A RECONSIDERATION OF THE EMPIRICAL IMPLICATIONS OF ADDITIVE PREFERENCES[1]

In the last twenty years, the theory of utility maximisation has had extensive application as a basis for deriving empirically estimable demand equations. This method of analysing and measuring demand has tended not to use general utility functions, but rather to depend upon specifications which assume that preferences are either directly or indirectly additive. Two models in particular have been used very widely; the linear expenditure system, first applied by Stone (1954), which assumes direct additivity, and the indirect addilog model, due to Leser (1941–42) and Houthakker (1960), which assumes indirect addivity. In addition to these, the directly additive models suggested by Frisch (1959) and by Powell (1966) have found practical application in a number of contexts, as have the more recent additive dynamic models of Houthakker and Taylor (1970) and of Phlips (1972). As estimation problems have been brought under control, these models have been applied to a wide range of data for different countries and today their use is part of the standard methodology of applied demand analysis.[2]

In this paper, it is shown that both the additivity assumptions imply approximate linear relationships between own-price and income elasticities; under direct additivity the ratio of own-price to income elasticity is approximately constant, while under indirect additivity the sum is approximately constant. These relationships are *a priori* implausible and there exists no empirical evidence in their favour. Consequently, the use of models based upon either of the additivity postulates seriously distorts the measurement of those responses in which demand analysis has the greatest interest, own-price and income elasticities. Clearly, this distortion is also at least partly to blame for the empirical rejections of additivity which have been found by several investigators.[3] However, since these rejections have until now been interpreted as demonstrating the failure of additive systems to model cross-price responses, which for many practical purposes are of the second-order of importance, our result adds considerable strength and plausibility to the earlier findings. The deficiencies of additive models should thus be taken

[1] This paper is a complete revision of "Additive Preferences and Pigou's Law," which was read at the European Meeting of the Econometric Society in Oslo, August 1973. My thanks are due to those who read and criticised the earlier paper, in particular Anton Barten, David Champernowne, Terence Gorman, Frank Hahn, Leif Johansen, Louis Phlips, Richard Stone and Henri Theil; none of the above is responsible for, nor necessarily even in agreement with, any of the conclusions of the paper.

[2] The literature is much too extensive to be quoted here, and is still expanding rapidly. For a review of much of it and for references see Brown and Deaton (1972), especially section IV.

[3] See, for example, Barten (1969), Byron (1970a) and (1970b), Theil (1971), Deaton (1974).

very seriously, and in spite of their enormous practical advantages, the whole question of their suitability for applied work should be reconsidered.

We proceed first by proving the results, secondly by assessing their empirical implications, and finally by considering the wider consequences for the methodology of applied demand analysis.

Following Houthakker (1960), directly additive utility functions may be written in the form

$$v(q) = \theta\left\{\sum_{k=1}^{n} v_k\,(q_k)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where v is utility, defined over R^n, the space of the n quantities q, the v_k are n sub-utility functions, each a function of q_k only, while $\theta\{\ \ \}$ is an arbitrary monotone increasing function. Indirectly additive functions are defined in terms of the n ratios of income, μ, to price, p, *i.e.*,

$$v(\mu, p) = \psi\left\{\sum_{k=1}^{n} v_k\left(\frac{\mu}{p_k}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\psi\{\ \ \}$ is an arbitrary function and each of the functions v_k is a function of μ/p_k alone.

Beginning with direct additivity, the first-order maximisation conditions may be written

$$\log \theta' + \log v_i' = \log \lambda + \log p_i \quad . \quad . \quad . \quad . \quad (3)$$

where λ is a Lagrange multiplier, and a prime denotes a first derivative. Differentiating with respect to $\log \mu$ and $\log p_j$ in turn,

