position to formally define $\mathbb{C}$ as the class of candidate critical value functions $\tilde{c}(F;\tilde{F})$ satisfying the following properties (A)-(D):

(A) (Decreasing) The function $\tilde{c}(F;\tilde{F})$ is decreasing in F on $(q_{1-\alpha}, \tilde{F})$.

(B) (Rejection probability of α, for $|\rho| = 1$, f_0 small) For $\bar{f}_0$ defined in (12),

$$\Pr_{|\rho|=1,f_0}\left[t^2 > \tilde{c}\left(F;\tilde{F}\right)\right] = \alpha, \quad \forall 0 < |f_0| < \bar{f}_0.$$

(C) (Bound on $\tilde{F}$) For each $0 < |f_0| \leq \bar{f}_0$,

$$\left\{f : \frac{f^2\left(f-f_0\right)^2}{f_0^2} \leq \tilde{c}\left(f^2;\tilde{F}\right)\right\} = [\underline{f}(f_0), \bar{f}(f_0)].$$

(D) (Class for uniqueness)

$$\tilde{c}(F;\tilde{F}) = \frac{q_{1-\alpha}^3}{F - q_{1-\alpha}} - b + o((F - q_{1-\alpha})^{-1/3}), \text{ as } F \downarrow q_{1-\alpha}$$

where $b = 3q_{1-\alpha} - \frac{q_{1-\alpha}^2}{2} + \frac{q_{1-\alpha}^3}{6}$.

It is clear that there is a wide range of critical value functions that satisfy Property (A), even for a fixed pair of values $\tilde{F}$ and $\underline{c}$. The same is true for Property (C), even though this property essentially places an upper bound on the point at which the plateau begins, depending on the shape of the decreasing segment. As an example, a critical value function that included the extended dashed line in Figure 7 does not satisfy Property (C), because when $f_0 = f_0'''$, the acceptance region in terms of f is no longer an interval.[62] We elaborate on the technical reason why we impose this property in a remark below. Here we simply note that the property effectively puts a bound on how far the decreasing segment extends. As the decreasing segment uniformly increases, so does that bound.

Similarly, Property (D) is not based on an *a priori* principle. Instead, it describes a range of critical value functions within which we will be able to say that for a

[62] Here, we are using Figure 7, which displays the tF critical value function, to illustrate this property, but in general other critical value functions could possess this property.

given $\tilde{F}$ there is only one decreasing function that satisfies Properties (A), (B), and (C). Note that Property (D), by itself, appears to describe a wide range of possible critical value functions and only restricts the behavior of the critical value function as it asymptotes, as $F \downarrow q_{1-\alpha}$. For example, consider the function $\frac{q_{1-\alpha}^3}{F-q_{1-\alpha}} - b + K_F$ with K_F some large positive or negative constant. Any such function would satisfy Property (D), and $K_F \to \pm\infty$ would even be allowed by (D) as long as $(F - q_{1-\alpha})^{1/3} K_F \to 0$.[63]

Clearly, for any fixed pair $(\tilde{F}, \underline{c})$, a critical value function is not uniquely determined by the combination of the three Properties (A), (C), and (D).

As we show in Appendix B.2, the additional restriction of (B) is not so restrictive to cause $\mathbb{C}$ to be empty (Proposition 1), and at the same time leads to a unique decreasing segment: any two members of the set $\mathbb{C}$ are identical on the interval in which both functions are decreasing (Proposition 2). Thus, the candidate critical value functions can be indexed by the value at which the plateau begins, $\tilde{F}$.

It is precisely within this class $\mathbb{C}$ that we obtain the tF critical value function (as described in Section III.B) – the member of $\mathbb{C}$ with the smallest value of plateau $\underline{c}$ (equivalently choosing the largest possible $\tilde{F}$, given that the nonconstant portion of the function is restricted to be decreasing), while still controlling size (rejection probabilities across the entire nuisance parameter space, as in (11), not only for $|\rho| = 1$, and small f_0).

Formally, we can define the tF critical value function described in Section III.B to be $c_\alpha(F) = \tilde{c}(F; F^*)$ where

$$F^* = \max\left\{\tilde{F} | \tilde{c}\left(F; \tilde{F}\right) \in \mathbb{C} \text{ and } \Pr_{\rho, f_0}\left[t^2 > \tilde{c}\left(F; \tilde{F}\right)\right] \leq \alpha,\ \forall \rho, f_0 \neq 0\right\}.$$

where $\underline{c} = \tilde{c}(F^*; F^*)$

Our computation of the tF critical value function is shown in Tables 3 Panel A and 3 Panel B. In order to find the minimized $\underline{c}$, we use numerical integration of the expression in (4) to compute the rejection probabilities under the null, as illustrated in Figure 2.[64] Even if there were some regions where rejection probabilities were larger for $|\rho| < 1$, they would be limited to that which could not be detected given the precision of numerical integration. For the 5 percent level, we compute $F^* = 104.7$ and $\sqrt{\underline{c}} = \sqrt{c_{0.05}(104.7)} = 1.96$ as labeled in Figure 6 and shown in Table 3 Panel A, while for the 1 percent level $F^* = 252.34$ and $\sqrt{\underline{c}} = \sqrt{c_{0.01}(252.34)} = 2.73$ as shown in Table 3 Panel B.

Remark. Property (C), by itself, accommodates a large range of critical value

[63]Property (D) is derived from uniqueness results in the dynamical systems literature, see Fefferman (2021) and the discussion in Appendix B.2.

[64]Stock and Yogo (2005) use Monte Carlo integration to evaluate the expression.

functions, but is not motivated by an *a priori* desirable characteristic. Instead, we make this restriction because it allows us to characterize the decreasing segment of the critical value function that satisfies Equation (5), which we have denoted in Section III.B as $\tilde{c}_{0.05}\left(f^2\right)$, as a continuous decreasing function that satisfies the system (6). In principle, one could attempt to extend the domain of $\tilde{c}_{0.05}\left(f^2\right)$, as illustrated by the dashed extension in Figure 7. However, doing so would imply that there would be some f_0 (in the figure, for example, f_0''') for which (5) holds, but that would *not* satisfy the system (6) – a different set of equations would be needed.

It turns out that for the case of $\alpha = 0.05$, the possibility of extending the function is irrelevant because $\tilde{c}_{0.05}\left(\bar{f}\left(f_0''\right)^2\right)$ is below 1.96^2, as shown in the figure; any plateau below 1.96^2 would lead to over-rejection for some large values of f_0, and such a plateau would not be considered due to the need to control size.

By contrast, when applied to the case of $\alpha = 0.01$, the restriction on the domain of $\tilde{c}_\alpha\left(f^2\right)$ does become relevant. As illustrated in Appendix Figure A3, the tF critical value function plateaus at $\bar{f}\left(f_0\right)^2 = 252.342$ with $c_{0.01}\left(\bar{f}\left(f_0\right)^2\right) = 2.726^2$ which is greater than the nominal value of 2.576^2.[65] Our numerical analysis indicates that when the plateau is set to 2.726^2, size continues to be controlled, which raises the possibility that an even lower plateau could be used if one extended $\tilde{c}_{0.01}\left(f^2\right)$ outside our restricted domain. One conjecture is that for $f_0 > f_0''$ in Appendix Figure A3 the t^2 function intersects $\tilde{c}_{0.01}\left(f^2\right)$ four times, leading to an acceptance region in f consisting of a union of two disjoint intervals. This conjecture is consistent with numerical analysis that we conducted for an earlier version of our paper, but given that it is also an unproven conjecture, here we take a conservative approach and instead rely exclusively on the $\tilde{c}_{0.01}\left(f^2\right)$ function restricted to the domain so that we have a formally proven characterization of $\tilde{c}_{0.01}\left(f^2\right)$ via the system (6).[66]

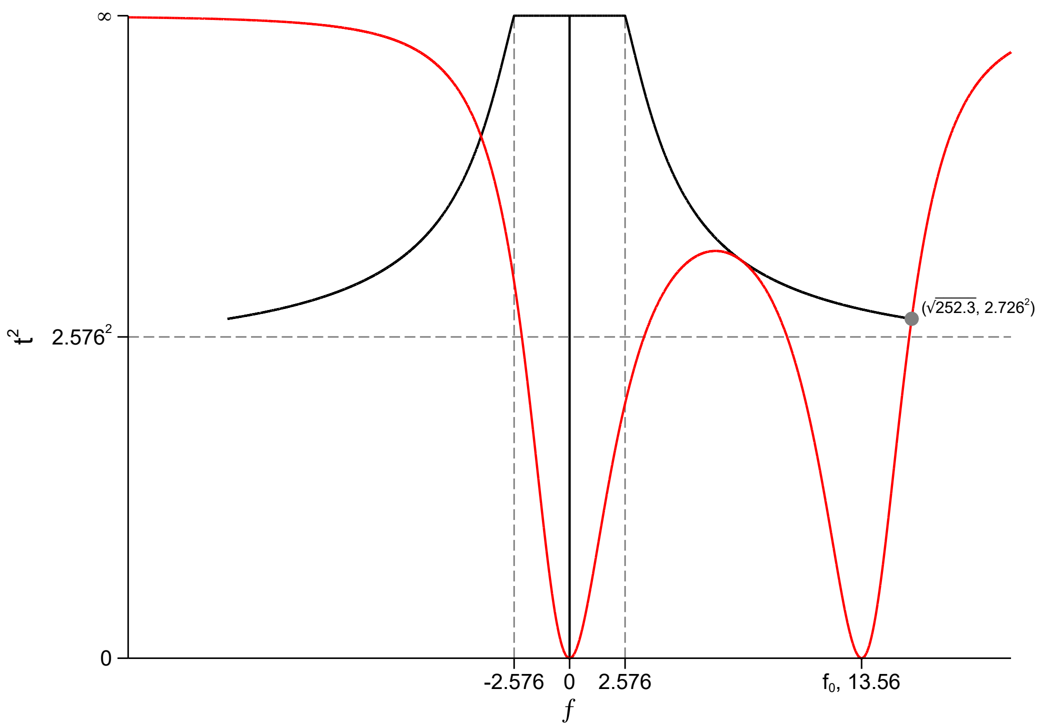
Remark. As added assurance of size control, in Appendix B.5 we present and prove a theoretical result that establishes that $\rho = \pm 1$ represents worst case rejection probabilities in a particular "corner" of the nuisance parameter space, when $|\rho|$ is in a neighborhood of 1 while f_0 is in a neighborhood of zero.

Remark. Finally, since the decreasing segment of the tF critical value function is driven by Property (B), it is natural to wonder whether there might exist an

[65] When we impose Property (C) for the cases of $\alpha = 0.005$ and $\alpha = 0.001$, and compute the decreasing segments used to construct the tF critical value functions, the lowest possible values on these computed segments are such that $\frac{\sqrt{\tilde{c}_{0.005}}}{2.807} \approx 1.075$ and $\frac{\sqrt{\tilde{c}_{0.001}}}{3.29} \approx 1.10$, (where 2.807 and 3.29 are the respective nominal thresholds for a two-tailed test), suggesting somewhat limited potential of efficiency gains from loosening the restriction of Property (C) for those significance levels.

[66] In Lee et al. (2020), we provide equations that use the conjecture of four intersection points.

