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THE ANALYSIS OF CONSUMER DEMAND IN THE UNITED KINGDOM, 1900–1970

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This paper considers the application of various models of consumer demand to United Kingdom time series from 1900 to 1970. As well as testing the various forms of the "Rotterdam" model, reparametrization of that system is carried out in order to test the linear expenditure system and the *direct* addilog system on an exactly comparable basis. A further variant of the Rotterdam model is also introduced; this is intermediate between symmetry and additivity and allows for the calculation of all cross price elasticities from information on own price and income elasticities alone. The results of testing these models on a nine commodity model using maximum likelihood estimation are presented and discussed. Unlike most previous work, and in spite of some anomalous results, the United Kingdom experience seems broadly consistent with neoclassical demand theory. However, all restrictions more stringent than those directly implied by the theory are rejected, though it is maintained that these may still be of considerable practical significance in particular instances.

1. INTRODUCTION

THIS PAPER PRESENTS an analysis, within the context of twentieth century British experience, of the way in which income and prices influence demand. To some extent we shall be concerned with repeating for the United Kingdom the experiments on the validity of demand theory carried out by Barten [**2**, **3**, and **4**] on Dutch data and with investigating whether his negative conclusions recur here. But whereas his work and this extension of it are concerned with the appropriateness of behavioral restrictions within a given model (the Rotterdam demand system), we shall be concerned with somewhat wider issues. We wish to be able to make judgements not only between different variants of the same model but also between different models and between models of different structure. For example, the question arises as to the relative appropriateness of additivity as imposed within the Rotterdam model on the one hand, and the linear expenditure system on the other; or whether it is better, given the necessity to impose strong restrictions in a practical context, to ignore the substitution effects of prices altogether or to impose additivity or some other constraint. These issues are likely to have real practical importance in situations for which degrees of freedom are scarce and strong a priori assumptions are necessary in order to allow price sensitivity at all. We thus wish to work with a general framework in which the full implications of different systems can easily be seen and which may be used to estimate the competing systems in a manner which will ensure the full comparability of the results.

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It is perhaps instructive to examine to what extent this purpose has already been accomplished and in what areas previous studies have not achieved the purpose intended here. Within a rapidly expanding literature, a number of authors have presented the results of comparisons of alternative demand models of quite different structure. An excellent example of this is Parks' paper [**15**]. Here, the same data is used for the efficient estimation of three systems, the linear expenditure system, Houthakker's indirect addilog model [**13**], and the Rotterdam model. This yields many insights into the powers and defects of these systems, and Parks uses Theil's informational measure [**18**] as a common criterion to discriminate between the models. But it is the selection of this criterion and its relationship to the estimation of the models which is the central problem in this work. Unless each model is given the same opportunity to maximize its chance of acceptability by the given criterion, a convincing test will not result. In this case, of course, there is no close relationship between the criterion maximized by the parameter estimators and that used in the selection of the system: this can only be accomplished by the use of a common model of the type discussed above.

The problem of selection is heightened by the basic dichotomy between utility and demand which runs through all of the empirical literature. Demand models may be derived either by the selection of a utility function or by the arbitrary specification of a system of equations which may then be modified according to the utility theory. If we choose the first course of action, we have the difficulty of the selection of functional form; if we choose the second, we have the difficulty of choice of variables to parametrize. In both cases, because of the mathematical complexity of a conversion from one to the other, a decision may have unexpected or untoward consequences. For example, it is well known that the use of linear demand functions satisfying the symmetry postulates involves acceptance of additive utility. Once again the use of a common framework for demand analysis, though in no way solving this problem, makes clearer the full ramifications of each system and thus enables us to discriminate more easily between alternative functional forms and parametrizations. Again to take an example, we may hope to discover, in the event additivity is proved unacceptable within the Rotterdam framework, whether this is due to the falsity of additivity in reality or alternatively merely to the choice of parametrization implicit in using that model.

Fortunately, there is no difficulty in finding a framework general enough to allow the full range of experiment. Any model which imposes no constraints on behavior may serve this purpose. However, the system must possess two properties: it must be possible to change the parametrization readily, and it must be possible to apply constraints explicitly within the model. It is the fulfillment of this second requirement that is the peculiar strength of the Rotterdam system. Many of the important constraints are linear, and the computational convenience of this has outweighed doubts on other scores, in particular non-integrability over all but infinitesimal time periods. However, the model is equally useful in respect of the first requirement. The parametrization normally used, that suggested by Theil [**18**] when the model was introduced, though convenient, is not immutable. There are many other possibilities, and in the next section we shall show how some of these can be used to compare the Rotterdam with other demand systems.