$$\frac{\partial \log v_i'}{\partial \log q_i} e_i = \frac{\partial \log (\lambda/\theta')}{\partial \log \mu} = \breve{\omega}, \text{ say} \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{\partial \log v_i'}{\partial \log q_i} e_{ij} = \frac{\partial \log (\lambda/\theta')}{\partial \log p_j} + \delta_{ij} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where e_i is the income elasticity of the ith good, e_{ij} is the (uncompensated) cross-price elasticity of good i with respect to the price of good j, and δ_{ij} is the Kronecker delta, equal to unity if $i = j$ and zero otherwise. The use of $\breve{\omega}$, which is Frisch's flexibility of the marginal utility of money, is justified, since if the utility function is explicitly additive, *i.e.*, $\theta\{x\} = x$, the quantity has this interpretation. Combining (4) and (5),

$$\breve{\omega} e_{ij} = e_i \frac{\partial \log (\lambda/\theta')}{\partial \log p_j} + e_i\,\delta_{ij} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This may be simplified by multiplying by w_i, the value share of the ith good, and summing over i to enforce the aggregation restriction, $\sum_i w_i e_{ij} + w_j = 0$; thus

$$-\breve{\omega}\, w_j = \frac{\partial \log (\lambda/\theta')}{\partial \log p_j} + w_j e_j \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which, substituted in (6), gives[1]

$$e_{ij} = \phi e_i \delta_{ij} - e_i w_j (1 + \phi e_j) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $\phi = \breve{\omega}^{-1}$.

Since the income elasticities are of the order of unity, $(\Sigma w_k e_k = 1)$, and since the average value shares w are of the order n^{-1}, the cross-elasticities are small relative to the own-price elasticity, and the latter is dominated by the first term on the right-hand side of (8), *i.e.*,

$$e_{ii} \simeq \phi e_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This relationship, of proportionality between income and own-price elasticities, was first put forward by Pigou in 1910 and may thus be known as "Pigou's Law."[2] Clearly, the approximation will only be close for a good which occupies a very small fraction of the budget and it will only be close for *all* goods if the level of disaggregation adopted is high. But the main interest in focusing attention on (9) is empirical; what is really important is the extent to which the approximation applies in empirical work using additive models, and whether or not there is evidence that proportionality, however approximate, distorts measurement. We shall see below, in the case of a particular additive system, that even for quite highly aggregated groupings of commodities, the relationship (9) is a remarkably close approximation.

For indirect additivity, we may proceed directly to the demand equations via Roy's theorem, *i.e.*,

$$\log q_i = \log \left(- \frac{\partial v}{\partial p_i} \right) - \log \left(\frac{\partial v}{\partial \mu} \right) \quad . \quad . \quad . \quad . \quad (10)$$

so that, substituting from (2),

$$\log q_i = \log v_i' + \log \mu - 2 \log p_i - \log \left(\sum_k v_k' p_k^{-1} \right) \quad . \quad (11)$$

Differentiating as before and remembering that for $\psi\{x\} = x$, $\partial v / \partial \mu = \lambda$, the marginal utility of money, we have

$$e_i = \frac{\partial \log v_i'}{\partial \log \mu} + 1 - \breve{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$e_{ij} = \frac{\partial \log v_i'}{\partial \log p_j} - 2\delta_{ij} - \frac{\partial \log \lambda}{\partial \log p_j} \quad . \quad . \quad . \quad . \quad (13)$$

But since v_i, and thus v_i', is a function of μ / p_i only,

$$\frac{\partial \log v_i'}{\partial \log p_j} = -\delta_{ij} \frac{\partial \log v_i'}{\partial \log \mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

[1] Equation (8) is well known; it may be derived directly by combination of equations (61) and (62) of Frisch (1959).

[2] Pigou's proof is rather different; the derivative on the right-hand side of (5) is assumed to be negligible from which (9) follows directly. Pigou's contribution has been largely ignored except for some critical comment, Friedman (1935), Samuelson (1942).

so that as a counterpart to equation (6), we have

$$e_{ij} = -\delta_{ij}(1 + e_i + \breve{\omega}) - \frac{\partial \log \lambda}{\partial \log p_j} \quad . \quad . \quad . \quad . \quad (15)$$

This may be simplified in a similar manner to (6), *i.e.*, by enforcing aggregaion, giving finally

$$e_{ij} = -\delta_{ij}(1 + e_i + \breve{\omega}) - w_j(e_j + \breve{\omega}) \quad . \quad . \quad . \quad (16)$$