Figure A3: Construction of the tF Critical Value Function, $\alpha = 0.01$



Note: To aid visualization of values, vertical axis uses the transformation $\frac{(t^2/2.576^2)}{1+(t^2/2.576^2)}$.

alternative valid critical value function that does not satisfy (B), also decreasing and similarly asymptoting to infinity, yet uniformly below $c_\alpha(F)$ in a neighborhood $(q_{1-\alpha}, q_{1-\alpha}+\varepsilon)$. Appendix B.4 shows that such an alternative does not exist. Thus, although one could imagine constructing size-controlling critical value functions that are uniformly below the tF function within intervals of F larger than $q_{1-\alpha}$ (and consequently above $c_\alpha(F)$ in other regions of F), one cannot construct one that is uniformly below $c_\alpha(F)$ within a neighborhood $(q_{1-\alpha}, q_{1-\alpha}+\varepsilon)$.

B.2 Existence and Uniqueness

From the system of equations (6), we can solve for f_0, reducing the system to a single functional equation (with $F = (\bar{f}(f_0))^2$) describing the behavior of $\tilde{c}_\alpha(\cdot)$ as it asymptotes:

$$\tilde{c}_\alpha\left([g(F)]^2\right) = \frac{g(F)^2\,[g(F)-h(F)]^2}{h(F)^2},$$

$$\text{where } h(F) = \frac{F}{\sqrt{\tilde{c}_\alpha(F)}+\sqrt{F}},$$
$$\text{and } g(F) = h(F) + \Phi^{-1}\left(\Phi\left(\sqrt{F}-h(F)\right)-(1-\alpha)\right).$$

More compactly, we have

$$\tilde{c}_\alpha\left(\left[\Phi^{-1}\left(\Phi\left(\sqrt{F}-\frac{F}{\sqrt{\tilde{c}_\alpha(F)}+\sqrt{F}}\right)-(1-\alpha)\right)+\frac{F}{\sqrt{\tilde{c}_\alpha(F)}+\sqrt{F}}\right]^2\right) \tag{13}$$
$$= \frac{\left[\Phi^{-1}\left(\Phi\left(\sqrt{F}-\frac{F}{\sqrt{\tilde{c}_\alpha(F)}+\sqrt{F}}\right)-(1-\alpha)\right)+\frac{F}{\sqrt{\tilde{c}_\alpha(F)}+\sqrt{F}}\right]^2}{\left[\frac{F}{\sqrt{\tilde{c}_\alpha(F)}+\sqrt{F}}\right]^2}$$
$$\cdot\left[\Phi^{-1}\left(\Phi\left(\sqrt{F}-\frac{F}{\sqrt{\tilde{c}_\alpha(F)}+\sqrt{F}}\right)-(1-\alpha)\right)\right]^2$$

Define $q_{1-\alpha} = \left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)^2$, $b = 3q_{1-\alpha} - \frac{q_{1-\alpha}^2}{2} + \frac{q_{1-\alpha}^3}{6}$.

Lemma 9. *There exists a function $\tilde{c}_\alpha(\cdot)$ satisfying (13) for $F \in (q_{1-\alpha}, q_{1-\alpha}+\delta]$ for some $\delta > 0$ with the following properties:*

(i) $\tilde{c}_\alpha(F) - \left(\frac{q_{1-\alpha}^3}{F-q_{1-\alpha}} - b\right) = O(\sqrt{F-q_{1-\alpha}})$ *as $F \downarrow q_{1-\alpha}$*

(ii) Let $\check{c}_\alpha$ satisfy (13) for $F \in (q_{1-\alpha}, q_{1-\alpha}+\check{\delta}]$ for some $\check{\delta} > 0$ with $\check{c}_\alpha(F) = \frac{q_{1-\alpha}^3}{F-q_{1-\alpha}} - b + o((F-q_{1-\alpha})^{-1/3})$ as $F \downarrow q_{1-\alpha}$. Then, $\tilde{c}_\alpha(F) = \check{c}_\alpha(F)$ for $F \in (q_{1-\alpha}, q_{1-\alpha}+\delta_1]$ and some $\delta_1 > 0$;

(iii) $\tilde{c}_\alpha \in C^\infty$ *on* $(q_{1-\alpha}, q_{1-\alpha}+\delta]$*;*

(iv) For any $k > 0$, there exists $\delta_2 > 0$ such that $\tilde{c}_\alpha(F) \geq k$ for $F \in (q_{1-\alpha}, q_{1-\alpha}+\delta_2]$, and $\tilde{c}_\alpha(F)$ is decreasing for $F \in (q_{1-\alpha}, q_{1-\alpha}+\delta_3]$ for some $\delta_3 > 0$.

PROOF: To show the desired existence, we will transform equation (13) to put it into canonical form for results from the dynamical systems literature. Once in canonical form, we find that (13) is a degenerate case to which the standard stable manifold theorem does not apply. New results from Fefferman (2021), Baldomá et al. (2007), and Baldomá, Fontich and Martín (2020) provide the desired existence and uniqueness.

Based on (13), define the map $T : (F, y) \mapsto (\nu, \eta)$ where

(14)

$$\nu(F,y) = \left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{y}+\sqrt{F}}\right) - (1-\alpha)\right) + \frac{F}{\sqrt{y}+\sqrt{F}}\right]^2$$

$$\eta(F,y) = \frac{\left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{y}+\sqrt{F}}\right) - (1-\alpha)\right) + \frac{F}{\sqrt{y}+\sqrt{F}}\right]^2 \left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{y}+\sqrt{F}}\right) - (1-\alpha)\right)\right]^2}{\left[\frac{F}{\sqrt{y}+\sqrt{F}}\right]^2}.$$

We will show existence of an invariant curve for the map T. In particular, a function $\tilde{c}_\alpha$ exists such that $T(\Pi) \subset \Pi$ where $\Pi = \{(F, c_\alpha(F)) \,|\, F \in (q_{1-\alpha}, q_{1-\alpha} + \delta]\}$ for some $\delta > 0$. Since $T(\Pi) \subset \Pi$,

$$\eta(F, \tilde{c}_\alpha(F)) = \tilde{c}_\alpha(\nu(F, \tilde{c}_\alpha(F))) \tag{15}$$

for all $F \in (q_{1-\alpha}, q_{1-\alpha} + \delta]$. Given the definitions of ν and η, (15) is exactly (13), so existence of the invariant curve for T yields a function $\tilde{c}_\alpha(\cdot)$ satisfying (13). We now turn to obtaining the desired invariant curve for T.

We will transform T to obtain an equivalent map with an approximation in canonical form.

$$h(t,z) = \frac{(t+q_{1-\alpha})\sqrt{t}}{\sqrt{z}+\sqrt{t(t+q_{1-\alpha})}}$$

$$gh(t,z) = \Phi^{-1}\left(\Phi\left(\sqrt{t+q_{1-\alpha}} - \frac{(t+q_{1-\alpha})\sqrt{t}}{\sqrt{z}+\sqrt{t(t+q_{1-\alpha})}}\right) - (1-\alpha)\right)$$

$$g(t,z) = gh(t,z) + h(t,z)$$

$$\xi(F,z) = [g(F-q_{1-\alpha},z)]^2$$

$$\zeta(F,z) = \frac{\left([g(F-q_{1-\alpha},z)]^2 - q_{1-\alpha}\right)[g(F-q_{1-\alpha},z)]^2\,[gh(F-q_{1-\alpha},z)]^2}{[h(F-q_{1-\alpha},z)]^2}$$

These functions define a dynamical system iterative map: $T^*: (F,z) \mapsto (\xi,\zeta)$ with a fixed point at $(q_{1-\alpha}, q_{1-\alpha}^3)$. Taking standard expansions in t and Lagrange remainders, we obtain

$$\begin{aligned}
h(t,z) &= \frac{q_{1-\alpha}}{\sqrt{z}}\sqrt{t} - \frac{q_{1-\alpha}^{3/2}}{z}t + \left[\frac{1}{\sqrt{z}} + \frac{q_{1-\alpha}^2}{z^{3/2}}\right]t^{3/2} - \left[\frac{3\sqrt{q_{1-\alpha}}}{2z} + \frac{q_{1-\alpha}^{5/2}}{z^2}\right]t^2 + r_h(t,z)t^{5/2} \\
gh(t,z) &= -\sqrt{q_{1-\alpha}} - \frac{q_{1-\alpha}}{\sqrt{z}}\sqrt{t} + \left[\frac{1}{2\sqrt{q_{1-\alpha}}} + \frac{q_{1-\alpha}^{3/2}}{z} - \frac{q_{1-\alpha}^{5/2}}{z}\right]t \\
&\quad + \left[\frac{q_{1-\alpha}-1}{\sqrt{z}} - \frac{q_{1-\alpha}^2(q_{1-\alpha}-1)^2}{z^{3/2}}\right]t^{3/2} \\
&\quad + \left[-\frac{1}{4\sqrt{q_{1-\alpha}}} - \frac{1}{8q_{1-\alpha}^{3/2}} + \frac{3\sqrt{q_{1-\alpha}}}{2z}(q_{1-\alpha}-1)^2 + \frac{q_{1-\alpha}^{5/2} - 3q_{1-\alpha}^{7/2}}{z^2} + \frac{11q_{1-\alpha}^{9/2}}{4z^2} - \frac{13q_{1-\alpha}^{11/2}}{12z^2}\right]t^2 \\
&\quad + r_{gh}(t,z)t^{5/2}
\end{aligned}$$

where the remainder terms $r_h(t,z)$ and $r_{gh}(t,z)$ can be bounded for t in a non-negative neighborhood of zero and z in a neighborhood of $q_{1-\alpha}^3$.

Corresponding expansions for $gh(t,z)$, $\xi(F,z)$, and $\zeta(F,z)$ follow. Re-centering the fixed point to the origin by the change of variables $\tau = \xi - q_{1-\alpha}$, $(t = F - q_{1-\alpha})$, $\mu = \zeta - q_{1-\alpha}^3$, $(u = z - q_{1-\alpha}^3)$, and then expanding in u in a neighborhood of zero yields:

$$\begin{aligned}
\tau(t,u) &= t - \frac{4}{q_{1-\alpha}}t^{3/2} - \frac{2}{q_{1-\alpha}^3}ut + R_\tau \\
\mu(t,u) &= -u + \left[-6q_{1-\alpha} + q_{1-\alpha}^2 - \frac{q_{1-\alpha}^3}{3}\right]t - \frac{2(2+q_{1-\alpha})}{q_{1-\alpha}}u\sqrt{t} + O(|(\sqrt{t},u)|^3)
\end{aligned}$$

where $R_\tau = \sum_{\iota=2}^4 \tilde{r}_\iota(t,u)(\sqrt{t})^\iota u^{4-\iota}$ and the terms $\tilde{r}_\iota(t,u)$ can be bounded for t in a non-negative neighborhood of zero and u in a neighborhood of zero. The form of the remainder R_τ allows t to be factored out in τ:

$$\tau = t\left[1 - \frac{4}{q_{1-\alpha}}t^{1/2} - \frac{2}{q_{1-\alpha}^3}u + \left(\sum_{\iota=0}^{2}\tilde{r}_\iota(t,u)(\sqrt{t})^\iota u^{2-\iota}\right)\right]. \tag{16}$$

Now, we can apply one more set of set of transformations $\tilde{X} = \frac{2}{q_{1-\alpha}}\sqrt{t}$, $\tilde{x} = \frac{2}{q_{1-\alpha}}\sqrt{\tau}$, $\tilde{Y} = u + bt$, and $\tilde{y} = \mu + b\tau$, where $b = 3q_{1-\alpha} - \frac{q_{1-\alpha}^2}{2} + \frac{q_{1-\alpha}^3}{6}$ is chosen to eliminate the $\tilde{X}^2$ term from the $\tilde{y}$ equation.

$$(17)\qquad \begin{aligned} \tilde{x} &= \tilde{X} - \tilde{X}^2 - \frac{1}{q_{1-\alpha}^3}\tilde{X}\tilde{Y} + O(|(\tilde{X},\tilde{Y})|^3) \\ \tilde{y} &= -\tilde{Y} - (2+q_{1-\alpha})\tilde{X}\tilde{Y} + O(|(\tilde{X},\tilde{Y})|^3) \end{aligned}$$

This mapping and its inverse:

$$(18)\qquad \begin{aligned} \tilde{X} &= \tilde{x} + \tilde{x}^2 - \frac{1}{q_{1-\alpha}^3}\tilde{x}\tilde{y} + O(|(\tilde{x},\tilde{y})|^3) \\ \tilde{Y} &= -\tilde{y} + (2+q_{1-\alpha})\tilde{x}\tilde{y} + O(|(\tilde{x},\tilde{y})|^3) \end{aligned}$$

are in form for direct application of the results in Fefferman (2021).

