

FUNCTIONAL FORMS FOR LABOR SUPPLY AND COMMODITY DEMANDS WITH AND WITHOUT QUANTITY RESTRICTIONS

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One of the most important current questions in economic analysis is whether or not labor markets clear in the short run. To answer this, it is necessary to be able to distinguish between restricted and unrestricted behavior by consumers supplying labor. This paper investigates the forms of preferences which lie behind linear models of labor supply, and derives the functional forms for commodity demands which accompany them, both with and without quantity restrictions in the labor market. Simple linkages between restricted and unrestricted demands are also considered as is the question of perfect aggregation over consumers in the presence of quantity restrictions.

INTRODUCTION

OUR MAIN AIM in this paper is to exploit the theory of rationing to propose plausible functional forms for commodity demand functions in the presence of quantity constraints in the labor market. Such functions, together with their unrationed counterparts, are essential for the analysis of cross-section or longitudinal data on labor supply and commodity demands in situations where some consumers face quantity constraints, for example, unemployment, or, equivalently nonparticipation, and others do not. Only if the demand functions for both are derived from a single common specification of preferences can efficient estimation be ensured. We show how this can be done using two sets of preferences for which the unconditional labor and income supply (i.e. wage times labor supply) functions are linear in the wage and in nonlabor income, and we provide a comparative discussion of the rationed and unrationed functional forms. Finally, we derive conditions under which, in general, commodity demand functions with quantity restrictions in the labor market can be thought of as unrestricted demands modified by an amount proportional to the difference between the enforced and desired labor supply. Such functions provide a simple tool for analyzing the interactions between markets when not all markets clear. A major concern throughout the paper is to consider, for each of the three classes of preferences, conditions under which the microeconomic supply and demand functions aggregate exactly to functions defined on the averages of the independent variables, whether there are wage rates, nonlabor incomes, or ration levels.

In the voluminous literature on labor supply, the most frequently estimated functional form is that in which labor supply is taken as linear in the wage rate and in nonlabor income. Examples are Boskin [5], Garfinkel [8], Greenberg and Kosters [9], Gronau [10], Hall [11], Ashenfelter and Heckman [4], Heckman [13, 14], Nakamura, Nakamura, and Cullen [19], and Layard, Barton, and Zabalza [16]. The second most popular form, which is also conveniently linear in the wage and in nonlabor income, is probably the labor supply function

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embodied in the linear expenditure system; see Abbot and Ashenfelter [1, 2] and Philips [21]. Because of the econometric convenience of both kinds of linearity, we briefly explore in Section 1 the implications for the corresponding preference classes. This analysis supplements that of Hausman [12], who derives the preferences that give labor supply linear in the wage, and that of Muellbauer [18] who, in the context of exact aggregation, finds those preferences that yield labor income supply and commodity expenditure demands which are jointly linear in the wage and nonlabor income. Section 2 provides the minimal necessary sketch of rationing theory—see Deaton and Muellbauer [7], Neary and Roberts [20], Deaton [6], and Slutsky [24] for further details—and then derives the “matched pairs” of rationed and unrationed demand functions from the preferences of Section 1. Section 3 derives the condition for the simple relationship between rationed and unrationed demands mentioned above. The functions used by Ashenfelter [3] to analyze the impact of unemployment on the allocation of aggregate commodity demands are a special but restrictive member of this class. One purpose of this paper is to make such an analysis possible with more general functional forms.

1. CONVENIENT FORMS FOR UNRATIONED LABOR SUPPLY FUNCTIONS

(a) *Linear Labor Supply Functions*

In empirical labor supply studies, a relationship of the following form is frequently estimated:

$$(1) \quad l = \beta_0 + \beta_1 \omega + \beta_2 \mu,$$

where l is hours worked, ω is the wage rate, and μ is unearned or transfer income. In theory, this must be the solution of the problem of maximizing utility u , given by

$$(2) \quad u = v(q_0, q)$$

subject to the budget constraint²

$$(3) \quad p \cdot q + \omega q_0 = \omega T + \mu = x, \quad \text{say,}$$

where q is a vector of commodities with prices p , q_0 is the amount of leisure, T is the time endowment (so that $l \equiv T - q_0$), and x is full income, $\omega T + \mu$. Corresponding to the problem (2) and (3), there exists a “full” cost function $c(u, \omega, p)$ defined as the minimum cost of reaching u at ω and p which, for a utility maximizing consumer, takes the value x , i.e.