As in the case of direct additivity, the cross-elasticities are small relative to the own-elasticities, and once again the latter may be approximated in a simple manner, *i.e.*,

$$e_{ii} \simeq -e_i - (1 + \breve{\omega}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which is again linear, but now with a slope of minus unity and a positive intercept. So that while, under direct additivity, it is the ratio of price to income elasticity which is approximately constant, under indirect additivity, it is their sum. If the marginal utility of money, and thus $\breve{\omega}$ and ϕ, are to have the same interpretation under both types of additivity, it is clear that the two approximations will only coincide when $\breve{\omega} = \phi = -1$. In this case, each expenditure is proportional to income, and this is the only behaviour which is consistent with simultaneous direct and indirect additivity defined in this way; see Samuelson (1965). However, for empirical purposes, there is no reason to suppose that the flexibility defined by direct additivity should be identical with the flexibility defined by indirect additivity. Indeed, a given demand system may be consistent with additive utility functions of both types, each representing the same *ordinal* preference ordering, although the *cardinal* levels of utility and of marginal utility will be different for each function. This is the case Samuelson calls *non-simultaneous* direct and indirect additivity; here the flexibility will be different for each of the approximations and the two lines (9) and (17) will intersect at a single point representing all the income and own-price elasticities of the system. Samuelson has derived the most general form of utility function consistent with both direct and indirect additivity, and the interested reader may check that for the resulting demand equations, the income elasticities are all unity and the own-price elasticities are all equal, apart from terms of order n^{-1}.

Neither of these special cases is of great empirical interest; however, the independence in theory of the two definitions of the flexibility is matched by the disparate interpretations provided by the two approximations. Under direct additivity, estimates of $\breve{\omega}$ will be derived from an appropriately weighted average of the ratios of income to price elasticities, and it is perhaps not too surprising that this ratio should demonstrate such stability across countries. Under indirect additivity, there is no reason to suppose that calculations embodying (17) will lead to similar estimates for $\breve{\omega}$ and although fewer estimates of the flexibility based on indirect additivity seem to have been published, those I can find[1] seem to be closer to -1 than to -2, this latter being the central estimate for directly additive models.

The validity of the two approximations (9) and (17) is not really a major point at issue; there is already a good deal of evidence against additivity *per se* and while the assumptions imply more than either of the relationships discussed, it would be extraordinary if the latter had nothing to do with the general invalidity of the hypothesis. There is also some direct evidence to support this view. In another context (Deaton, 1974), the author tested a version of the Rotterdam demand system intermediate between the symmetric and directly additive versions of the model; this model retains the structure of the additive substitution matrix while abandoning the link between own-price and income elasticities. This separates the proportionality effects of additivity from its other, more general restrictions. On long-run United Kingdom data, both sets of constraints were rejected; thus, while proportionality itself is invalid, it is not the only aspect of additivity which is contrary to the evidence.

The most obvious way of independently assessing the relationship between income and price elasticities without imposing either (9) or (17) is to estimate a set of loglinear demand equations by regressing the log of quantity demanded against the log of real income and the log of price relative to a price index. This methodology, though deficient from a theoretical point of view, is simple and it gives estimates of elasticities directly; for these reasons it is still often used in demand analysis.[2] Since the quality of the two approximations (9) and (17), depends upon the degree of commodity disaggregation, experiments were carried out on two different versions of the same data. For the first set, data on purchases of thirty-seven distinct non-durable commodities from 1954 to 1970 were taken from C.S.O. (1971), while for the second set, the goods were aggregated into eight broad groups. The resulting estimates[3] of the price and income elasticities are plotted against one another in Fig. 1; the points are for the 37-commodities, the crosses for the 8-commodities. Clearly, there is no visual evidence for either of the relationships required by direct or indirect additivity; indeed, the correlation between the points is actually *positive*. Note also that the broad groups do not appear to conform any more than do the more detailed commodities. The visual impression may be confirmed by imposing, say, the proportionality relationship for different values of ϕ and computing F-ratios for the restriction; as expected, this was rejected for a substantial number of commodities, confirming the general inapplicability of such a relationship.