Applying the above series of transformations directly to the map $T:(F,y)\mapsto(\nu,\eta)$ in (14) yields the mapping $\Psi:(\tilde{X},\tilde{Y})\mapsto(\tilde{x},\tilde{y})$ given by

$$(19)\qquad \begin{aligned} \tilde{x} &= \frac{2}{q_{1-\alpha}}\sqrt{\nu\left(\frac{q_{1-\alpha}^2}{4}\tilde{X}+q_{1-\alpha},\frac{4}{q_{1-\alpha}^2\tilde{X}^2}(\tilde{Y}+q_{1-\alpha}^3)-b\right)-q_{1-\alpha}} \\ \tilde{y} &= \left[\nu\left(\frac{q_{1-\alpha}^2}{4}\tilde{X}+q_{1-\alpha},\frac{4}{q_{1-\alpha}^2\tilde{X}^2}(\tilde{Y}+q_{1-\alpha}^3)-b\right)-q_{1-\alpha}\right] \\ &\quad\cdot\left[\eta\left(\frac{q_{1-\alpha}^2}{4}\tilde{X}+q_{1-\alpha},\frac{4}{q_{1-\alpha}^2\tilde{X}^2}(\tilde{Y}+q_{1-\alpha}^3)-b\right)+b\right]-q_{1-\alpha}^3 \end{aligned}$$

So Ψ is the mapping approximated by (17) and the inverse $\Psi^{-1}:(X,Y)\mapsto(x,y)$ is approximated in (18).

By Fefferman (2021) Theorem 1.1, there exists a function $\bar{c}$ that:
(a) generates an invariant curve for Ψ, for $\bar{\Gamma}=\{(\tilde{x},\bar{c}(\tilde{x}))\,|\,\tilde{x}\in[0,\bar{\delta}]\}$, $\Psi(\bar{\Gamma})\subset\bar{\Gamma}$;
(b) is tangent to the x-axis near the fixed point at the origin, $\bar{c}(\tilde{x})=O(\tilde{x}^3)$ as $\tilde{x}\downarrow 0$; and
(c) is infinitely differentiable on $[0,\bar{\delta}]$ for some $\bar{\delta}>0$.
This theorem also delivers uniqueness in the following sense. Let $\breve{c}$ be a function such that $\tilde{x}^{-\frac{2}{3}}\breve{c}(\tilde{x})\to 0$ as $\tilde{x}\downarrow 0$ and define $\breve{\Gamma}=\{(\tilde{x},\breve{c}(\tilde{x}))\,|\,\tilde{x}\in[0,\breve{\delta}]\}$ for $\breve{\delta}>0$. If $\Psi(\breve{\Gamma})\subset\breve{\Gamma}$, then $\breve{c}=\bar{c}$ on $[0,\tilde{\delta}]$ for some $\tilde{\delta}>0$.

Given the function $\bar{c}$ that defines an invariant curve for Ψ, we define a corresponding function for T:

$$\tilde{c}_{\alpha}(F) = \frac{\bar{c}\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right) + q_{1-\alpha}^3}{F-q_{1-\alpha}} - b. \tag{20}$$

for $F > q_{1-\alpha}$ such that $\bar{c}\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right)$ is well-defined. Then, $\tilde{c}_{\alpha}$ will inherit the smoothness properties of $\bar{c}$ on this domain proving (iii). Consider F such that $q_{1-\alpha} < F \le q_{1-\alpha} + \frac{q_{1-\alpha}^2}{4}\bar{\delta}^2$, and define $y = \tilde{c}_{\alpha}(F)$. Now apply the map T yielding (ν, η) as given by (14). To show that $\tilde{c}_{\alpha}$ defines an invariant curve for T, we need to show that $\eta = \tilde{c}_{\alpha}(\nu)$. Let $\tilde{X} = \frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}$ and $\tilde{Y} = (y+b)(F-q_{1-\alpha}) - q_{1-\alpha}^3$. By the definition of $\tilde{c}_{\alpha}$, $\tilde{Y} = \bar{c}(\tilde{X})$ and $\tilde{X} \in (0, \bar{\delta}]$. Define $(\tilde{x}, \tilde{y}) = \Psi(\tilde{X}, \tilde{Y})$ as in (19). Then, the result in Fefferman (2021) shows that $\tilde{y} = \bar{c}(\tilde{x})$ and $\tilde{x} \in (0, \bar{\delta}]$. Notice that $\nu = \frac{q_{1-\alpha}^2}{4}\tilde{x} + q_{1-\alpha} \in (q_{1-\alpha}, q_{1-\alpha} + \frac{q_{1-\alpha}^2}{4}\bar{\delta}^2]$, and

$$\begin{aligned}
\eta &= \frac{4}{q_{1-\alpha}^2\tilde{x}^2}(\tilde{y} + q_{1-\alpha}^3) - b = \frac{4}{q_{1-\alpha}^2\tilde{x}^2}(\bar{c}(\tilde{x}) + q_{1-\alpha}^3) - b \\
&= \frac{\bar{c}\left(\frac{2}{q_{1-\alpha}}\sqrt{\nu - q_{1-\alpha}}\right) + q_{1-\alpha}^3}{\nu - q_{1-\alpha}} - b = \tilde{c}_{\alpha}(\nu),
\end{aligned}$$

as desired. This invariance shows that $\tilde{c}_{\alpha}$ satisfies (13) for $F \in (q_{1-\alpha}, q_{1-\alpha} + \frac{q_{1-\alpha}^2}{4}\bar{\delta}^2]$. Also, note that by the definition of $\tilde{c}_{\alpha}$ in (20), $\bar{c}(\tilde{x}) = O(\tilde{x}^3)$ directly implies $\tilde{c}_{\alpha}(F) - \left(\frac{q_{1-\alpha}^3}{F-q_{1-\alpha}} - b\right) = O(\sqrt{F-q_{1-\alpha}})$ as $F \downarrow q_{1-\alpha}$ proving (i).

Next, we show uniqueness of $\tilde{c}_{\alpha}$. Consider a function

$$\check{c}_{\alpha} \in \left\{ c \,\middle|\, (F-q_{1-\alpha})^{1/3}\left[c(F) - \left(\frac{q_{1-\alpha}^3}{F-q_{1-\alpha}} - b\right)\right] \to 0 \text{ as } F \downarrow q_{1-\alpha} \right\}$$

such that $T(\Pi) \subset \Pi$ where $\Pi = \{(F, \check{c}_{\alpha}(F)) \,|\, F \in (q_{1-\alpha}, q_{1-\alpha} + \frac{q_{1-\alpha}^2}{4}\check{\delta}^2]\}$ for some $\check{\delta} > 0$. Set $\check{c}(\tilde{x}) = \left[\check{c}_{\alpha}\left(\frac{q_{1-\alpha}^2}{4}\tilde{x}^2 + q_{1-\alpha}\right) + b\right]\left(\frac{q_{1-\alpha}^2}{4}\tilde{x}^2\right) - q_{1-\alpha}^3$. Similar to the argument above, $T(\Pi) \subset \Pi$ implies that $\Psi(\check{\Gamma}) \subset \check{\Gamma}$ for $\check{\Gamma} = \{(\tilde{x}, \check{c}(\tilde{x})) \,|\, \tilde{x} \in (0, \check{\delta}]\}$. By the uniqueness result in Fefferman (2021), it follows that $\check{c} = \bar{c}$ on $(0, \tilde{\delta}]$ for some $\tilde{\delta} > 0$ and hence $\check{c}_{\alpha} = \tilde{c}_{\alpha}$ on $(q_{1-\alpha}, q_{1-\alpha} + \frac{q_{1-\alpha}^2}{4}\tilde{\delta}^2]$, which shows (ii).

Now, we show that $\tilde{c}_{\alpha}(F)$ is decreasing for $F \in (q_{1-\alpha}, q_{1-\alpha} + \delta_3]$ for some $\delta_3 > 0$. Since $\bar{c}'(\tilde{x})$ is continuous on $[0, \bar{\delta}]$, it is also bounded. In particular, $\bar{c}'(\tilde{x}) < k$

on $[0,\bar{\delta}]$, for some $k>0$. Since $\bar{c}(\tilde{x})=O(\tilde{x}^3)$, there exists $\bar{\delta}_1\in(0,\bar{\delta}]$ such that $\bar{c}(\tilde{x})>-\frac{q_{1-\alpha}^3}{3}$ for $\tilde{x}\in[0,\bar{\delta}_1]$, and hence $-q_{1-\alpha}\bar{c}\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right)<\frac{d^4}{3}$ for $F\in(q_{1-\alpha},q_{1-\alpha}+\frac{q_{1-\alpha}^2}{4}\bar{\delta}_1^2]$. Let $\delta_3=\min\{\frac{q_{1-\alpha}^2}{4}\bar{\delta}_1,\frac{q_{1-\alpha}^8}{9k^2}\}$. Then, $F\in(q_{1-\alpha},q_{1-\alpha}+\delta_3]$ implies $\sqrt{F-q_{1-\alpha}}<\frac{q_{1-\alpha}^4}{3k}$ and $\bar{c}'\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right)\sqrt{F-q_{1-\alpha}}<\frac{q_{1-\alpha}^4}{3}$. Then, for $F\in(q_{1-\alpha},q_{1-\alpha}+\delta_3]$,

$$\begin{aligned}\tilde{c}_\alpha'(F) &= \frac{\bar{c}'\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right)\sqrt{F-q_{1-\alpha}}-q_{1-\alpha}\bar{c}\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right)-q_{1-\alpha}^4}{q_{1-\alpha}(F-q_{1-\alpha})^2}\\ &< \frac{\frac{q_{1-\alpha}^4}{3}+\frac{q_{1-\alpha}^4}{3}-q_{1-\alpha}^4}{q_{1-\alpha}(F-q_{1-\alpha})^2} = -\frac{q_{1-\alpha}^3}{3(F-q_{1-\alpha})^2}<0.\end{aligned}$$

Lastly, take any $k>0$. Set $\delta_2=\min\left\{\frac{2q_{1-\alpha}^3}{3(k+b)},\frac{q_{1-\alpha}^2}{4}\bar{\delta}_1^2\right\}$. For $F\in(q_{1-\alpha},q_{1-\alpha}+\delta_2]$, $0<\frac{2}{d}\sqrt{F-q_{1-\alpha}}\le\bar{\delta}_1$ which implies $\bar{c}\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right)>-\frac{q_{1-\alpha}^3}{3}$, and $F-q_{1-\alpha}<\frac{2q_{1-\alpha}^3}{3(k+b)}$ implies $-q_{1-\alpha}^3+(k+b)(F-q_{1-\alpha})<-\frac{q_{1-\alpha}^3}{3}$. Hence, $\bar{c}\left(\frac{2}{q_{1-\alpha}}\sqrt{F-q_{1-\alpha}}\right)>-q_{1-\alpha}^3+(k+b)(F-q_{1-\alpha})$ which can be re-arranged to yield $\tilde{c}_\alpha(F)>k$, so (iv) is proven. ⊠

Existence. Now we show that a candidate critical value function in $\mathbb{C}$ exists.