$$(4) \quad c(u, \omega, p) = \min_{q_0, q} \{ \omega q_0 + p \cdot q; v(q_0, q) \geq u \} = x.$$

²If the typically static form of the budget constraint is to be made consistent with intertemporal choice, we need to assume intertemporal separability of preferences and take μ to be unearned or transfer income minus saving.

By the derivative property of the cost function, the derivatives of $c(u, \omega, p)$ with respect to ω and p_i , denoted $c_0(u, \omega, p)$ and $c_i(u, \omega, p)$ are the compensated or Hicksian demands for q_0 and q_i respectively. If u is then written as a function of x , ω and p by inverting (4) to give the indirect utility function $\psi(x, \omega, p)$, substitution in the Hicksian demands will give the Marshallian demands

$$(5) \quad q_0 = g_0(x, \omega, p) = c_0\{\psi(x, \omega, p), \omega, p\},$$

$$(6) \quad q_i = g_i(x, \omega, p) = c_i\{\psi(x, \omega, p), \omega, p\},$$

and the first of these must be consistent with equation (1).

Note first that since $l = T - q_0$ and T is a constant, the linearity in l is equivalent to the linearity in q_0 . Second, the theoretical variables in the Marshallian demands are full income and the wage, not transfer income and the wage. However, $x = \omega T + \mu$, so that linearity in ω and μ implies linearity of (5) in ω and x . Hence for (5) to be consistent with (1), $\partial g_0 / \partial x$ and $\partial g_0 / \partial \omega$ must each be independent of *both* x and ω . To translate this into a condition on $c(u, \omega, p)$, note from (5) that

$$(7) \quad \frac{\partial g_0}{\partial x} = \frac{c_{0u}}{c_u} \quad \text{and} \quad \frac{\partial g_0}{\partial \omega} = -\frac{c_{0u}c_0}{c_u} + c_{00},$$

where double subscripts denote double differentiation, $c_u = \partial c / \partial u = (\partial \psi / \partial x)^{-1}$ and we have used Roy's identity that $\partial \psi / \partial \omega = -c_0 / c_u$. Each of the expressions in (7) must thus be a function of p alone and we can write this, without loss of generality, as

$$(8) \quad \frac{c_{0u}}{c_u} = \frac{1}{\beta(p)}$$

and

$$(9) \quad c_{00} - \frac{1}{\beta(p)} c_0 = -\frac{\alpha(p)}{\beta(p)}$$

for suitable choice of functions $\alpha(p)$ and $\beta(p)$. The integration of (8) and (9) is straightforward and leads to

$$(10) \quad c(u, \omega, p) = -\eta(u, p)\exp(\omega/\beta(p)) + \omega\alpha(p) + \epsilon(p)$$

where $\alpha(p)$ and $\beta(p)$ are the original functions, which must be positive and homogeneous of degrees zero and unity respectively, $\eta(u, p)$ is a positive decreasing function of u and is homogeneous of degree one in p as is the function $\epsilon(p)$. This is essentially the same result as in Hausman [12] although Hausman does not make explicit how prices enter or what are the implications for the commodity demand functions. The indirect utility function from (10) is the solution to

$$(11) \quad \eta(u, p) = -\{x - \omega\alpha(p) - \epsilon(p)\}\exp\{-\omega/\beta(p)\}.$$