However, too much emphasis can be placed upon these particular

[1] Baschet and Debreu (1971) and Solari (1971) provide estimates of the indirect addilog model for a number of countries.

[2] For example, Houthakker (1965), Goldberger and Gamaletsos (1970).

[3] The estimating equation is $\log q_i = \alpha_i + \beta_i \log (\mu/\pi) + \gamma_i \log (p_i/\pi)$, where π is the implicit price deflator of consumers' expenditure. Time trends were allowed in the β and γ parameters; the figures plotted are for the year 1963.

results; the model is very crude, and is subject to many obvious objections. Even so, such calculations do give some idea of the size of distortions likely to follow from the imposition of either of the additivity assumptions. That this impression is not misleading can be confirmed by seeing what happens when the same data are used to estimate an additive model. For this purpose, compare Fig. 1 with Fig. 2; the latter shows the income and price elasticities

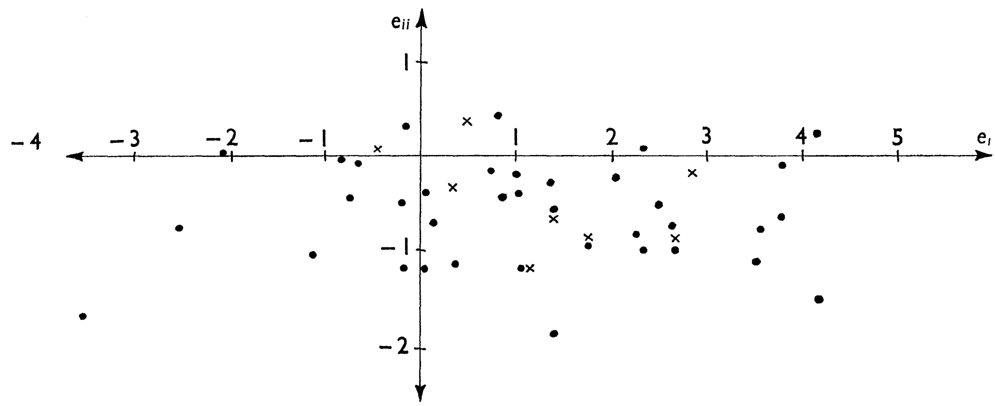

FIG. 1. Income and price elasticities for double-log model.

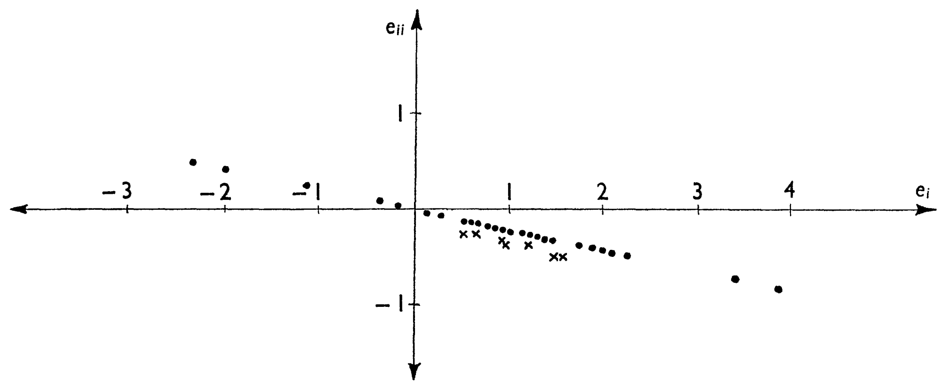

FIG. 2. Income and price elasticities for a directly additive model.

calculated for the year 1963 from parameter estimates of the linear expenditure system.[1] Clearly, the proportionality approximation is very close, even for the broad groups. The kind of distortion which is induced by this can best be assessed with reference to a hypothetical, though not unrealistic, example. Consider a good which is normal though not particularly income elastic, but which has a high own-price elasticity; there is no theoretical reason to suppose that such may not exist. Suppose further that on average,

[1] Details of estimation are discussed fully in a forthcoming monograph by the author; this will also discuss much more fully other aspects of these results.