Proposition 1. *($\mathbb{C}$ is nonempty) There exists a critical value function $\tilde{c}\left(F;\tilde{F}\right)$ satisfying Properties (A), (B), (C), and (D).*

Proof: To see this, using the notation from Lemma 9, which states the existence of some $\tilde{c}_\alpha(F)$, we construct the critical value function $\tilde{c}\left(F;\tilde{F}\right)$ as follows.

Since $\tilde{c}_\alpha(F)$ asymptotes as $F\downarrow q_{1-\alpha}$, we can choose $\tilde{F}\in(q_{1-\alpha},q_{1-\alpha}+\delta_3]$ such that $\frac{\tilde{F}}{\sqrt{\tilde{c}_\alpha(\tilde{F})+\sqrt{\tilde{F}}}}\le q_{1-\alpha}$, for δ_3 as defined in Lemma 9. Define

$$\tilde{c}\left(F;\tilde{F}\right)\equiv\begin{cases}\tilde{c}_\alpha(F) & \text{for } F\in(q_{1-\alpha},\tilde{F}]\\ \tilde{c}_\alpha\left(\tilde{F}\right) & \text{for } F>\tilde{F}\end{cases}.$$

By this definition and Lemma 9 (iii) and (iv), $\tilde{c}\left(F;\tilde{F}\right)$ is a continuous plateau function satisfying property (A).

Recall $\bar{f}_0$ as defined in (12). Take any f_0 such that $0<f_0\le\bar{f}_0$ (the case $0>f_0>-\bar{f}_0$ follows similarly). We next establish convexity of the acceptance region as in property (C). Choice of $\tilde{F}$ ensures that $\frac{f^2(f-f_0)^2}{f_0}$ is strictly increasing for $f > \sqrt{q_{1-\alpha}}$ and strictly decreasing for $f < 0$. Also, note that $\tilde{c}\left(f^2;\tilde{F}\right)$ is weakly decreasing for $f > \sqrt{q_{1-\alpha}}$ and weakly increasing for $f < 0$. By the asymptoting behavior of $\tilde{c}\left(f^2;\tilde{F}\right)$ as $f \downarrow \sqrt{q_{1-\alpha}}$, there exists $f > \sqrt{q_{1-\alpha}}$ such that $\tilde{c}\left(f^2;\tilde{F}\right) > \frac{f^2(f-f_0)^2}{f_0}$. Also note $\frac{\tilde{F}\left(\sqrt{\tilde{F}}-f_0\right)^2}{f_0} \geq \frac{\tilde{F}\left(\sqrt{\tilde{F}}-\bar{f}_0\right)^2}{\bar{f}_0} = \tilde{c}_\alpha\left(\tilde{F}\right) = \tilde{c}\left(\tilde{F};\tilde{F}\right)$. It follows that $\frac{f^2(f-f_0)^2}{f_0}$ intersects $\tilde{c}\left(f^2;\tilde{F}\right)$ exactly once for $f > \sqrt{q_{1-\alpha}}$. Moreover, the intersection point occurs for $f \in (\sqrt{q_{1-\alpha}}, \sqrt{\tilde{F}}]$ which is the decreasing part of $\tilde{c}\left(f^2;\tilde{F}\right)$. Similarly $\frac{f^2(f-f_0)^2}{f_0}$ intersects $\tilde{c}\left(f^2;\tilde{F}\right)$ exactly once for $f < -\sqrt{q_{1-\alpha}}$ and the intersection occurs for $f \in [-\sqrt{\tilde{F}}, -\sqrt{q_{1-\alpha}})$. This establishes property (C).

Now we turn to property (B). Again take any f_0 such that $0 < f_0 \leq \bar{f}_0$. As argued above, there exists $f^U \in (\sqrt{q_{1-\alpha}}, \sqrt{\tilde{F}}]$ such that $\frac{(f^U)^2\left(f^U-f_0\right)^2}{f_0} = \tilde{c}\left((f^U)^2;\tilde{F}\right)$. Solving for f_0 in terms of f^U, we get $f_0 = \frac{(f^U)^2}{\sqrt{\tilde{c}\left((f^U)^2;\tilde{F}\right)}+f^U}$. Notice that $f^U \in (\sqrt{q_{1-\alpha}}, \sqrt{\tilde{F}}]$, so (13) holds with $F = (f^U)^2$ and $\tilde{c}_\alpha(F) = \tilde{c}\left((f^U)^2;\tilde{F}\right)$. It follows that the next expression is well defined. Let

$$\begin{aligned} f^L &= \Phi^{-1}\left(\Phi(f^U - f_0) - (1-\alpha)\right) + f_0 \\ &= \Phi^{-1}\left(\Phi\left(f^U - \frac{(f^U)^2}{\sqrt{\tilde{c}\left((f^U)^2;\tilde{F}\right)} + f^U}\right) - (1-\alpha)\right) + \frac{(f^U)^2}{\sqrt{\tilde{c}\left((f^U)^2;\tilde{F}\right)} + f^U} \end{aligned}$$

With this definition, (13) can be restated as $\tilde{c}\left((f^L)^2;\tilde{F}\right) = \frac{(f^L)^2(f^L-f_0)^2}{f_0^2}$. That is, f^L is the other point of intersection between the functions $\frac{f^2(f-f_0)^2}{f_0}$ and $\tilde{c}\left(f^2;\tilde{F}\right)$. By property (C), $[f^L, f^U]$ forms the acceptance region under f_0. Also, notice that rearranging the expression for f^L, we have

$$\Phi(f^U - f_0) - \Phi(f^L - f_0) = 1 - \alpha.$$

Hence, property (B) holds. And Property (D) holds by Lemma 9 (ii). ⊠

Uniqueness. Having established that $\mathbb{C}$ is nonempty, we characterize the extent to which the decreasing segment of any member of $\mathbb{C}$ is unique for a given $\tilde{F}$.

Lemma 10. *Suppose $c\left(F;\tilde{F}\right) \in \mathbb{C}$. Then for $F \in (q_{1-\alpha}, \tilde{F}]$, $c\left(F;\tilde{F}\right)$ satisfies the functional equation (13).*

PROOF: Continuity and decreasingness of the function from Property (A), as well as the restriction on the size of $\tilde{F}$ from Property (C), implies that when $|\rho| = 1$ the t^2 function (a quartic in f) will intersect the critical value function two times when $|f_0| < \bar{f}_0$. Thus, Property (B) implies that the system of equations in (6), which implies $c\left(F;\tilde{F}\right)$ must satisfy the functional equation (13). ☒

Proposition 2. ***(Uniqueness)****. Suppose $c_1\left(F;\tilde{F}_1\right), c_2\left(F;\tilde{F}_2\right) \in \mathbb{C}$. Let $\tilde{F}_{min} = \min\left\{\tilde{F}_1, \tilde{F}_2\right\}$. Then $c_1\left(F;\tilde{F}_1\right) = c_2\left(F;\tilde{F}_2\right)$ for $F \in (q_{1-\alpha}, \tilde{F}_{min}]$.*

PROOF: We will prove by contradiction: suppose that for some $F_A \in (q_{1-\alpha}, \tilde{F}_{min}]$, $c_1\left(F_A;\tilde{F}_1\right) \neq c_2\left(F_A;\tilde{F}_2\right)$. By Lemma 10, c_1 and c_2 satisfy the functional equation on $(q_{1-\alpha}, \tilde{F}_{min}]$. By Lemma 9, $\exists \delta_1 > 0$ such that $c_1\left(F;\tilde{F}_1\right) = c_2\left(F;\tilde{F}_2\right)$ for $F \in (q_{1-\alpha}, q_{1-\alpha}+\delta_1]$. From the supposition, $q_{1-\alpha} < q_{1-\alpha}+\delta_1 < F_A$. Let $\bar{\delta} = \sup\{\delta : c_1\left(F;\tilde{F}_1\right) = c_2\left(F;\tilde{F}_2\right), \forall F \in (q_{1-\alpha}, q_{1-\alpha}+\delta]\}$. By continuity of the functions from Property (A), the supremum is equal to the maximum. Also, by continuity of the functions from Property (A), $\exists \varepsilon > 0$ such that $c_1\left(F;\tilde{F}_1\right) \neq c_2\left(F;\tilde{F}_2\right)$ for all $F \in (q_{1-\alpha}+\bar{\delta}, q_{1-\alpha}+\bar{\delta}+\varepsilon)$ and $q_{1-\alpha}+\bar{\delta}+\varepsilon < \tilde{F}_{min}$.

When $|\rho| = 1$, the t^2 function of f is quartic and due to the continuity and decreasingness in Property (A) and the restriction of Property (C), it intersects exactly twice for each $0 < |f_0| < \bar{f}_0$ for both c_1 and c_2.[67] Consider $\tilde{f}_U = \sqrt{q_{1-\alpha}+\bar{\delta}}$, and let $\tilde{f}_0 = \frac{\tilde{f}_U^2}{\sqrt{c_1\left(\tilde{f}_U^2;\tilde{F}_1\right)}+\tilde{f}_U}$. Then, $\tilde{f}_U$ is one point of intersection of $c_1(f^2;\tilde{F}_1)$ and $\frac{f^2(f-\tilde{f}_0)^2}{\tilde{f}_0^2}$. Let $\tilde{f}_L$ denote the other point of intersection. Since $c_1(f^2;\tilde{F}_1)$ is symmetric about zero, and $\frac{f^2(f-\tilde{f}_0)^2}{\tilde{f}_0^2}$ is symmetric about $\tilde{f}_0/2$, $\tilde{f}_L^2 < \tilde{f}_U^2 = q_{1-\alpha}+\bar{\delta}$. For any $F_U \in (q_{1-\alpha}+\bar{\delta}, q_{1-\alpha}+\bar{\delta}+\varepsilon)$, note that $\sqrt{F_U}$ is one point of intersection between $c_1(f^2;\tilde{F}_1)$ and $\frac{f^2(f-f_0)^2}{f_0^2}$ where $f_0 = \frac{F_U}{\sqrt{c_1\left(F_U;\tilde{F}_1\right)}+\sqrt{F_U}}$. Let $f_L = -\sqrt{F_L}$ denote the other point of intersection. By continuity, we can choose F_U close enough to $\tilde{f}_U^2 = q_{1-\alpha}+\bar{\delta}$ such that $F_L < q_{1-\alpha}+\bar{\delta}$. By Properties (A), (B), (C), the acceptance region for $\rho = 1$ and f_0 as given above, must be $[f_L, \sqrt{F_U}]$ and

$$\Phi\left(\sqrt{F_U}-f_0\right) - \Phi\left(f_L - f_0\right) = 1-\alpha. \tag{21}$$

Since c_1 and c_2 coincide for $F \leq q_{1-\alpha}+\bar{\delta}$, the point f_L must also be a point of intersection between $c_2(f^2;\tilde{F}_2)$ and $\frac{f^2(f-f_0)^2}{f_0^2}$ using f_0 as above. By Properties (A),

[67] Note that $\bar{f}_0$ in this statement would generally take different values for c_1 and c_2 based on (12).