Sufficiency of (10) and (11) for (1) is easily checked by differentiation with respect to ω and substitution. Hence

$$(12) \quad l = \{T - \alpha(p) + \epsilon(p)/\beta(p)\} - \{1/\beta(p)\}\mu - \{(T - \alpha(p))/\beta(p)\}\omega$$

which gives β_0 , β_1 , and β_2 in terms of the parameters of preferences. Although the compensated commodity demand functions corresponding to (10) can be derived by differentiation, an explicit solution for the Marshallian demands requires a more restrictive specification of the function $\eta(u, p)$. Here we adopt the form

$$(13) \quad \eta(u, p) = \rho(u)\theta(p)$$

for a linear homogeneous function $\theta(p)$, but more general forms are possible. Given (13) the commodity demands corresponding to linear labor supply are

$$(14) \quad q_i = \epsilon_i(p) + \alpha_i(p)\omega + \left\{ \frac{\theta_i(p)}{\theta(p)} - \frac{\omega}{\beta(p)} \frac{\beta_i(p)}{\beta(p)} \right\} \\ \times \{ \mu + \omega(T - \alpha(p)) - \epsilon(p) \}.$$

These functions are quadratic in ω and contain a term in $\mu\omega$ as well as in μ . This is essentially because, with linear labor supply, earned income must be quadratic in the wage as well as containing a term in $\mu\omega$ so that these terms are forced into the commodity demand functions through the budget constraint. On cross-section data, with p constant from household to household, they could be estimated by linear regression of q_i on (a constant), ω , μ , ω^2 , and $\omega\mu$ under the restriction that the ratio of the coefficients on the last two should be the same for all i .

Although this would present no great practical difficulty, note that (14) is *not* a general quadratic form in ω and μ . There is no term in μ^2 and there are other restrictions on the coefficients, e.g. that the coefficients on $\omega\mu$ and ω^2 must be in a fixed ratio for all commodities. Hence, in spite of the extra terms, there is little extra generality in (14) over, say, a linear functional form, at least for the analysis of unrationed demands.

The fact that (12) aggregates exactly across individuals with the same preferences is convenient for working with grouped or aggregate data on per capita hours, wage rates, and nonlabor incomes. However, the corresponding commodity demand functions do not share this convenience because of the presence of quadratic and interaction terms. From this point of view, the next set of preferences we consider is more attractive.

(b) *Linear Income Supply Functions*

An alternative specification which is equally simple for the analysis of labor supply is to assume that earned income, ωl , is a linear function of ω and μ . Once

again, ωl must be a linear function of ω and x , so that

$$(15) \quad \omega l = \alpha_0 + \alpha_1 \omega + \alpha_2 \mu$$

in contrast to (1). The equations (7), (8), and (9) are modified to

$$(16) \quad \frac{\omega c_{0u}}{c_u} = \delta(p)$$

and

$$(17) \quad \omega c_{00} + c_0 - \delta(p)c_0 = \{1 - \delta(p)\}b(p)$$

for functions $\delta(p)$ and $b(p)$ of prices alone. Once more integration is straightforward leading to the cost function

$$(18) \quad c(u, \omega, p) = t(u, p)\omega^{\delta(p)} + \omega b(p) + d(p)$$

where $\delta(p)$ and $b(p)$ are homogeneous of degree zero, $d(p)$ is homogeneous of degree one, and $t(u, p)$, which is positive and increasing in u , is homogeneous of degree $\{1 - \delta(p)\}$ in p . Note the close relationship between the two sets of preferences (10) and (18); indeed, replacement of ω in the first term on the right hand side of (10) by $\log \omega$ leads to a form apparently identical to (18). However, the various functions in (10) and (18) have quite different properties, especially as regards homogeneity, and these differences have important consequences, as we shall see below.