the price of the good is rising relative to that of other commodities, and that, as a result, the quantity consumed is falling in spite of increases in real income. This is not unlikely; the consumption of rail travel in the United Kingdom can be explained quite adequately in this way. If this good is to be modelled by an additive demand system, in order to have a high price elasticity, it must have a much higher income elasticity, since the ratio of the two must be the same as for all other goods. But this would involve a trend *increase* in consumption, and this does not fit the data. Consequently, the model, in order to match the secular decline in consumption, must make the good inferior and this has the final absurd consequence, via Pigou's Law, of rendering the commodity *positively* price elastic, *i.e.*, a Giffen good.[1] Thus, if there are any cyclical price effects, the predicted expenditures of the model, though following the trend correctly, will reverse the cycle. In the application of the linear expenditure system to the thirty-seven commodity data, almost a quarter of the goods were classed as inferior Giffen goods, an indication of how widespread the difficulties are. Detailed analysis of each commodity confirms this and reveals that the range of price and income responses required to explain the evidence is not exaggerated by the contrast between Figs. 1 and 2. Clearly, aggregation to broad groups reduces the incidence of extreme cases, and certainly the hypothetical example is the worst possible type of behaviour for additivity to model, but the basic point remains; additive models will only work well when the ratio of price to income elasticities is genuinely equal for all commodities and there is no reason to suppose that this is more likely for broad groups than for more differentiated commodities.

The above argument, though put in terms of direct additivity, could easily be recast to deal with the indirectly additive case. It is hardly necessary to estimate the indirect addilog model and go through a similar analysis to show that, in this case too, very similar difficulties are bound to arise.

These results clearly add substance to the considerable body of evidence against additivity which already exists. However, it is possible to go further and to argue that the reinterpretation of the causes of invalidity make the case for not using additivity a much more compelling one. For, if additivity is regarded primarily as a means of dealing with cross-price responses in a simple and theoretically plausible manner, its rejection on the evidence is not likely to be of crucial importance. Let us see why this is so and why the results of the paper alter the position. As commonly interpreted, the restrictions of additivity are taken as linking the off-diagonal terms of the price substitution matrix to the corresponding income responses, see, *e.g.*, Houthakker (1960). Since in econometric work, the whole range of cross-price effects is almost never measurable without prior information, and

[1] This could of course be prevented by restricting the parameters of an additive system so that inferiority cannot arise; this only hides the difficulty, and does so at the cost of making the fit impossibly bad.

since such terms are likely to be of limited importance in any case, one or other of the additivity assumptions is enormously helpful, allowing as it does the estimation of a complete system on very limited information. If the assumption is found to be invalid because there exists some cross-price behaviour which cannot be allowed for under additivity, this is neither very unexpected nor very serious. The difficulty of modelling cross-price behaviour adequately is, after all, very great compared to the reward for doing so, since the latter is likely to be only a minor increase in the verisimilitude of secondary effects. Alternative methodologies do exactly the same; loglinear models treat cross-price effects by subsuming them into a general price index in a way which is inconsistent even with Walras' law, although these defects may well be offset by other more practical advantages. However, the results proved above and the evidence on distortion seriously challenge the usefulness of additivity as a simplifying assumption of this sort. Convenience and ease of estimation are purchased at the cost of severe distortion of those effects which it is most desirable to measure accurately. On this argument, the extent to which additivity has been used in applied work seriously over-states its real usefulness.

The principal alternative models for demand analysis, if additivity is to be abandoned, are the more " pragmatic " systems, for example the loglinear or Rotterdam models. Although these are both fundamentally inconsistent with utility maximisation, the theory may still find a role in the provision of restrictions to aid estimation and improve precision. These models, though lacking the theoretical plausibility of systems such as the linear expenditure or addilog models, are extremely flexible and may be used to model a wide range of non-additive price behaviour. In any case, in view of the recent negative results by Sonnenschein (1972) and (1973), and Debreu (1973), consistency with utility maximisation may be but a dubious requirement for aggregate models,[1] and these pragmatic systems remain the most promising tools of demand analysis currently available. If utility function based models are to compete with these, they must use non-additive functions and very few of these exist. The oldest, the quadratic utility function, is unsatisfactory from a number of points of view,[2] while the one of the newest, the extension of the linear expenditure system suggested by Nasse (1970) is promising, but relatively untried.[3] One superficially attractive possibility, Brown and Heien's (1972) *S*-branch utility tree, is unfortunately much less flexible than it appears and embodies the proportionality law in a weaker, but still unpalatable, form; this system is briefly discussed in the Appendix.