(B), (C), there is one other point of intersection, which we denote by $f_{U,2} = \sqrt{F_{U,2}}$, and we must have $\Phi(f_{U,2} - f_0) - \Phi(f_L - f_0) = 1 - \alpha$. Given (21) and invertibility of $\Phi(\cdot)$, we must have $F_{U,2} = F_U$. That is, the other points of intersection for c_1 and c_2 must also be the same. Recall that we chose $F_U \in (q_{1-\alpha} + \bar{\delta}, q_{1-\alpha} + \bar{\delta} + \varepsilon)$, so that $c_1\left(F_U; \tilde{F}_1\right) \neq c_2\left(F_U; \tilde{F}_2\right)$. Hence, $\frac{f^2(f-f_0)^2}{f_0^2}$ cannot intersect both c_1 and c_2 at F_U. This contradiction means the supposition does not hold and the conclusion of the proposition follows. ⊠

B.3 Numerical Recipe for tF Critical Values

We now describe the process of computing the critical values in Tables 3 Panel A and 3 Panel B.

1. For a range of values in the $(q_{1-\alpha}, q_{1-\alpha} + \varepsilon]$ interval, with ε as small as numerically feasible, compute $c_\alpha(F)$ as

   $$c_\alpha(F) = \frac{q_{1-\alpha}^3}{F - q_{1-\alpha}} - \left(3q_{1-\alpha} - \frac{q_{1-\alpha}^2}{2} + \frac{q_{1-\alpha}^3}{6}\right)$$

   which comes from Lemma 9.

2. Using the resulting pairs $(F, c_\alpha(F))$, extend the function by generating new values F' and $c_\alpha(F')$ using the inverse map T^{-1}, which can be broken down into the following intermediate steps:

   $$f_0 = \frac{F}{\sqrt{c_\alpha(F)} - \sqrt{F}}$$
   $$\sqrt{F'} = f_0 + \Phi^{-1}\left((1-\alpha) + \Phi\left(-\sqrt{F} - f_0\right)\right)$$
   $$c_\alpha(F') = \frac{F'\left(\sqrt{F'} - f_0\right)^2}{f_0^2}$$

3. Using the updated function, re-apply the inverse map as above and iterate to extend the domain as much as possible, with each iteration verifying that the function is decreasing over the extended domain. Note that, by construction, this segment of the function satisfies property (D).

4. At the end of the iterative process, truncate the domain (if necessary) so that Property (C) holds. Note that as long as Property (C) holds and the non-plateau function is still decreasing, then Property (B) will hold.

5. Consider different critical value functions based on plateaus, departing at different points on the decreasing segment, as specified in Property (A).

6. Use numerical integration to find the lowest plateau that controls size for all ρ and f_0.

We implement this algorithm using an extremely fine grid of values on the initial interval in the first step, with ε on the order of 0.0001.

B.4 Non-existence of alternative critical value function that is uniformly below $c_\alpha(F)$ in a neighborhood $(q_{1-\alpha}, q_{1-\alpha}+\delta)$

Proposition 3. *Consider any alternative continuous function $k(F)$ (with plateau structure) satisfying Property (A) such that $k(F) \leq c_\alpha(F)$ for all $F \leq q_{1-\alpha}+\delta$, $0 < \delta \leq F^* - q_{1-\alpha}$, with $k(F_1) < c_\alpha(F_1)$ for some value $F_1 < q_{1-\alpha}+\delta$. Then $k(F)$ cannot control size to be α.*

PROOF: We prove that if in a neighborhood, $k(F)$ is uniformly below $c_\alpha(F)$, then there will be a data generating process (in particular, one with $\rho = 1$) that will over-reject.

Define

$$f_0^* = \frac{F_1}{\sqrt{c_\alpha(F_1)} + \sqrt{F_1}}.$$

And recall that when $\rho = 1$, $t^2(F) = \frac{F\left(\sqrt{F}-f_0\right)^2}{f_0^2}$. Fixing $f_0 = f_0^*$, where by supposition $t^2(F_1) = c_\alpha(F_1) > k(F_1)$, and c_α and k are continuous and non-increasing at F_1, and t^2 is continuous and strictly increasing at F_1 (since $F_1 > f_0^*$). It follows by continuity that there exists an ε with $0 < \varepsilon < F_1 - q_{1-\alpha}$ such that for $F \in [F_1-\varepsilon, F_1)$ we have $k(F) < t^2(F) < c_\alpha(F)$. By this definition of the small region $[F_1-\varepsilon, F_1)$, it must be true that $[F_1-\varepsilon, F_1) \subset \{F \,|\, t^2(F) \geq k(F)\}$. At the same time, by the definition of f_0^*, we have $[F_1-\varepsilon, F_1) \cap \{F \,|\, t^2(F) \geq c_\alpha(F)\} = \emptyset$. Since $k(F) \leq c_\alpha(F)$ for all $F \leq_{1-\alpha} +\delta$ and because k is decreasing in F,then $\{F \,|\, t^2(F) \geq c_\alpha(F)\} \subset \{F \,|\, t^2(F) \geq k(F)\}$. So both $\{F \,|\, t^2(F) \geq c_\alpha(F)\}$ and $[F_1 - \varepsilon, F_1)$ are non-intersecting and subsets of $\{F \,|\, t^2(F) \geq k(F)\}$. Thus, $\{F \,|\, t^2(F) \geq$

$c_\alpha(F)\} \cup [F_1-\varepsilon, F_1) \subset \{F \,|\, t^2(F) \geq k(F)\}$. This means that

$$\begin{aligned}\alpha &= \Pr_{f_0^*,\rho=1}(t^2(F) \geq c_\alpha(F)) \\ &\leq \Pr_{f_0^*,\rho=1}(t^2(F) \geq k(F)) - \Pr_{f_0^*,\rho=1}(F \in [F_1-\varepsilon, F_1)) \\ &< \Pr_{f_0^*,\rho=1}(t^2(F) \geq k(F))\end{aligned}$$

But this contradicts k controlling size at level α. ⊠

B.5 tF: Size control for $|\rho|$ near 1, small f_0

Proposition 4. *Under the null hypothesis, for any arbitrarily small departure from $|\rho| = 1$ there exists a neighborhood of values f_0 near $f_0 = 0$ such that all rejection probabilities $\Pr_{\rho,f_0}\left[t^2 > c_\alpha(F)\right]$ are smaller than the intended significance level α.*

Proof: Below, the proof involves focusing on small f_0, using the change of variables $\varrho = \sqrt{1-\rho^2}$ and considering the derivative of the rejection probability with respect to ϱ, evaluated at $\varrho = 0$. We find that the first derivative is zero for f_0 small. We therefore compute the second derivative at $\varrho = 0$, and then take a Taylor series expansion of this second derviative expression to find that in a neighborhood of $f_0 = 0$, this second derivative is negative, which implies that when one departs slightly from $|\rho| = 1$, then the rejection probability will decline, leading to size control in this "corner" of the nuisance parameter space. Below, we suppress α to simplify notation.

We begin with our relationship

$$t^2 = \frac{f^2 t_{AR}^2}{f^2 - 2\rho t_{AR} f + t_{AR}^2}$$

which expresses t^2 as a function of t_{AR}, f, and correlation ρ.

Under the tF procedure, rejection occurs in the event that

$$f^2 t_{AR}^2 - \left(f^2 - 2\rho t_{AR} f + t_{AR}^2\right) c_\alpha\left(f^2\right) > 0$$

where $c_\alpha\left(f^2\right)$ is our critical value function, and where f and t_{AR} are bivariate normal with unit variances and mean vector $(f_0, 0)$ (under the null hypothesis), with correlation ρ.

We do a change of variables

$$x = f - \rho t_{AR}$$

and note that x and t_{AR} are by construction uncorrelated and therefore, by bivariate normality, independent. x has mean f_0 and variance $1-\rho^2$.

Substituting, we now have rejection occurring when

$$(22)\quad (x+\rho t_{AR})^2 t_{AR}^2 - \left((x+\rho t_{AR})^2 - 2\rho t_{AR}(x+\rho t_{AR}) + t_{AR}^2\right) c_\alpha\left((x+\rho t_{AR})^2\right) > 0$$

We now have

$$\begin{aligned}\Pr\left[t^2 > c_\alpha\left(f^2\right)\right] = \int_{-\infty}^{\infty} & [1-\Phi(r_4(\rho,z)) \\ & +\Phi(r_1(\rho,z)) \\ & + 1[|z|>\bar{z}]\{\Phi(r_3(\rho,z)) - \Phi(r_2(\rho,z))\}] \frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{z-f_0}{\sqrt{1-\rho^2}}\right)dz\end{aligned}$$

where r_1, r_2, r_3, r_4 are functions of x and ρ that are implicitly defined by the r_j that satisfy

$$(x+\rho r_j)^2 r_j^2 - \left((x+\rho r_j)^2 - 2\rho r_j(x+\rho r_j) + r_j^2\right) c_\alpha\left((x+\rho r_j)^2\right) = 0$$

r_j gives the t_{AR} coordinate of any point on the critical value boundaries, as a function of ρ and x. Since the equation defines a (near) quartic polynomial in r_j, we can expect up to four roots of the equation. z is the variable of integration for the random variable x.

We now do two changes of variables

$$U = \frac{x-f_0}{\sqrt{1-\rho^2}}$$
$$\varrho = \sqrt{1-\rho^2}$$

where we will be focusing on a neighborhood, without loss of generality, of $\rho = 1$ (and equivalently a neighborhood of $\varrho = 0$).

t_{AR} and U are also independent; U is a standard normal random variable. With this change of variables we substitute and now have

$$\begin{aligned}\Pr\left[t^2 > c\left(f^2\right)\right] = \int_{-\infty}^{\infty} & [1-\Phi(r_4^*(\varrho,u,f_0)) \\ & +\Phi(r_1^*(\varrho,u,f_0)) \\ & + 1[|f_0+\varrho u|>\bar{z}]\{\Phi(r_3^*(\varrho,u,f_0)) - \Phi(r_2^*(\varrho,u,f_0))\}]\phi(u)\,du\end{aligned}$$

where we have $r_j^*(\varrho,u,f_0)=r_j\left(\sqrt{1-\varrho^2},f_0+\varrho\,u\right)$ for $j=1,2,3,4$, and $\bar{z}$ is defined as the value of u that separates the regions where there are 4 or 2 roots. Note that, using the change of variables, each of the r_j^* also satisfy the equation

$$F\left(\varrho,r_j^*,u,f_0\right)=\left(f_0+\varrho\,u+\sqrt{1-\varrho^2}r_j^*\right)^2\left(r_j^*\right)^2$$
$$-\left(\left(f_0+\varrho\,u+\sqrt{1-\varrho^2}r_j^*\right)^2-2\sqrt{1-\varrho^2}r_j^*\left(f_0+\varrho\,u+\sqrt{1-\varrho^2}r_j^*\right)+\left(r_j^*\right)^2\right)$$
$$c_\alpha\left(\left(f_0+\varrho\,u+\sqrt{1-\varrho^2}r_j^*\right)^2\right)=0$$

Derivatives: first and second derivatives

We now take both the first and second derivative of the rejection probability with respect to ϱ, evaluated at $\varrho=0$, and with f_0 "sufficiently small". Here, "sufficiently small" corresponds to small enough f_0 so that the derivative terms below associated with r_2^* and r_3^* will be zero.