The labor supply functions from (18) are given by

$$(19) \quad \omega l = \delta(p)d(p) - \delta(p)\mu + \{1 - \delta(p)\}\omega\{T - b(p)\}$$

so that $\delta(p)$ has the interpretation of the amount by which earned income is reduced for a one unit increase in unearned income. Derivation of the commodity demands requires some restriction on the function $t(u, p)$. Analogously to (13) we adopt

$$(20) \quad t(u, p) = u\{a(p)\}^{(1 - \delta(p))},$$

where $a(p)$ is homogeneous of degree one. This yields commodity demands of the form

$$(21) \quad q_i = d_i(p) + \omega b_i(p) + \left\{ [1 - \delta(p)] \frac{a_i(p)}{a(p)} + \delta_i(p) \log \frac{\omega}{a(p)} \right\} \\ \times \{ \omega(T - b(p)) + \mu - d(p) \}$$

which is comparable to (14) and of a similar degree of generality. Like (14), (21) does not permit exact linear aggregation across individuals. However, unlike (14), there is a not too restrictive simplification which does give exact linear aggregation for the commodity demands as well as income supply. Since the function

$\delta(p)$ in (18) is homogeneous of degree zero, let us assume it to be a constant, δ say, in which case $\delta_i(p)$ is zero. If so, (21) simplifies to

$$(22) \quad q_i = \left\{ d_i(p) - (1 - \delta) \frac{a_i(p)}{a(p)} d(p) \right\} \\ + \left\{ b_i(p) + (1 - \delta) \frac{a_i(p)}{a(p)} [T - b(p)] \right\} \omega + (1 - \delta) \frac{a_i}{a} \mu$$

so that, in cross-section studies, the commodity demand functions, like the income supply function (19), can be estimated by linear regression on the wage rate and unearned income. This makes the two sets of functions (19) and (22) a particularly simple and attractive basis for econometric analysis and accounts for the link between the present results and those required for exact linear aggregation. With the specification (20) and with δ constant, the cost function (18) is identical to the case which Muellbauer [18] shows to be necessary and sufficient for exact linear aggregation.

We note finally that even with $\delta(p)$ taken to be constant and under the specification (20), the cost function (18) is still a second-order flexible functional form provided $a(p)$, $b(p)$, and $d(p)$ are all first-order flexible functional forms and at least one is a second-order flexible functional form. Hence, provided the functions are suitably chosen (19) and (22) make a suitable vehicle for the analysis of time-series and longitudinal data as well as of cross-section data.

2. FUNCTIONAL FORMS FOR RESTRICTED DEMANDS

If labor supply is predetermined outside the consumer's control, then commodity demand functions are conditional on income (or total expenditure) and, in the absence of separability between leisure and goods, on the amount of employment, rather than depending, as in Section 1, on the wage rate and transfer income separately. However, both sets of commodity demand functions are derived from the same set of preferences, the only difference being the existence of the additional labor market constraint. This can be handled according to the theory of rationing and the effects of the quantity constraint can be analyzed from the first-order conditions for utility maximization as originally laid out by Tobin and Houthakker [25]. However, as is well known, this methodology only characterizes the rationed demand function *locally*, giving the derivatives of the constrained demands in the neighborhood of the point where the constraint only just begins to bind. It cannot yield global formulae for rationed and unrationed demands. To do this, we must take a dual approach. Here, we give only the briefest possible summary, translated to our context, of those results in Neary and Roberts [20] which are required for our derivations.

Let z be the amount to which q_0 is constrained (e.g. T for involuntary unemployment). Then define the *restricted* cost function $c^*(u, \omega, p, z)$ by

$$(23) \quad c^*\{u, \omega, p, z\} = \min_q \{\omega z + p \cdot q; v(z, q) \geq u\}.$$

Clearly, (23) can be rewritten as

$$(24) \quad c^*\{u, \omega, p, z\} = \omega z + \gamma(u, p, z)$$

where $\gamma(u, p, z) = \min\{p \cdot q; v(z, q) \geq u\}$ and does not depend on ω . Now, if preferences are convex, which for convenience we assume, there will always exist, for each u and p , some wage rate ω which will make the ration z optimal. Such a wage is what Slutsky [24] and Neary and Roberts [20], following Rothbarth [23], call a “virtual price.” Denote this wage ω^* and write it

$$(25) \quad \omega^* = \zeta(u, z, p)$$

where $\zeta(u, z, p)$ is obtained by setting the Hicksian demand for leisure to z , i.e. it is implicitly defined by