In particular we shall concentrate on two other models of demand. The first of these is the linear expenditure system. This is a familiar tool of demand analysis, and no comparison of models would be complete without it. The second case, Houthakker's *direct* addilog system [**13**], is perhaps less universally familiar; nevertheless, it provides an excellent example of the advantages of working within a general framework. In the past, this system has proved analytically intractable, and its inconvenient form has made it difficult either to use or to compare with other models. Nevertheless, by the appropriate selection of parameters, this model, like the linear expenditure system, may be expressed within the Rotterdam framework. Once this is done, its empirical and theoretical properties can be compared directly with the other models in a way not otherwise possible.

The field of comparison thus contains four variants of the Rotterdam system (free, homogeneous, symmetric, and additive) together with the direct addilog model and the linear expenditure system, making six models in all. To this we add two more. The first is the trivial model yielded by ignoring all substitution effects of prices. It is important to test this first since, if the substitution matrix is of little importance in explaining demand, the study of the appropriate constraints on its terms is not an interesting exercise. Secondly, we suggest a fifth variant of the Rotterdam model. Since three of our models so far are additive, and since, even when compared with symmetry, this is a very strong assumption, it is desirable to have some model more restrictive than symmetry but not as restrictive as additivity. An "intermediate" system is proposed to fill this gap. This model also allows the calculation of all cross price elasticities from income and own price elasticities only. It thus requires more information than Frisch's method [**10**] but is hopefully more accurate.

We shall compare these eight models using United Kingdom time series 1900 to 1970; all parameters will be estimated by maximum likelihood, and the concentrated likelihood values for each of the systems will provide the basis for the comparisons between them.

In Section 3, the maximum likelihood estimators used are described. There is now considerable familiarity with the methods appropriate in these cases, and only a brief restatement of the principles involved will be given together with the formulae used to calculate the estimates. This section concludes with a brief description of the data. Section 4 presents the results. All parameter estimates are given but the discussion centers around the properties of the data as revealed by the models. It is worth anticipating some of the conclusions here. Contrary to the results for the Netherlands reported by Barten [**4**] and Byron [**6** and **7**], the theory appears to be broadly acceptable. The homogeneity postulate holds for all but two of the commodity groups, and the much stronger symmetry restrictions are also consistent with the evidence. The symmetric substitution matrix derived from these results proves not to be negative semi-definite, thus violating the convexity requirements. Nevertheless, this violation seems not to be significant, and experiments showed that *all* the relative prices must change before any non-convex behavior can be induced; no simpler price change could induce such effects. Restrictions beyond symmetry were uniformly rejected. Even so the intermediate model performed considerably better than the additive systems: these latter would be rejected even as a specialization of the former. Nevertheless, additivity was always preferable to the model without any form of price substitution. Finally, the results have some bearing on the question of how the marginal utility of money varies with the level of income and we shall have some evidence for slight modification of the numerical values for the income flexibility of money suggested by Frisch [**10**].

2. THE DEVELOPMENT OF THE MODELS

The Rotterdam system which we take as our starting point takes the form[2] (see, e.g. [**4**])

$$(1) \qquad \hat{w}\, d\log q = b\, d\log \bar{\mu} + C\, d\log p.$$

The vector w represents the average value shares; q is a vector of quantities of each commodity; and p is a vector of prices. The term $d\log\bar{\mu}$ is an index number of the change in real income and is given by

$$(2) \qquad d\log\bar{\mu} = w'\, d\log q \approx d\log\mu - w'\, d\log p$$

where μ is total money expenditure. The vector b and matrix C are the parameters of the system and are given by

$$(3) \qquad b = \hat{p}\frac{\partial q}{\partial \mu}; \qquad C = \mu^{-1}\hat{p}S\hat{p}; \qquad S = \frac{\partial q}{\partial p} + \frac{\partial q}{\partial \mu}q';$$

where S is the Slutsky substitution matrix. The vector b is thus a vector of marginal budget shares while C measures the contribution to the value shares of compensated price changes. Each of the matrices and vectors in equations (1) to (3) are of order n, the number of commodities; the time suffix implied by the equations will usually be omitted.

Note that there is no strong a priori reason why b and C should be held constant. Nevertheless, some decision must be taken, and though it would be desirable only to parametrize those quantities which could conceivably be the parameters of some underlying utility function, this is not in general possible. Indeed, short of specifying a *particular* utility function, which is exactly what this approach manages to avoid, there is no way of recognizing these parameters from the demand functions alone. However, as indicated in the previous section, it is the great strength of this choice of parameters that the constraints of demand theory can be directly applied to these constants.