Thus, with sufficiently small f_0, the first derivative of the rejection probability is

$$\frac{\partial\Pr\left[t^2>c\left(f^2\right)\right]}{\partial\,\varrho}=\int_{-\infty}^{\infty}\left[-\phi\left(r_4^*\right)\frac{\partial r_4^*}{\partial\,\varrho}\right.$$
$$\left.+\,\phi\left(r_1^*\right)\frac{\partial r_1^*}{\partial\,\varrho}\right]\phi\left(u\right)du$$

and the second derivative is

$$\frac{\partial^2\Pr\left[t^2>c\left(f^2\right)\right]}{\partial\,\varrho^2}=\int_{-\infty}^{\infty}\left[r_4^*\phi\left(r_4^*\right)\left(\frac{\partial r_4^*}{\partial\,\varrho}\right)^2-\phi\left(r_4^*\right)\frac{\partial^2 r_4^*}{\partial\,\varrho^2}\right.$$
$$\left.-r_1^*\phi\left(r_1^*\right)\left(\frac{\partial r_1^*}{\partial\,\varrho}\right)^2+\phi\left(r_1^*\right)\frac{\partial^2 r_1^*}{\partial\,\varrho^2}\right]\phi\left(u\right)du$$
$$=\int_{-\infty}^{\infty}\left[\phi\left(r_4^*\right)\left\{r_4^*\left(\frac{\partial r_4^*}{\partial\,\varrho}\right)^2-\frac{\partial^2 r_4^*}{\partial\,\varrho^2}\right\}\right.$$
$$\left.-\phi\left(r_1^*\right)\left\{r_1^*\left(\frac{\partial r_1^*}{\partial\,\varrho}\right)^2-\frac{\partial^2 r_1^*}{\partial\,\varrho^2}\right\}\right]\phi\left(u\right)du.$$

We then take the following steps:

1. Using implicit differentiation, obtain the first and second derivatives of r_j^*

with respect to ϱ. These expressions will be functions of $r_j^*, \varrho, u, f_0, c_\alpha(\cdot)$, and $c'_\alpha(\cdot)$.

2. Evaluate these derivatives at $\varrho = 0$. The expressions will be functions of $r_j^*, u, f_0, c_\alpha(\cdot)$, and $c'_\alpha(\cdot)$

3. Because $\varrho = 0$ is equivalent to $\rho = 1$, we can replace $r_j^* = f_j^* - f_0$, where f_j^* is the corresponding f-coordinate on the critical value boundary. This substitution results in functions that involve $f_j^*, u, f_0, c_\alpha(\cdot)$, and $c'_\alpha(\cdot)$

4. We use the fact that at $\varrho = 0$, that for every associated f_0 there are f_j^* that satisfy $f_0 = \frac{\left(f_j^*\right)^2}{\sqrt{\left(f_j^*\right)^2 + \sqrt{c\left(\left(f_j^*\right)^2\right)}}}$, substituting this in leaves expressions that involve $f_j^*, u, c_\alpha(\cdot)$, and $c'_\alpha(\cdot)$

5. We make another substitution: $\zeta = c_\alpha\left(\left(f_j^*\right)^2\right)\left[\left(f_j^*\right)^2 - q_{1-\alpha}\right]$ which implies that $c'_\alpha\left((f^*)^2\right) = \frac{\zeta'\left((f^*)^2\right)}{(f^*)^2 - q_{1-\alpha}} - \frac{\zeta\left((f^*)^2\right)}{\left((f^*)^2 - q_{1-\alpha}\right)^2}$. This substituion leads to expressions that are functions of $f_j^*, u, \zeta(\cdot)$, and $\zeta'(\cdot)$

6. Another change of variables, using $\tau > 0$ as $f_4^* = \sqrt{\tau^2 + q_{1-\alpha}}$ and $f_1^* = -\sqrt{\tau^2 + q_{1-\alpha}}$, leaving expressions that involve τ, u, ζ, ζ', and $q_{1-\alpha}$.

7. Collect powers of u, integrate out u (noting U is standard normal) so that $\int u^2 \phi(u)\, du = 1$. This leaves expressions that involve τ, ζ, and ζ', and $q_{1-\alpha}$. At this step, we have found that the first derivative of the rejection probability for f_0 sufficiently small as described above is equal to zero.

8. Take a first order taylor series expansion in τ around $\tau = 0$, and note from property (i) from Lemma 9 that as τ tends to zero, ζ tends to $q_{1-\alpha}^3$ and ζ' tends to $-\left(3q_{1-\alpha} - \frac{q_{1-\alpha}^2}{2} + \frac{q_{1-\alpha}^3}{6}\right)$. This means that the linear approximation for the second derivative (with respect to ϱ) is a linear function with constants and linear coefficient depending on $q_{1-\alpha}$ only.

9. Specifically, the second derivative is

$$\phi\left(\sqrt{q_{1-\alpha}}\right)\left[-2\left(\sqrt{q_{1-\alpha}} + q_{1-\alpha}^{\frac{3}{2}}\right) - \frac{4\left(1 + q_{1-\alpha}\right)}{\sqrt{q_{1-\alpha}}}\tau\right] \tag{23}$$

This means that we can always find a small enough $\tau = \sqrt{F - q_{1-\alpha}}$ so that the second derivative is negative. Since we are at $\varrho = 0$, for each of these small values of τ, there is a corresponding f_0

$$f_0 = \frac{\tau^2 + q_{1-\alpha}}{\sqrt{c_\alpha(\tau^2 + q_{1-\alpha})} + \sqrt{\tau^2 + q_{1-\alpha}}}$$

(f_0 and τ are one-to-one with sufficiently small τ, because

$$\frac{df_0}{d\tau} = \frac{2\tau\left(\sqrt{c_\alpha(\tau^2 + q_{1-\alpha})} + \sqrt{\tau^2 + q_{1-\alpha}}\right) - (\tau^2 + q_{1-\alpha})\left[\frac{c'_\alpha}{2\sqrt{c_\alpha}}2t - \frac{2\tau}{2\sqrt{\tau^2 + q_{1-\alpha}}}\right]}{\left(\sqrt{c_\alpha(\tau^2 + q_{1-\alpha})} + \sqrt{\tau^2 + q_{1-\alpha}}\right)^2} > 0$$

for all small postive values of τ, since c'_α is negative).

So this means that you can always find a neighborhood $(0, \tau_0)$ such that for all values of τ in the neighborhood, the second derivative will be negative, and therefore, you can always find a neighborhood $(0, f_0^*)$ such that for all f_0 in the neighborhood, the second derivative will be negative. We have cross-checked the expression in (23) by numerically computing rejection probabilities for ρ values close to 1 and $f_0 = 0$. ⊠

C Conditional Expected Length: *AR* and *tF*

C.1 Limiting Distribution of *AR* and *tF* confidence sets

Derivation of inflation factor $\frac{\sqrt{1-\frac{q}{\hat{F}}(1-\hat{\rho}^2)}}{1-\frac{q}{\hat{F}}}$

To derive how much we inflate the 2SLS confidence interval to obtain the *AR* interval length, we use the relationship

$$\hat{t}_{AR}^2 = \frac{\hat{t}^2 \hat{f}^2}{\hat{f}^2 + 2\hat{\rho}\sqrt{\hat{F}}\hat{t} + \hat{t}^2}$$

and solve

$$\frac{\hat{t}^2 \hat{f}^2}{\hat{f}^2 + 2\hat{\rho}\sqrt{\hat{F}}\hat{t} + \hat{t}^2} < q$$

for $\hat{t}$.

$$\hat{t}^2\hat{f}^2 - q\left(\hat{f}^2 + 2\hat{\tilde{\rho}}\sqrt{\hat{F}}\hat{t} + \hat{t}^2\right) < 0$$
$$\hat{t}^2\hat{f}^2 - q\hat{f}^2 - q2\hat{\tilde{\rho}}\sqrt{\hat{F}}\hat{t} - q\hat{t}^2 < 0$$
$$\left(\hat{f}^2 - q\right)\hat{t}^2 - \left(2q\hat{\tilde{\rho}}\sqrt{\hat{F}}\right)\hat{t} - q\hat{f}^2 < 0$$

which is a convex function in $\hat{t}$ when $\hat{f}^2 > q$. So the $\hat{t}$ that satisfies the inequality is an interval in this case, with endpoints

$$\frac{\left(2q\hat{\tilde{\rho}}\sqrt{\hat{F}}\right) \pm \sqrt{\left(2q\hat{\tilde{\rho}}\sqrt{\hat{F}}\right)^2 + 4\left(\hat{f}^2 - q\right)q\hat{f}^2}}{2\left(\hat{f}^2 - q\right)} =$$
$$\frac{\left(q\hat{\tilde{\rho}}\sqrt{\hat{F}}\right) \pm \sqrt{q}\sqrt{\hat{F}}\sqrt{q\hat{\tilde{\rho}}^2 + \hat{f}^2 - q}}{\left(\hat{f}^2 - q\right)} =$$
$$\frac{\left(q\hat{\tilde{\rho}}\sqrt{\hat{F}}\right) \pm \sqrt{q}\sqrt{\hat{F}}\sqrt{\hat{f}^2 - q\left(1 - \hat{\tilde{\rho}}^2\right)}}{\left(\hat{f}^2 - q\right)} =$$
$$\frac{\left(\frac{q\hat{\tilde{\rho}}}{\sqrt{\hat{F}}}\right) \pm \sqrt{q}\sqrt{1 - \frac{q\left(1-\hat{\tilde{\rho}}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}$$

Since

$$\hat{t} = \frac{\hat{\beta} - \beta}{\sqrt{\hat{V}_N\left(\hat{\beta}\right)}}$$

then the *AR* interval is given by

$$\hat{\beta} + \frac{-\left(\frac{q\hat{\tilde{\rho}}}{\sqrt{\hat{F}}}\right) + \sqrt{q}\sqrt{1 - \frac{q\left(1-\hat{\tilde{\rho}}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{\hat{V}_N\left(\hat{\beta}\right)} \geq \beta \geq \hat{\beta} + \frac{-\left(\frac{q\hat{\tilde{\rho}}}{\sqrt{\hat{F}}}\right) - \sqrt{q}\sqrt{1 - \frac{q\left(1-\hat{\tilde{\rho}}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{\hat{V}_N\left(\hat{\beta}\right)}$$
$$\hat{\beta} + \frac{-\sqrt{\frac{q}{\hat{F}}}\hat{\tilde{\rho}} - \sqrt{1 - \frac{q\left(1-\hat{\tilde{\rho}}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{q}\sqrt{\hat{V}_N\left(\hat{\beta}\right)} \leq \beta \leq \hat{\beta} + \frac{-\sqrt{\frac{q}{\hat{F}}}\hat{\tilde{\rho}} + \sqrt{1 - \frac{q\left(1-\hat{\tilde{\rho}}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{q}\sqrt{\hat{V}_N\left(\hat{\beta}\right)}$$

Since the half-length of the 2SLS confidence interval is $\sqrt{q}\sqrt{\hat{V}_N\left(\hat{\beta}\right)}$, then the inflation factor to obtain the half-length of the AR interval is

$$\frac{\sqrt{1-\frac{q}{\hat{F}}\left(1-\hat{\tilde{\rho}}^2\right)}}{1-\frac{q}{\hat{F}}}$$