$$(26) \quad c_0\{u, \zeta(u, z, p), p\} = z$$

where $c(u, \omega, p)$ is the *unrestricted* cost function and, as before, c_0 is the partial derivative of c with respect to ω . Now, for any z , a wage of ω^* will render z optimal so that, for this level of ω , the values of the restricted and unrestricted cost functions must coincide. Hence, as an *identity* in u, z , and p , we have, from (24),

$$(27) \quad c(u, \zeta(u, z, p), p) = z\zeta(u, z, p) + \gamma(u, p, z),$$

so that, by comparison, $c^*(u, \omega, p, z)$ can always be obtained from

$$(28) \quad c^*(u, \omega, p, z) = \{\omega - \zeta(u, z, p)\}z + c\{u, \zeta(u, z, p)p\}.$$

This result is the central formula linking rationed to unrationed demands and can be used to provide global generalizations of the Tobin-Houthakker results; see Deaton and Muellbauer [7, Chapter 4.3] and Neary and Roberts [20]. Note that it will not always be possible to derive $c^*(u, \omega, p, z)$ explicitly and, in particular, the solution of (26) for the function $\zeta(u, z, p)$ may not be possible and this would preclude the analytical derivation of (28). However, if this can be overcome, and for the cost functions of Section 1 there are no problems, then differentiation of the restricted and unrestricted cost functions in turn will yield a “matched pair” of restricted and unrestricted demand functions, each derived from identical preferences.

(a) *Restricted Demand Functions Corresponding to Linear Labor Supply*

Beginning with the linear labor supply cost function (10), we differentiate with respect to ω , set equal to z , and rearrange to obtain

$$(29) \quad \omega^* = \zeta(u, z, p) = \beta(p) \log \left\{ \frac{\beta(p)(\alpha(p) - z)}{\eta(u, p)} \right\},$$

a function which, as is generally true, is positive linear homogeneous in p , increasing in u (recall $\eta_u < 0$) and decreasing in z . Equation (29) can now be

applied to obtain the restricted cost function

$$(30) \quad c^*(u, \omega, p, z) = \omega z + \epsilon + \beta(\alpha - z) \left[\log \left\{ \frac{\beta(\alpha - z)}{\eta(u, p)} \right\} - 1 \right].$$

If $\eta(u, p)$ takes the specific form (13), and if we treat z as a fixed number, the cost function (30) is a Gorman polar form, as rearrangement will show. In this case, the demand functions may be written

$$(31) \quad q_i = (\epsilon_i + \alpha_i \beta) + \left(\frac{\beta_i}{\beta} - \frac{\theta_i}{\theta} \right) \beta(\alpha - z) + \left[\frac{\beta_i}{\beta} + \frac{\alpha_i}{\alpha - z} \right] (y - \epsilon)$$

where y is the total earned and unearned income ($= \omega(T - z) + \mu$). Note that if (31) is interpreted as a linear Engel curve from the Gorman polar form, the ration z —or at least $(\alpha - z)$ —affects both the intercept and the marginal propensity to consume. On cross-section data, the quantities $\alpha_1, \beta_1, \delta_1, \alpha, \beta, \gamma$, and δ can be treated as parameters, although unlike the previous equations, (31) would require nonlinear estimation provided $\alpha_i \neq 0$. In situations where we have information on whether or not individual households are or are not constrained on the labor market, equations (12), (14), and (31) can be estimated as a set with common parameters appearing in both.

(b) Restricted Demand Functions Corresponding to Linear Income Supply

The linear income supply cost function is treated similarly. From (18), the shadow price function is

$$(32) \quad \omega^* = \left\{ \frac{t(u, p)\delta(p)}{z - b(p)} \right\}^{1/(1-\delta(p))}$$

Hence, the restricted cost function is

$$(33) \quad c^*(u, \omega, p, z) = z\omega + d(p) + \{1 - \delta(p)\}\delta(p)^{\delta(p)/(1-\delta(p))} \\ \times \{t(u, p)\}^{1/(1-\delta(p))} \{z - b(p)\}^{-(\delta(p))/(1-\delta(p))}.$$