In particular we have (again, see [**4**])

(4.1a) Engel aggregation $\quad b'\iota = 1,$

(4.1b) Cournot aggregation $\quad C'\iota = 0,$

[2] Notationally, we use Greek letters for scalar quantities, small Roman letters for vectors, and capital Roman letters for matrices. The derivatives of vectors with respect to vectors, e.g. $(\partial q/\partial p)$ should of course be read as matrices. The "hat" notation, e.g. $\hat{p}$, denotes a matrix with the vector p as its leading diagonal and with zeros elsewhere.

(4.2)	Homogeneity	$C\iota = 0,$
(4.3)	Symmetry	$C = C',$
(4.4)	Convexity	$x'Cx \leqslant 0 \quad \text{for all } x,$
(4.5)	Additivity	$C = \phi(\hat{b} - bb').$

Now equation (1) was derived by Theil by algebraic manipulation of a first order Taylor linearization of a quite general demand function. Thus (1) itself can be regarded as an expansion of the same sort and with the same degree of generality. Given this, it might seem at first glance that the testing of equations (4.1)–(4.5) is all that must be done to confirm or reject the postulates of utility theory. But as Goldberger [**11**] has shown, if (1) is used over a longer than infinitesimal time span, it loses its generality with the breakdown of the linearization and may only adequately represent systems where the quantities parametrized by (3) are truly constant over time. And as he has demonstrated elsewhere (Goldberger [**12**]), the only utility mapping consistent with (1) is a degenerate case of the additive function underlying the linear expenditure system (see (6) below) implying that all price elasticities are zero. In other words, over a sufficiently small time period, demand system (1) may be integrated into any utility function, but when taken as valid over periods long enough to be of interest from an econometric point of view, it is integrable only in a trivial and uninteresting sense.

It would be a mistake to reject the Rotterdam system on these grounds alone in the same way that it is a mistake to claim that it is useless because infinitesimal changes cannot be observed in practice. A deductive system is *only* of practical significance to the extent that it *can* be applied to concrete phenomena and it is too easy to protect demand theory from empirical examination by rendering its variables unobservable and hence its postulates unfalsifiable. Furthermore, and from a more pragmatic viewpoint, the model remains an excellent vehicle for a purely empirical analysis of the way in which price behavior may be constrained and one would expect the conclusions to be relatively robust with respect to alternative parametrizations. Nevertheless the qualifications are important in that they make it necessary to compare the Rotterdam with alternative models, for we may no longer be sure that our conclusions are derived from the data alone and not from our choice of parameters. It is thus in this spirit that we go on to change the parameters of (1) to render it comparable with other models of demand analysis while preserving this as our basic model for the comparisons. We turn first to the linear expenditure system.

The model is usually written in the form

$$\hat{p}q = \hat{p}c + b(\mu - p'c), \qquad b'\iota = 1, \tag{5}$$

where the vectors b and c are the parameters of the system. The b's have the same interpretation as in the Rotterdam model, i.e., marginal budget shares, while the c's have the dimension of quantities and are sometimes interpreted as quantities to which the consumer is in some sense committed (though there is no presumption that they should be positive). The system may be derived from any monotonically increasing transformation of the additive utility function,

$$u(q) = b' \log (q - c), \qquad b'\iota = 1. \tag{6}$$

If we take first differences of equation (5), we have

$$\hat{p}\, dq = (\hat{c} - \hat{q})\, dp + b(d\mu - c'\, dp).$$

Defining $v = \mu^{-1}\hat{c}p$, i.e., the "committed" budget shares, and using the transform $dx = \hat{x}\, d \log x$, we have, after division by μ,

$$\hat{w}\, d \log q = (\hat{v} - \hat{w})\, d \log p + b(d \log \mu - w'\, d \log p) + b(w - v)'\, d \log p.$$

But from (5), $w = v + b(1 - v'\iota)$, and writing $\phi = -(1 - v'\iota)$, we have $v - w = b\phi$. Thus, after substitution, we may write the linear expenditure system,

$$\hat{w}\, d \log q = b\, d \log \bar{\mu} + \phi(\hat{b} - bb')\, d \log p, \tag{7}$$
$$\phi = -1 + \mu^{-1}p'c, \qquad b'\iota = 1.$$

The quantity ϕ is the inverse of the elasticity of the marginal utility of money and is the reciprocal of Frisch's "money flexibility" [**10**]. This result is well known (see, for example, [**11**]) and may easily be proved directly from the constrained maximization of the utility function (6).