Derivation of limiting distributions of the $(1-\alpha)$ **confidence intervals** $\hat{L}_{IV}, \hat{L}_{AR}, \hat{L}_{tF}$

(24)
$$\hat{L}_{IV} \xrightarrow{d} L_{IV} \equiv 2\sqrt{q_{1-\alpha}}\sqrt{1-2\rho\frac{t_{AR}(\beta)}{f}+\frac{t_{AR}^2(\beta)}{f^2}}\frac{1}{\sqrt{F}}\sqrt{V_\Omega}$$
$$\hat{L}_{AR} \xrightarrow{d} L_{AR} \equiv \frac{\sqrt{F}\sqrt{F-q_{1-\alpha}\left(1-\tilde{\rho}^2\right)}}{F-q_{1-\alpha}}L_{IV}$$
$$\hat{L}_{tF} \xrightarrow{d} L_{tF} \equiv \frac{\sqrt{c_\alpha(F)}}{\sqrt{q_{1-\alpha}}}L_{IV}$$

where

$$\tilde{\rho}^2 = \frac{\left(-t_{AR}(\beta)+\rho f\right)^2}{\left(f^2-2\rho t_{AR}(\beta)f+t_{AR}^2(\beta)\right)}$$
$$V_\Omega = \frac{AV\left(\widehat{\pi\beta}\right)-2\beta AC\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2 AV(\hat{\pi})}{AV(\hat{\pi})}$$

Limiting Distribution of $\hat{L}_{IV}$

Throughout this proof, when we consider the statistics $\hat{t}$ and $\hat{t}_{AR}$, they have $\left(\hat{\beta}-\beta\right)$, the estimator minus the true value of the parameter β, in the numerator.

By definition we have

$$\hat{L}_{IV} = 2\sqrt{q}\sqrt{\frac{\hat{V}_N\left(\widehat{\pi\beta}\right)-2\hat{\beta}C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\hat{\beta}^2\hat{V}_N(\hat{\pi})}{\hat{\pi}^2}}$$

We first note that

$$\frac{\hat{V}_N\left(\widehat{\pi\beta}\right)-2\hat{\beta}C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\hat{\beta}^2\hat{V}_N\left(\hat{\pi}\right)}{\hat{\pi}^2}=\frac{\left(\hat{\beta}-\beta\right)^2}{\hat{t}^2}$$

$$=\frac{1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}}{\hat{t}_{AR}^2}\left(\hat{\beta}-\beta\right)^2$$

$$=\frac{1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}}{\hat{t}_{AR}^2}\frac{\hat{t}_{AR}^2}{\hat{\pi}^2}\left(\hat{V}_N\left(\widehat{\pi\beta}\right)-2\beta C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2\hat{V}_N\left(\hat{\pi}\right)\right)$$

$$=\left(1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}\right)\frac{\hat{V}_N\left(\hat{\pi}\right)}{\hat{V}_N\left(\hat{\pi}\right)}\frac{\left(\hat{V}_N\left(\widehat{\pi\beta}\right)-2\beta C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2\hat{V}_N\left(\hat{\pi}\right)\right)}{\hat{\pi}^2}$$

$$=\left(1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}\right)\frac{1}{\hat{F}}\frac{\left(\hat{V}_N\left(\widehat{\pi\beta}\right)-2\beta C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2\hat{V}_N\left(\hat{\pi}\right)\right)}{\hat{V}_N\left(\hat{\pi}\right)}$$

The result follows under Weak-IV asymptotics, by the continuous mapping theorem.

Limiting Distribution of $\hat{L}_{tF}$

By definition,

$$\hat{L}_{tF}=\frac{\sqrt{c_\alpha\left(\hat{F}\right)}}{\sqrt{q_{1-\alpha}}}\hat{L}_{IV}$$

The result follows under Weak-IV asymptotics, by the continuity of $c_\alpha(\cdot)$, and the continuous mapping theorem.

Limiting Distribution of $\hat{L}_{AR}$

We have shown above that the $(1-\alpha)$ AR confidence set is an interval if and only if $\hat{F}>q_{1-\alpha}$. If $\hat{F}<q_{1-\alpha}$, then the confidence set is the whole real line except for an interval of length $\hat{L}_{AR}$.

We have shown above that $\hat{L}_{AR}$ is related to $\hat{L}_{IV}$ by the relationship

$$\hat{L}_{AR}=\frac{\sqrt{1-\frac{q_{1-\alpha}}{\hat{F}}\left(1-\hat{\hat{\rho}}^2\right)}}{1-\frac{q_{1-\alpha}}{\hat{F}}}\hat{L}_{IV}$$

Note that, by definition

$$\hat{\hat{\rho}}=\frac{\hat{C}\left(Z\hat{u},Z\hat{v}\right)}{\sqrt{\hat{V}\left(Z\hat{u}\right)\hat{V}\left(Z\hat{v}\right)}}$$

where we recall that $\hat{u}$ and $\hat{v}$ are the IV and first-stage residuals.

From Lemma 5 we have

$$\hat{t}_{AR}^2 = \frac{\hat{t}^2}{1+2\hat{\rho}\frac{\hat{t}}{\sqrt{\hat{f}^2}}+\frac{\hat{t}^2}{\hat{f}^2}} = \frac{\hat{t}^2\hat{f}^2}{\hat{f}^2+2\hat{\rho}\sqrt{\hat{F}}\hat{t}+\hat{t}^2}$$

Using this equation, we solve for $\hat{\hat{\rho}}$ and take its square, to obtain

$$\hat{\hat{\rho}}^2 = \frac{\left(\hat{t}^2\hat{f}^2-\hat{t}_{AR}^2\hat{t}^2-\hat{t}_{AR}^2\hat{f}^2\right)^2}{\left(2\hat{t}_{AR}^2\right)^2\hat{F}\hat{t}^2}$$

We can now substitute in the numerical relationship

$$\hat{t}^2 = \frac{\hat{t}_{AR}^2}{1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}} = \frac{\hat{f}^2\hat{t}_{AR}^2}{\hat{f}^2-2\hat{\rho}\hat{f}\hat{t}_{AR}+\hat{t}_{AR}^2}$$

and with some simplification, one obtains

$$\hat{\hat{\rho}}^2 = \frac{\left(-\hat{t}_{AR}+\hat{\rho}\hat{f}\right)^2}{\left(\hat{f}^2-2\hat{\rho}\hat{t}_{AR}\hat{f}+\hat{t}_{AR}^2\right)}$$

which, under Weak-IV asymptotics and the continuous mapping theorem, converges in distribution to

$$\tilde{\rho}^2 = \frac{\left(-t_{AR}+\rho f\right)^2}{\left(f^2-2\rho t_{AR} f+t_{AR}^2\right)}$$

So $\hat{L}_{AR}$ converges in distribution to

$$L_{AR} = \frac{\sqrt{1-\frac{q_{1-\alpha}}{F}\left(1-\tilde{\rho}^2\right)}}{1-\frac{q_{1-\alpha}}{F}}L_{IV} = \frac{\sqrt{F}\sqrt{F-q_{1-\alpha}\left(1-\tilde{\rho}^2\right)}}{F-q_{1-\alpha}}L_{IV}$$

C.2 $E\left[L_{AR}|F>q_{1-\alpha}\right]=\infty$

Conditional Expected Length of AR interval. Let

$$\Omega = \mathbf{plim}N\begin{pmatrix} \hat{V}_N\left(\widehat{\pi\beta}\right) & C\hat{O}V\left(\hat{\pi},\widehat{\pi\beta}\right) \\ C\hat{O}V\left(\hat{\pi},\widehat{\pi\beta}\right) & \hat{V}_N\left(\hat{\pi}\right) \end{pmatrix},$$

the asymptotic variance-covariance matrix of the reduced form and first-stage coefficients, be positive definite. Then $E\left[L_{AR}|F>q\right]=\infty$

From above, we have

$$L_{AR} = \frac{\sqrt{F}\sqrt{F - q_{1-\alpha}(1-\tilde{\rho}^2)}}{F - q_{1-\alpha}} 2\sqrt{q_{1-\alpha}} \sqrt{1 - 2\rho\frac{t_{AR}}{f} + \frac{t_{AR}^2}{f^2}} \frac{1}{\sqrt{F}} \sqrt{V_\Omega}$$
$$= 2\sqrt{q_{1-\alpha}} \frac{\sqrt{f^2 - q_{1-\alpha}(1-\tilde{\rho}^2)}}{f^2 - q_{1-\alpha}} \sqrt{\frac{f^2 - 2\rho t_{AR} f + t_{AR}^2}{f^2}} \sqrt{V_\Omega}$$

with

$$\begin{pmatrix} t_{AR} \\ f \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ \frac{\pi}{\sqrt{AV(\hat{\pi})}} \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$
$$V_\Omega = \frac{(1,-\beta)'\Omega(1,-\beta)}{AV(\hat{\pi})}$$

We will show that

$$E\left[2\sqrt{q_{1-\alpha}}\frac{\sqrt{f^2-q_{1-\alpha}(1-\tilde{\rho}^2)}}{f^2-q_{1-\alpha}}\sqrt{\frac{f^2-2\rho t_{AR}f+t_{AR}^2}{f^2}}\sqrt{V_\Omega}|F>q_{1-\alpha}\right]= \tag{25}$$

$$\frac{1}{\Pr[F>q_{1-\alpha}]}\int_{-\infty}^{\infty}\int_{\left(-\infty,-\sqrt{q_{1-\alpha}}\right)\cup\left(\sqrt{q_{1-\alpha}},\infty\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{x^2-q_{1-\alpha}\left(1-\tilde{\rho}(x,y)^2\right)}}{x^2-q_{1-\alpha}}$$
$$\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}\sqrt{V_\Omega}\phi_{f_0,\rho}(x,y)\,dxdy\geq$$

$$\frac{1}{\Pr[F>q_{1-\alpha}]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{x^2-q_{1-\alpha}\left(1-\tilde{\rho}(x,y)^2\right)}}{x^2-q_{1-\alpha}}$$
$$\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}\sqrt{V_\Omega}\phi_{f_0,\rho}(x,y)\,dxdy>$$

$$\frac{1}{\Pr[F>q_{1-\alpha}]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{x^2-q_{1-\alpha}(1-k_1)}}{x^2-q_{1-\alpha}}k_2k_3\sqrt{V_\Omega}dxdy\geq$$

$$\frac{1}{\Pr[F>q_{1-\alpha}]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{q_{1-\alpha}-q_{1-\alpha}(1-k_1)}}{x^2-q_{1-\alpha}}k_2k_3\sqrt{V_\Omega}dxdy=$$
$$\frac{1}{\Pr[F>q_{1-\alpha}]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{q_{1-\alpha}-q_{1-\alpha}(1-k_1)}}{\left(x-\sqrt{q_{1-\alpha}}\right)\left(x+\sqrt{q_{1-\alpha}}\right)}k_2k_3\sqrt{V_\Omega}dxdy\geq$$
$$\frac{1}{\Pr[F>q_{1-\alpha}]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{1}{\left(x-\sqrt{q_{1-\alpha}}\right)}\frac{\sqrt{q_{1-\alpha}k_1}}{2\sqrt{q_{1-\alpha}}+\varepsilon}k_2k_3\sqrt{V_\Omega}dxdy=\infty$$

where $\phi_{f_0,\rho}$ is the bivariate normal density with mean $(f_0,0)$, unit variances and correlation ρ, and $\tilde{\rho}(x,y)\equiv\frac{(-x+\rho y)^2}{\left(x^2-2\rho xy+y^2\right)}$, with both $\varepsilon>0$ and $\underline{y}$ chosen below.