Once again, under the specific assumption (20), with δ constant, and with z treated as a fixed number, (33) is a Gorman polar form, but in contrast to (30), the hours variable z appears only in the term multiplying utility. The demand functions (under the assumption that $\delta(p) = \delta$) take the form

$$(34) \quad q_i = d_i(p) + \left\{ \frac{a_i(p)}{a(p)} + \sigma \frac{b_i(p)}{z - b(p)} \right\} \{y - d(p)\}$$

where $\sigma = \delta/(1 - \delta)$. Hence, it is only the marginal propensities to consume which are affected by changes in z . The effect on the Engel curve of an increase

in z for a commodity which is complementary with leisure ($b_i < 0$) is a rotation of the Engel curve around the "subsistence" point where $y = d$. This is substantially less general than in equation (31) where independent shifts of both slope and intercept are possible.

(c) *Exact Aggregation of Restricted Demand Functions*

Both (31) and (34) are linear in income given z , so that both permit exact linear aggregation of commodity demands over workers all of whom are rationed to the same level. However, only for (34) do the corresponding unrestricted commodity demands aggregate linearly. Suppose one had aggregate consumption data generated by a population of individuals some of whom could freely choose their labor supply according to (19) while the rest were unemployed with $z = T$ for each. Then, given knowledge of the average wage and unearned income for each of the two groups, average consumption is given by the weighted average of (22), averaged for the employed, and (34), averaged for the unemployed, the weights being the fractions of individuals employed and unemployed respectively. These functions are the most general which permit this exact aggregation. Ashenfelter's [3] study is of this type although he is forced to assume that both sets of individuals have the same nonlabor income less savings. Also the linear expenditure system which he uses is a simplified form of (22) and (34) that assumes additive preferences both between goods and between goods and leisure.

For (31) and (34) to yield restricted demand functions which aggregate linearly over individuals constrained to different employment levels, further simplification is required. Equation (31) is linear in z and μ if $\alpha_i = 0$, i.e. if $\alpha(p) = \text{constant}$ while (34) is linear in y and hence in z and μ if $b_i = 0$, i.e. $b(p) = \text{constant}$. In the latter case, $b(p) = \text{constant}$ implies separability between goods and leisure so that the employment constraint only exercises an income effect. Note, however, that it is only the combination of linearity and the aggregation requirement which gives the separability result. In general, exact aggregation with different incomes and ration levels will be guaranteed if only the restricted commodity demand functions are linear in both y and z . Such demands are yielded by the preferences arising out of the analysis of the next section.

3. ON A SIMPLE RELATIONSHIP BETWEEN RATIONED AND UNRATIONED DEMANDS

In the absence of specific functional forms for rationed demand functions, a number of authors (see e.g. Ito [15], Portes [22], and Muellbauer [17]) have worked with a simple linear relationship linking rationed and unrationed demands. Here, for rationed and unrationed Marshallian demands, $g_i(x, \omega, p)$ and $g_i^*(x, \omega, p, z)$, the hypothesis in its most general form is that

$$(35) \quad g_i^*(x, \omega, p, z) = g_i(x, \omega, p) + \delta_i(x, \omega, p)\{z - g_0(x, \omega, p)\},$$

i.e. the effect of the ration on unrationed demands is proportional to the difference between the ration and the notional demand for the rationed good. Note that the constant of proportionality, although allowably a function of x, ω , and p , is *not* a function of z . Although (35) has a degree of superficial plausibility, it is not obvious whether or not it is consistent with rationing and preference theory. In particular, note that although ω can only have income effects on the left-hand side of (35), its effects on the unrationed demands on the right-hand side are apparently unrestricted. As we shall demonstrate, this limits the class of preferences for which (35) is globally valid.