Note the additive structure of (7); indeed substitution of (4.5), the additivity postulate, into (1) gives a system identical to (7) but for the specification of ϕ. Thus the additive Rotterdam model can be minimally altered to give rise to the linear expenditure system. On empirical grounds, it becomes clear that the difference between them will be resolved by finding out how indeed ϕ does vary. The linear expenditure system in the form (7) can be estimated with respect to the parameters b and c. The extent to which the likelihood increases over the additive Rotterdam model will measure how much the extra flexibility of ϕ is required by the data.

Our third model is Houthakker's direct addilog system [**13**]. This may be derived from the utility function

$$u(q) = \sum_k \alpha_k q_k^{\beta_k} \tag{8}$$

where α and β are parameters. To derive the demand functions we form the Lagrangian

$$\Phi = \sum_k \alpha_k q_k^{\beta_k} + \lambda(\mu - p'q),$$

giving after differentiation and apart from the constraint, the first order conditions

$$\alpha_i \beta_i q_i^{\beta_i - 1} - \lambda p_i = 0 \quad \text{for all } i. \tag{9}$$

Thus, dividing the ith equation by the jth and taking logarithms, we have

$$(\beta_i - 1) \log q_i - (\beta_j - 1) \log q_j = \log p_i - \log p_j - \log \frac{\alpha_i \beta_i}{\alpha_j \beta_j}. \tag{10}$$

These equations plus the budget constraint define the system; though the idea of demand functions defined only in terms of relationships between pairs of commodities may be hardly appealing, it is perfectly acceptable. To the extent that the system has been used for empirical work, equation (10) has been used and presumably the technique developed by Parks [**15**] for the indirect addilog system could be applied here. Nevertheless the form remains clumsy and it is not easy to see how the system relates to the other models.

In order to write this system in the Rotterdam format, we proceed by deriving expressions for ϕ and the elasticities of the model. This was done first in Deaton and Wigley [**9**] but the derivation there is inelegant and an alternative is given below. Differentiating equation (9) with respect to income, we have

$$\frac{\partial \lambda}{\partial \mu} = \frac{\alpha_i \beta_i(\beta_i - 1)}{p_i} q_i^{\beta_i - 2} \frac{\partial q_i}{\partial \mu}.$$

Thus

$$\phi^{-1} = \frac{\partial \lambda}{\partial \mu} \cdot \frac{\mu}{\lambda} = (\beta_i - 1)\frac{\partial q_i}{\partial \mu} \cdot \frac{\mu}{q_t} = (\beta_i - 1)e_i,$$

where e_i is the income elasticity of the ith good. Defining $\gamma_i = (1 - \beta_i)^{-1}$, we may write $-\gamma = e\phi$, and thus from Engel aggregation ($w'e = 1$),

(11) $\quad \phi = -w'\gamma.$

Differentiating (10) first with respect to $\log \mu$ and then with respect to $\log p_k$, and writing e_{ij} for the typical price elasticity, we have

(12.1) $\quad (\beta_i - 1)e_i - (\beta_j - 1)e_j = 0,$

and

(12.2) $\quad (\beta_i - 1)e_{ik} - (\beta_j - 1)e_{jk} = \delta_{ik} - \delta_{jk}, \quad$ all i, j, k.

Applying Engel aggregation to (12.1) gives

(13.1) $\quad e_i = -\phi^{-1}\gamma_i,$

and applying Cournot aggregation ($\Sigma_k w_k e_{ki} + w_i = 0$) to (12.2),

(13.2) $\quad e_{ij} = \phi^{-1}\gamma_i w_j(1 - \gamma_j) - \gamma_i \delta_{ij}.$

From equations (13.1)–(13.2) we may now calculate the expressions corresponding to the b and C parameters of the Rotterdam model. We have

$$b_i = p_i \partial q_i/\partial \mu = w_i e_i = -\phi^{-1} w_i \gamma_i, \quad \text{and}$$

$$C_{ij} = w_i(e_{ij} + e_i w_j) = -w_i w_j \gamma_i \gamma_j \phi^{-1} - w_i \gamma_i \delta_{ij}.$$

Thus we may write the addilog system in the form

(14) $\quad \hat{w}\, d\log q = b\, d\log \bar{\mu} + \phi(\hat{b} - bb')\, d\log p, \qquad b = -\hat{w}\gamma\phi^{-1},$

$\quad \phi = -w'\gamma.$

Note that it is the parameters γ which are now constant. Thus the marginal budget shares are now functions of the average budget shares and will behave differently from those of the Rotterdam or linear expenditure systems. Similarly, for this system, estimation will be with respect to the γ parameters, but once again the value of the likelihood function will enable us to compare the performance of this with the other models.