In (25), the first equality (lines 1 and 2) holds by definition. The first inequality (lines 2 and 3) holds because the region of integration in the third line is a subset of the region for the second line. Deferring the second inequality momentarily, the third inequality (lines 4 and 5) holds because $\sqrt{x^2-q_{1-\alpha}(1-k_1)}>\sqrt{q_{1-\alpha}-q_{1-\alpha}(1-k_1)}$ because $x^2>q_{1-\alpha}$ in the region of integration. We expand a term in the denominator from lines 5 to 6 and the final inequality follows because $\frac{1}{x+\sqrt{q_{1-\alpha}}}\geq\frac{1}{2\sqrt{q_{1-\alpha}}+\varepsilon}$ when $x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)$. The final line holds because we will show it is equal to a positive constant multiplied by the integral

$\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{1}{\left(x-\sqrt{q_{1-\alpha}}\right)} dx$, which is infinite.

What remains is to show that the second inequality (lines 3 and 4) holds. Note first that $\sqrt{V_\Omega} > 0$ due to the positive definiteness of Ω. Furthermore, we will show that there always exists an integrating region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}} + \varepsilon\right) \times \left(\underline{y}, \underline{y} + \varepsilon\right)$ that lead to lower bounds $k_1, k_2, k_3 > 0$ for $\tilde{\rho}(x,y)^2, \sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}$ and $\phi_{f_0,\rho}(x,y)$, respectively, on the region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}} + \varepsilon\right) \times \left(\underline{y}, \underline{y} + \varepsilon\right)$.

1. $\tilde{\rho}(x,y)^2 > k_1 > 0$. Consider the quantity

$$\tilde{\rho}(x,y)^2 = \frac{(-y+\rho x)^2}{(x^2-2\rho yx+y^2)}$$

We seek a region of x, y space that satisfies

$$\frac{(-y+\rho x)^2}{(x^2-2\rho yx+y^2)} \geq k_1 > 0$$

We restrict x to be in the interval $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}} + \varepsilon\right)$. We can keep the denominator positive by restricting

$$y > \sup_{x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x + \sqrt{4\rho^2x^2-4}}{2}$$

In addition, we are seeking the values of y that satisfy

$$\begin{aligned}
(-y+\rho x)^2 - k_1\left(x^2-2\rho yx+y^2\right) &> 0 \\
y^2 - 2\rho yx + \rho^2x^2 - k_1\left(x^2-2\rho yx+y^2\right) &> 0 \\
y^2(1-k_1) - 2\rho yx(1-k_1) + x^2\left(\rho^2-k_1\right) &> 0
\end{aligned}$$

which is a quadratic inequality in y. We can choose $0 < k_1 < 1$ so that the function in the last line is convex in t_{AR}. So we can additionally restrict

$$y > \sup_{x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x(1-k_1) + \sqrt{4\rho^2x^2(1-k_1)^2 - 4(1-k_1)x^2\left(\rho^2-k_1\right)}}{2(1-k_1)}$$

So by setting

$$\underline{y} = \max\left(\sup_{x \in \left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x + \sqrt{4\rho^2 x^2 - 4}}{2}, \right.$$

$$\left. \sup_{x \in \left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x (1-k_1) + \sqrt{4\rho^2 x^2 (1-k_1)^2 - 4(1-k_1) x^2 (\rho^2 - k_1)}}{2(1-k_1)} \right),$$

then for any $y > \underline{y}$, and $x \in \left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right)$ we have $\tilde{\rho}(x,y)^2 \geq k_1 > 0$, as desired.

2. $\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}} > k_2 > 0$. We established above that for $y > \underline{y}$, the numerator in the square root is positive. In the integrating region, the denominator is positive as well. Let k_2 be the infimum of $\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}$ over the region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right) \times \left(\underline{y}, \underline{y}+\varepsilon\right)$.

3. $\phi_{f_0,\rho}(x,y) > k_3 > 0$. The bivariate density is strictly positive. Let k_3 be the infimum of $\phi_{f_0,\rho}(x,y)$ over the region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right) \times \left(\underline{y}, \underline{y}+\varepsilon\right)$

⊠

C.3 $E\left[L_{tF} | F > q_{1-\alpha}\right] < \infty$

Conditional Expected Length of tF interval: $E\left[L_{tF} | F > q_{1-\alpha}\right] < \infty$.

As shown above

$$\begin{aligned} L_{tF} &\equiv \frac{\sqrt{c_\alpha(F)}}{\sqrt{q_{1-\alpha}}} 2\sqrt{q_{1-\alpha}} \sqrt{1 - 2\rho\frac{t_{AR}}{f} + \frac{t_{AR}^2}{f^2}} \frac{1}{\sqrt{F}} \sqrt{V_\Omega} \\ &= \sqrt{c_\alpha(F)} 2 \sqrt{1 - 2\rho\frac{t_{AR}}{f} + \frac{t_{AR}^2}{f^2}} \frac{1}{\sqrt{F}} \sqrt{V_\Omega} \end{aligned}$$

The conditional expectation of interest is

$$E\left[2\sqrt{c_\alpha(F)} \sqrt{1 - 2\rho\frac{t_{AR}}{f} + \frac{t_{AR}^2}{f^2}} \frac{1}{\sqrt{F}} \sqrt{V_\Omega} \middle| F > q_{1-\alpha} \right]$$

We start by considering the conditional expectation

$$E\left[2\sqrt{c_{\alpha}\left(F\right)}\sqrt{1-2\rho\frac{t_{AR}}{f}+\frac{t_{AR}^{2}}{f^{2}}}\frac{1}{\sqrt{F}}\sqrt{V_{\Omega}}\middle|F=F'\right]$$

Since $c_{\alpha}\left(F\right)$ only depends on F, this is equivalent to

$$2\sqrt{c_{\alpha}\left(F'\right)}E\left[\sqrt{1-2\rho\frac{t_{AR}}{f}+\frac{t_{AR}^{2}}{f^{2}}}\frac{1}{\sqrt{F}}\sqrt{V_{\Omega}}\middle|F=F'\right]$$

Let us consider the expectation conditional on F

$$E\left[\sqrt{\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^{2}}{F^{2}}}\sqrt{V_{\Omega}}\middle|F=F'\right]$$

Consider the conditional expectation of the random variable inside the square root:

$$E\left[\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^{2}}{F^{2}}\middle|F=F'\right]$$

which can be expressed as

$$\frac{1}{\left(F'\right)^{2}}\left[F'-2\rho\sqrt{F'}E\left[t_{AR}|F=F'\right]+E\left[t_{AR}^{2}|F=F'\right]\right]$$

Consider that

$$\begin{aligned}
E\left[t_{AR}|F=F'\right] &= \left(\rho\left(-\sqrt{F'}-f_{0}\right)\right)\phi\left(-\sqrt{F'}\right)+\left(\rho\left(\sqrt{F'}-f_{0}\right)\right)\phi\left(\sqrt{F'}\right)\\
E\left[t_{AR}^{2}|F=F'\right] &= \left(1-\rho^{2}+\left(\rho\left(-\sqrt{F'}-f_{0}\right)\right)^{2}\right)\phi\left(-\sqrt{F'}\right)\\
&\quad+\left(1-\rho^{2}+\left(\left(\rho\left(\sqrt{F'}-f_{0}\right)\right)\right)^{2}\right)\phi\left(\sqrt{F'}\right)
\end{aligned}$$

Since each of these expressions is bounded on $F'>q_{1-\alpha}$, $E\left[\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^{2}}{F^{2}}|F=F'\right]$ is thus bounded on $F'>q_{1-\alpha}$ by some constant $\bar{F}$. Due to Jensen's inequality, we obtain

$$2\sqrt{c_{\alpha}(F')}E\left[\sqrt{\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^{2}}{F^{2}}}\sqrt{V_{\Omega}}\middle|F=F'\right]\leq 2\sqrt{c_{\alpha}(F')}\bar{F}$$

Therefore, for $F'>q_{1-\alpha}$, the function $E\left[L_{tF}|F=F'\right]$ is bounded above by the function $2\sqrt{c_{\alpha}(F')}\bar{F}$. Therefore,

$$\begin{aligned}E\left[L_{tF}|F>q_{1-\alpha}\right]&=\frac{1}{\Pr\left[F>q_{1-\alpha}\right]}\int_{q_{1-\alpha}}^{\infty}E\left[L_{tF}|F=F'\right]\omega\left(F'\right)dF'\\&\leq\frac{2\bar{F}}{\Pr\left[F>q_{1-\alpha}\right]}\int_{q_{1-\alpha}}^{\infty}\sqrt{c_{\alpha}(F')}\omega\left(F'\right)dF'\end{aligned}$$

where $\omega(\cdot)$ is the density of F. To complete the proof, we need to show that the last integral in the display above is finite. To do so, we will break the integral into two pieces. The first piece will be from $q_{1-\alpha}$ to $q_{1-\alpha}+\delta$, and the second will be from $q_{1-\alpha}+\delta$ to ∞.

For the first piece, from Lemma 9 we know that there exists a $\delta>0$ such that $c_{\alpha}(F)(F-q_{1-\alpha})$ is continuous on $(q_{1-\alpha},q_{1-\alpha}+\delta]$. Moreover, Lemma 9 implies we can extend the function $c_{\alpha}(F)(F-q_{1-\alpha})$ continuously at $F=q_{1-\alpha}$. Doing so extends the definition of $c_{\alpha}(F)(F-q_{1-\alpha})$ to the compact set $[q_{1-\alpha},q_{1-\alpha}+\delta]$, implying that it is uniformly continuous on this set by the Heine-Cantor theorem and hence bounded above by some finite M on that set. The density $\omega(\cdot)$ is also bounded above by some finite K in the same interval. Therefore

$$\begin{aligned}\int_{q_{1-\alpha}}^{q_{1-\alpha}+\delta}\sqrt{c_{\alpha}(F')}\omega\left(F'\right)dF' &= \int_{q_{1-\alpha}}^{q_{1-\alpha}+\delta}\sqrt{\frac{c_{\alpha}(F')(F-q_{1-\alpha})}{F-q_{1-\alpha}}}\omega\left(F'\right)dF'\\&\leq K\sqrt{M}\int_{q_{1-\alpha}}^{q_{1-\alpha}+\delta}\frac{1}{\sqrt{F-q_{1-\alpha}}}dF'=K\sqrt{M}\int_{0}^{\delta}\frac{1}{\sqrt{x}}dx\end{aligned}$$

and the last integral in the display above is finite by the integral p-test.

For the second piece, since $c_{\alpha}(\cdot)$ is decreasing, $\sqrt{c_{\alpha}(F')}$ is bounded above by $M'=\sqrt{c_{\alpha}(q_{1-\alpha}+\delta)}$ which is finite. But then since $\omega(\cdot)$ is a density,

$$\int_{q_{1-\alpha}+\delta}^{\infty}\sqrt{c_{\alpha}(F')}\omega\left(F'\right)dF'<M'\int_{q_{1-\alpha}+\delta}^{\infty}\omega\left(F'\right)dF'\leq M'$$

completing the proof. ⊠