Since ω only affects $g_i^*(\cdot)$ through the income effect, it is generally true that

$$(36) \quad g_i^*(x, \omega, p, z) = f_i(x - \omega z, p, z)$$

for suitable functions $f_i(\cdot)$. If, however, z is set at its unrationed level $g_0(x, \omega, p)$, the unrationed and rationed demands must coincide. Hence, from (36)

$$(37) \quad g_i(x, \omega, p) = f_i\{x - \omega g_0(x, \omega, p), p, g_0(x, \omega, p)\}.$$

Hence, by comparison of (36) and (37), and taking a Taylor expansion about the unrationed point, we have the *local* approximation

$$(38) \quad g_i^*(x, \omega, p, z) = g_i(x, \omega, p) + (f_{iz} - f_{ix}\omega)\{z - g_0(x, \omega, p)\}$$

so that the original equation (35) can always be justified as a local approximation if the ration level is not too far from the unrationed demand.

The linear relationship will only be globally valid however if $(f_{iz} - f_{ix}\omega)$ is independent of z , or equivalent if g_i^* is independent of z . But, from (36), since z appears not only independently but also always as part of $x - \omega z$, and since ω cannot appear elsewhere in (36), linearity in z also implies linearity in $x - \omega z$. Hence (35) is globally valid if and only if the *rationed* demand functions are linear in both $x - \omega z$ and z . Hence, for suitable functions $a_i^*(p)$, $b_i^*(p)$, and $d_i^*(p)$,

$$(39) \quad g_i^*(x, \omega, p, z) = a_i^*(p) + b_i^*(p)z + d_i^*(p)(x - \omega z).$$

We can solve this for the *rationed* cost function from (24) by writing $x - \omega z = \gamma(u, p, z)$ and $g_i^*(\cdot) = \partial \gamma(u, p, z) / \partial p_i$ and solving the resulting system of (linear) partial differential equations. Hence

$$(40) \quad c^*(u, \omega, p, z) = \omega z + a(p) - b(p)z + d(p)\pi(u, z)$$

where $a(p), b(p)$ and $d(p)$ are linearly homogeneous in p and $\pi(u, z)$ is increasing in u and convex in z . It is easily checked that $a_i^*(p) = a_i - d_i a/d$, $b_i^*(p) = -(b_i - b d_i/d)$, and $d_i^*(p) = d_i/d$.

Finally, the underlying unrestricted preferences allowing the result can be retrieved from (40) using the envelope property of restricted and unrestricted cost functions, viz.

$$(41) \quad c(u, \omega, p) = \min_z c^*(u, \omega, p, z).$$

Hence, defining the function $\xi(u, x)$ implicitly by

$$(42) \quad \pi_2\{u, \xi(u, a)\} = \alpha$$

for all α , the unrestricted cost function must take the form

$$(43) \quad c(u, \omega, p) = a(p) - \{b(p) - \omega\} \xi\left(u, \frac{b(p) - \omega}{d(p)}\right) \\ + d(p) \pi\left\{u, \xi\left(u, \xi\left(u, \frac{b(p) - \omega}{d(p)}\right)\right)\right\}.$$

Equations (42) and (43) can be used to generate preferences which yield rationed and unrationed demands satisfying (35) exactly.

The final functional form is perhaps surprisingly general, the only real restriction being that preferences should be such as to generate linearity in the rationed demands. Note in particular that separability is *not* required. The preferences represented by (43) are not weakly separable between goods and leisure except in the case where $b(p) = 0$. However, *given* weak separability, the commodity branch of utility must take the Gorman polar form and exhibit linear (group) Engel curves. An example which does so is the linear expenditure system. In general, however, weak separability is essentially irrelevant in the construction of (35), either locally or globally. Note, finally, in reference to the claim at the end of the previous section, that (39) permits exact linear aggregation over individuals with differing ration levels z and incomes ex-ration, $x - \omega z$, i.e. y above. It is clearly also the most general set of functions to do so, so that (40) and (43) represent the most general form of restricted and unrestricted preferences allowing exact aggregation of the rationed demands.