If we compare the Rotterdam additive model (equations (1) and (4.5)), the linear expenditure system (7), and the direct addilog model (14), we see that in each case the additivity relation, $C = \phi(\hat{b} - bb')$, is enforced. The difference between the systems lies only in the specifications of ϕ and b; thus, though for each system the constraint holds at any given moment of time, the way in which the price responses relate to the income responses over the time span of the data varies from model to model. We shall thus be able to see whether the validity of the additivity assumption is dependent upon the selection of parameters or functional form.

But no matter how additivity is specified, it remains a very strong assumption, and it is worth considering its effects in relation to empirical work with time series data. It is a characteristic of most information of this type, and it is certainly true of the data used in this study, that there is considerable collinearity between quantities purchased, the prices, and total money expenditure. In consequence, most, though not all, of the information contained in the series relates to relative rates of growth of the various categories. Furthermore, the information contained in the price series which is independent of income is of the second order of importance even over long time periods. Thus, the marginal budget shares, which determine the relationship between demand and income, are always very well and precisely determined by the relatively abundant income information in the data. Parameters determining price effects are less well determined and play a subsidiary role. So, when additivity is introduced, *all* price information is absorbed into the single quantity ϕ, and the whole structure of the *price* substitution matrix is determined solely by the way in which expenditures relate to *income*. This is just as true of the own price substitution effects as it is of the cross terms, and it is perhaps this that would seem to be less acceptable than any of the other consequences of additivity.

In the light of these arguments we now suggest an intermediate system which, while preserving many of the assumptions of additivity introduces considerably more flexibility. Working again within the Rotterdam framework, we see from (4.5) that one of the consequences of additivity is that the ratio c_{ik}/c_{jk} is independent of k if $k \neq i,j$. This property extends to the compensated price elasticities, i.e., the ratio of the cross compensated price elasticities of any two goods is independent of the price being varied. Now this in itself, while a consequence of additivity, is a weaker condition, and the intermediate system will have this property alone in common with additivity. Thus while the additive structure of the substitution matrix is preserved, it will no longer be linked to the income terms.

Consider the model defined by

$$(15) \qquad \hat{w}\, d\log q = b\, d\log \bar{\mu} + \chi(\hat{c} - cc')\, d\log p, \qquad b'\iota = c'\iota = 1, \qquad \chi < 0,$$

where c is a vector of n parameters. It is immediately clear that system (15) not only satisfies the constraints (4.1) to (4.5) but that the independence discussed above is achieved. For $c_{ik}/c_{jk} = c_i/c_j$, $k \neq i, j$, which is independent of k. Thus (15) is sufficient for this weak independence. We now show that it is necessary. First it is clear that the matrix C, apart from the diagonal, must have all its rows proportional to one another. Thus, we may write

$$(16) \qquad C = \hat{d} + ef'.$$

However, C must be symmetric, implying $ef' = fe'$, $\therefore e = \kappa f$, for some constant κ. Define $c = f/f'\iota$; then if $\chi = -\kappa(f'\iota)^2$, we have

$$(17) \qquad C = \hat{d} - \chi cc'.$$

Finally, we must impose homogeneity and aggregation: $C\iota = d - \chi c = 0$, giving $d = \chi c$, and

$$(18) \qquad C = \chi(\hat{c} - cc'), \qquad c'\iota = 1.$$

Clearly C is negative semi-definite if χ is negative, thus establishing the necessity of (15). Note finally that additivity is a special case of (15) with $b = c$.

We conclude this section by listing the models which will be tested in the remainder of this paper. Each system may be written in the form $\hat{w}\, d \log q = b\, d \log \bar{\mu} + C\, d \log p$; the particular forms of b and C are given for each case below:

MODEL 1. *Rotterdam System Unconstrained*: b, C parametrized; $b'\iota = 1$, $\iota'C = 0'$.

MODEL 2. *Rotterdam System with Homogeneity*: b, C parametrized; $b'\iota = 1$, $\iota'C = 0'$, $C\iota = 0$.

MODEL 3. *Rotterdam System with Symmetry*: b, C parametrized; $b'\iota = 1$, $\iota'C = 0'$, $C = C'$.

MODEL 4. *Rotterdam System: Intermediate*: b, c, χ parametrized; $b'\iota = 1$, $C = \chi(\hat{c} - cc')$.

MODEL 5. *Rotterdam System with Additivity*: b, ϕ parametrized; $b'\iota = 1$, $C = \phi(\hat{b} - bb')$.

MODEL 6. *Linear Expenditure System*: b, c parametrized; $b'\iota = 1$, $C = -(1 - p'c\mu^{-1})(\hat{b} - bb')$.