[1] Oddly enough, not even the results quoted can rescue the Rotterdam or loglinear models at a theoretical level since the former is inconsistent with any set of differentiable demand functions while the latter contradicts Walras' Law, the only property which " market " excess demand functions *must* possess.

[2] See Goldberger (1967).

[3] For a discussion see Brown and Deaton (1972).

Conclusion

At a time when consumer demand theory has been shown to be of only limited usefulness in the study of general equilibrium, the case still remains to be made for it in its alternative application, to the practical analysis of observed consumer behaviour. However, although non-additive preference orderings may yet yield important insights into observed phenomena, the main argument of this paper is that *the assumption of additive preferences is almost certain to be invalid in practice and the use of demand models based on such an assumption will lead to severe distortion of measurement.* So that if the price to be paid for the theoretical consistency of demand models is the necessity of assuming additive preferences, then the price is too high.

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Appendix

The S-Branch Utility Tree: A Generalisation of the Linear Expenditure System, Proposed by M. Brown and D. Heien

This appendix is designed to justify briefly the statements in the text to the effect that the above model is of insufficient generality to overcome the objections to additivity in general and to the linear expenditure system in particular.

The goods are partitioned into S groups or branches; the utility function is then written

$$v(q) = \sum_{s=1}^{S}\left\{\alpha_s\left[\sum_{i\epsilon S} \beta_{si}(q_i - \gamma_{si})^{\rho_s}\right]^{\frac{\rho}{\rho_s}}\right\}^{\frac{1}{\rho}}$$

where α, β, γ, ρ are parameters. First, note that this utility function is *strongly*, *i.e.*, additively, separable; thus, with respect to the S branches, the utility function is additive. Consequently, everything said in the text about additivity, including the proportionality law and its invalidity, apply here to the way the model treats broad groups of goods.

The model does however extend the linear expenditure system for within-group behaviour. As Brown and Heien state, complements are now technically permitted, although this is unlikely to be of much practical help since it can only occur if all goods within a branch are complementary to one another. Of more interest is the more general treatment of own-price and income elasticities and the modifications which are induced to Pigou's Law. Own price elasticity is given by[1]

$$e_{ii} = -\left(\frac{q_i - \gamma_{si}}{q_i}\right)\{w_{si} + \sigma\,(W_{si} - w_{si}) + \sigma_s\,(1 - W_{si})\}$$

[1] There is an error in the equation as printed in the original *Econometrica* article.

where
$$w_{si} = p_i(q_i - \gamma_{si})/(\mu - \sum_s \sum_k p_k \gamma_{sk})$$
$$W_{si} = p_i(q_i - \gamma_{si})/(\mu_s - \sum_{k \epsilon S} p_k \gamma_{sk})$$

μ_s is expenditure on the sth group, and σ_s and σ are equal to $(1 - \rho_s)^{-1}$ and $(1 - \rho)^{-1}$ respectively. Income elasticity is given by

$$e_i = \left(\frac{q_i - \gamma_{si}}{q_i}\right) \frac{\mu}{\mu - \sum_s \sum_k p_k \gamma_{sk}}$$

so that, to the same degree of approximation used in the main text above

$$e_{ii} = \phi_s e_i \, ; \phi_s = -\sigma_s \left(\mu - \sum_s \sum_k p_k \gamma_{sk}\right) \mu^{-1}$$

Thus, the universal proportionality approximation (9) is replaced by a series of such relationships, one for each group. Clearly, this is less restrictive than the original version and the S-branch model can thus be expected to fit the data better than the linear expenditure system. On the other hand, in an absolute sense, these new restrictions are just as implausible and as unlikely to be valid as the old. The S-branch is thus not general enough to overcome the principal objections to strict additivity.

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