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REFERENCES

- [1] ABBOTT, M., AND O. ASHENFELTER: "Labour Supply, Commodity Demand and the Allocation of Time," *Review of Economic Studies*, 43(1976), 389-411.
- [2] ———: "Labour Supply, Commodity Demand and the Allocation of Time: Correction," *Review of Economic Studies*, 46(1979), 567-569.
- [3] ASHENFELTER, O.: "Unemployment as Disequilibrium in a Model of Aggregate Labor Supply," *Econometrica*, 48(1980), 547-564.
- [4] ASHENFELTER, O., AND J. HECKMAN: "The Estimation of Income and Substitution Effects in a Model of Family Labor Supply," *Econometrica*, 42(1974), 73-86.
- [5] BOSKIN, M. J.: "The Economics of Labor Supply," in *Income Maintenance and Labor Supply*, ed. by G. G. Cain and H. W. Watts. New York: Academic Press, 1973.
- [6] DEATON, A. S.: "Theoretical and Empirical Approaches to Consumer Demand under Rationing" in *Essays in the Theory and Measurement of Consumer Behavior*, ed. by A. S. Deaton. New York: Cambridge University Press, 1981.

- [7] DEATON, A. S. AND J. MUELLBAUER: *Economics and Consumer Behavior*. New York: Cambridge University Press, 1980.
- [8] GARFINKEL, I.: "On Estimating the Labor Supply Effects of a Negative Income Tax" in *Income Maintenance and Labor Supply*, ed. by G. Cain and H. Watts. New York: Academic Press, 1973.
- [9] GREENBERG, D. H. AND M. KOSTERS: "Income Guarantees and the Working Poor" in *Income Maintenance and Labor Supply*, ed. by G. Cain and H. Watts. New York: Academic Press, 1973.
- [10] GRONAU, R.: "The Effects of Children on the Housewife's Value of Time," *Journal of Political Economy*, 81(1973), S168–199.
- [11] HALL, R. E.: "Wages, Income and Hours of Work in the U.S. Labor Force," in *Income Maintenance and Labor Supply*, ed. by G. Cain and H. Watts. New York: Academic Press, 1973.
- [12] HAUSMAN, J. A.: "The Effect of Wages, Taxes and Fixed Costs on Women's Labor Force Participation," mimeo, MIT, 1979.
- [13] HECKMAN, J.: "Shadow Prices, Market Wages and Labor Supply," *Econometrica*, 42(1974), 679–694.
- [14] ———: "Sample Selection Bias as a Specification Error," *Econometrica*, 47(1979), 153–161.
- [15] ITO, T.: "Methods of Estimation for Multi-Market Disequilibrium Models," *Econometrica*, 48(1980), 97–125.
- [16] LAYARD, R., M. BARTON, AND M. ZABALZA: "Married Women's Participation and Hours," *Econometrica*, 47(1980), 51–72.
- [17] MUELLBAUER, J.: "Continuous vs. Discrete Macroeconometrics," mimeo, Birkbeck College, 1979.
- [18] ———: "Linear Aggregation in Neoclassical Labour Supply," *Review of Economic Studies*, 48(1981), 21–36.
- [19] NAKAMURA, M., A. NAKAMURA, AND D. CULLEN: "Job Opportunities, the Offered Wage and the Labor Supply of Married Women," *American Economic Review*, 69(1979), 787–805.
- [20] NEARY, J. P., AND K. W. S. ROBERTS: "The Theory of Household Behaviour under Rationing," *European Economic Review*, 13(1980), 25–42.
- [21] PHILIPS, L.: "The Demand for Leisure and Money," *Econometrica*, 46(1978), 125–143.
- [22] PORTES, R. D.: "Effective Demand and Spillovers in an Empirical Two Market Disequilibrium Model," Harvard Institute of Economic Research, Discussion Paper 595, 1977.
- [23] ROTHBARTH, E.: "The Measurement of Changes in Real Income under Conditions of Rationing," *Review of Economic Studies*, 8(1941), 100–107.
- [24] SLUTSKY, S.: "Preferences and Demands in Constrained Situations," Working Paper 192, Economics Department, Cornell University, 1979.
- [25] TOBIN, J., AND H. S. HOUTHAKKER: "The Effects of Rationing on Demand Elasticities," *Review of Economic Studies*, 18(1958), 140–153.