MODEL 7. *Direct Addilog System*: γ parametrized; $\phi = -w'\gamma$, $b = -\hat{w}\gamma\phi^{-1}$, $C = \phi(\hat{b} - bb')$.

MODEL 8. *No Substitution Model*: b parametrized; $b'\iota = 1$, $C = 0$.

3. ESTIMATION METHODS AND DATA

Estimation

We may write the models of the previous section

$$(19) \qquad y_t = f(x_t, \beta) + \varepsilon_t$$

where y_t is a vector of n independent variables, $\hat{w}\, d \log q$, x_t is the vector of independent variables, i.e. $x_t' = (d \log \bar{\mu}_t, d \log p_t')$, β is a vector or matrix of parameters, and ε_t is a vector of residuals. The stochastic specification for all models will be taken to be

$$(20) \qquad \mathscr{E}(\varepsilon_{ti}) = 0 \quad \text{for all } t, i,$$
$$\mathscr{E}(\varepsilon_{ti}, \varepsilon_{t'j}) = \delta_{tt'}\omega_{ij} \quad \text{for all } t, t', i, j.$$

In other words, only contemporaneous covariances are permitted between the residuals of the various commodities. The matrix whose i,jth term is ω_{ij} will be denoted Ω.

Now each of the systems is subject to an exact non-stochastic constraint, either satisfied automatically or imposed via restrictions on the parameters. In every case this ensures that

$$(21) \qquad (y_t - f)'\imath = \varepsilon_t'\imath = 0 \quad \text{for all } t,$$

i.e., the predicted values for each commodity add exactly to the actuals. Thus, from (20) and (21), we have

$$\sum_i \omega_{ij} = \sum_i \mathscr{E}(\varepsilon_{it}, \varepsilon_{jt}) = \mathscr{E}\left(\sum_i \varepsilon_{it}, \varepsilon_{jt}\right) = \mathscr{E}(0, \varepsilon_{jt}) = 0.$$

This singularity of Ω implies that, on the assumption of a multivariate normal distribution for ε, unless we drop one of the equations of the system, the likelihood function of the sample is not defined. Barten [**4**] has shown that we may avoid this asymmetry by writing the likelihood function

$$L_1 = n^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}(n-1)}|\Omega + ii'|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\varepsilon_t'(\Omega + ii')^{-1}\varepsilon_t\},$$

where i is a normalized vector of units, i.e., $i = \imath/\sqrt{n}$. Thus, given T observations, the logarithmic likelihood function is

$$(22) \qquad \log L = \tfrac{1}{2}T(\log n - (n-1)\log 2\pi) - \tfrac{1}{2}T\log|\Omega + ii'|$$
$$- \tfrac{1}{2}\sum_t \varepsilon_t'(\Omega + ii')^{-1}\varepsilon_t.$$

Since Ω is unknown, we derive the concentrated likelihood function by maximizing (22) with respect to the elements of Ω subject to the constraint $\Omega\imath = 0$. This may be done in the usual way giving an estimator of Ω, $\tilde{\Omega}$:

$$(23) \qquad \tilde{\Omega} = \frac{1}{T}\sum_t \varepsilon_t\varepsilon_t' = \frac{1}{T}E'E,$$

where E is the matrix whose T rows are made up of the vectors ε_t. If we substitute (23) in (22), we find that the last term reduces to $-\frac{1}{2}T(n-1)$, and we have a concentrated likelihood function

$$(24) \qquad \log L^* = \tfrac{1}{2}T\log n - \tfrac{1}{2}T(n-1)(1 + \log 2\pi) - \tfrac{1}{2}T\log\left|\frac{1}{T}\sum_t \varepsilon_t\varepsilon_t' + ii'\right|.$$

Thus the estimation of the model (19) becomes equivalent to that of minimizing the determinant

$$(25)\qquad \left|\frac{1}{T}\sum_t \varepsilon_t\varepsilon_t' + \iota\iota'\right| \quad \text{subject to } \varepsilon_t'\iota = 0 \quad \text{for all } t.$$

How this is best done varies from model to model. There are four groups which must be treated separately:

(i) linear models with restriction (models 1 and 8);

(ii) linear models without restrictions which are within equations and are the same for each equation (model 2);

(iii) linear models with restrictions running across equations (model 3);

(iv) non-linear models with or without linear constraints (models 4, 5, 6 and 7).

Only the first two give rise to maximum likelihood estimates which may be calculated without iteration. In this paper we shall confine ourselves to stating the estimator for cases (i) to (iv) and indicating briefly how they are derived in each case from rule (25).

Case (i): The model may be written

$$(26)\qquad Y = X\beta + E$$

where Y and X are the matrices formed from the vectors y_t and x_t respectively. Given the adding up constraint, $Y\iota = Xa_1$, where a_1 is a vector of which the first element is unity and the others are zero, we must have $\beta\iota = a_1$. Writing

$$(27)\qquad V = \frac{1}{T}E'E + \iota\iota',$$

the minimization problem (25) becomes that of minimizing log det V subject to $\beta\iota = a_1$. The resulting estimator is the OLS estimator

$$(28)\qquad \hat{\beta} = (X'X)^{-1}X'Y.$$

Case (ii): Within equation identical constraints are easily fitted into this framework. We now have an additional constraint (or constraints) of the form $\beta'a_2 = 0$; in the case of homogeneity, a_2 has a zero for its first element and ones elsewhere. The estimator in this case is derived by adding one more constraint to the Lagrangian of case (i) and we finally reach an estimator

$$(29)\qquad \tilde{\beta} = \left\{I - \frac{(X'X)^{-1}a_2a_2'}{a_2'(X'X)^{-1}a_2}\right\}\hat{\beta}.$$

Like (28) this estimator may be evaluated directly and is in fact no more than the constrained OLS estimator.

Case (iii): Case (i) must now be modified by the addition of q independent constraints, each involving elements from more than one of the columns of β.

In the case of symmetry there are $\frac{1}{2}n(n-1)$ of these restrictions. The easiest way to write these restrictions is in the form

$$\sum_{i=1}^{n} R_i\beta_i = 0,$$

where the β_i are the n columns of the matrix β, and there are n matrices R_i each of dimension $q \times n$. Once again an estimator may be derived from consideration of the appropriate Lagrangian; in this case we may derive

$$\tilde{\beta}_i = \hat{\beta}_i - \sum_j \tilde{\omega}_{ij}(X'X)^{-1}R_j'\left\{\sum_j\sum_k \tilde{\omega}_{jk}R_j(X'X)^{-1}R_k'\right\}^{-1} \sum_k R_k\hat{\beta}_k \tag{30}$$

where $\tilde{\omega}_{ij}$ is the typical element of $\tilde{\Omega}$ as defined by (23). Now in the case where Ω is known, (30), like (29) and (28), defines an MLE which can be directly evaluated; furthermore, it may be shown (see Deaton [**8**]) that these three estimators are also best linear unbiased. However, when Ω is defined only by (23), $\tilde{\beta}$ is involved on both the right and left hand sides of (30), and we must thus use iterative methods to derive the estimator. One possibility is to linearize an expression for $\tilde{\Omega}$; alternatively, starting from some estimate of Ω, a sequential process of re-evaluation of $\tilde{\beta}$ according to (30) and (23) in turn may be followed. This latter, though unlikely to be powerfully convergent, is convenient and will be adopted here.[3]

Case (iv): In the non-linear models, the adding-up constraint is linear within a non-linear framework except in the case of the addilog model where no constraint is necessary; any set of parameters will yield a properly singular estimate of the variance-covariance matrix. Dealing with this case first, we simply seek to minimize log det V with respect to the vector β of the parameters. If we define the matrix B_t as follows:

$$\{B_t\}_{ij} = \frac{\partial f_i(x_t, \beta)}{\partial \beta_j},$$

the minimization conditions are simply

$$\sum_t \{B_t'V^{-1}(y_t - f_t)\} = 0.$$

Here we shall use the linearization $f_t = f(x_t, \beta^0) + B_t\delta\beta$, giving a Gauss-Newton iterative procedure defined by

$$\delta\hat{\beta} = \left\{\sum_t B_t'V^{-1}B_t\right\}^{-1}\left\{\sum_t B_t'V^{-1}(y - f^0)_t\right\}. \tag{31}$$

[3] For arguments for replacing $\tilde{\Omega}$ by $\hat{\Omega}$, the OLS estimator, and thus directly evaluating (30), see Byron [7] and Barten [4] and for further counterarguments see Deaton [**8**]. In the light of the results presented in the next section, I would now argue that additional effort required to iterate on (30) is justified because it ensures that the likelihood function is indeed maximized and guarantees against rejections of the hypothesis arising from underevaluation as compared with other models.

This linearization has the advantage of yielding estimators with obvious similarities to Aitken estimators; note however that not only f^0 but also V must be recalculated at each step.

Use was made of the Marquandt [**14**] minimization algorithm; this algorithm, though designed for least squares minimization, was modified fairly easily to deal with determinental minimization. In cases other than the addilog, a further modification to (31) was necessitated by the linear constraints of the form $R\beta = r$. This was done by selecting starting values for β satisfying the constraint and applying $R\delta\beta = 0$ at each step. It is easy to show that the appropriate estimators are given by

$$\delta\tilde{\beta} = \delta\hat{\beta} - \left\{\sum_t B_t' V^{-1} B_t\right\}^{-1} R' \left[R \left\{\sum_t B_t' V^{-1} B_t\right\}^{-1} R'\right]^{-1} R\,\delta\hat{\beta}. \tag{32}$$

For the four non-linear systems, the definitions of β and B are given below; see the end of the previous section for the definitions of f.

MODEL 4. *Intermediate System*:

$$\beta' = (b', c', \chi),$$

$$B = (I\, d\log\mu, \chi(d\,\widehat{\log p} - c\, d\log p'), (\hat{c} - cc')\, d\log p).$$

MODEL 5. *Additive Rotterdam System*:

$$\beta' = (b', \phi),$$

$$B = \{I(d\log\mu - \phi b'\, d\log p) + \phi(d\log p - b\, d\log p'), (\hat{b} - bb')\, d\log p\}.$$

MODEL 6. *Linear Expenditure System*:

$$\beta' = (b', c'),$$

$$B = \{I(d\log\mu - \phi b'\, d\log p) + \phi(d\,\widehat{\log p} - b\, d\log p'), (\hat{b} - bb')\, d\log p\mu^{-1}p'\}$$

where $\phi = -1 + \mu^{-1}p'c$.

MODEL 7. *Addilog System*:

$$\beta = \gamma,$$

$$B = \sigma\{\hat{w}/w'\gamma - \hat{w}\gamma w'/(w'\gamma)^2\} - \hat{w}\, d\,\widehat{\log p} + \hat{w}\gamma\, d\log p'\hat{w}/w'\gamma,$$

and

$$\sigma = d\log\mu + d\log p'\hat{w}\gamma.$$

The Data

The observations relate to the United Kingdom personal sector for the years 1900–1970; the earliest years are based on Prest [16], the inter-war years on Stone and Rowe [17], and post-war years have been added from time to time by members of the Department of Applied Economics, Cambridge. The series, as it stands now, is consistent with the latest national income estimates in the United Kingdom Central Statistical Office [20], though inevitably not all the categories have precisely the same definitions. The years affected by wars and rationing have been excluded, namely 1914 to 1921 and 1939 to 1953 inclusive. After this deletion and the loss of three observations to first differencing, we are left with forty-five observations in all.

The basic information relates to some forty commodities, thirty-six after the exclusion of durable goods. This is considerably too many for the type of experimental work being done here; iterative estimation is expensive enough without dealing with over one hundred parameters. In consequence the commodities were aggregated into nine groups; these with their components were as follows:

1. *Food*: bread and cereal; meat and bacon; fish; oils and fats; sugar and confectionery; dairy products; fruit; potatoes and vegetables; beverages; and other manufactured food.
2. *Footwear and Clothing*: consisting of these two categories alone.
3. *Housing and Household*: rents, rates, and water charges; household maintenance and improvements; household textiles and hardware; matches, soap, and cleaning materials; domestic service.
4. *Fuel and Light*: coal and coke; electricity; gas; other fuels.
5. *Drink and Tobacco*: beer; wines and spirits, cigarettes, other tobacco.
6. *Travel and Communication*: postal charges; telephone and telegraph; running costs of vehicles; railway travel; other travel; consumers' expenditure abroad.
7. *Entertainment*: books and magazines, newspapers, other entertainment.
8. *Other Goods.*
9. *Other Services.*

This aggregation means that, even in the worst case and symmetry apart, it is not necessary to estimate by non-linear means more than twenty-eight parameters. And even though some of the interesting detail is lost by this aggregation, some of the hypotheses, e.g., additivity, are often claimed to be more appropriate for broad classifications of this kind.

Each series was deflated by mid-year population. The infinitesimals of the foregoing analysis were replaced by forward differences and, following previous work, the value shares were approximated by the average of the observed value shares in successive periods.

4. ANALYSIS AND RESULTS

The parameter estimates are presented in Tables I to V. Each model was estimated twice, once with and once without intercepts. These constants, though

TABLE I

ROTTERDAM MODEL UNRESTRICTED WITH INTERCEPTS

	B	C.1	C.2	C.3	C.4	C.5	C.6	C.7	C.8	C.9
1. Food	0.135072	−0.103773	−0.009268	0.016353	0.001008	−0.001308	−0.034550	0.014085	−0.006625	0.078700
	(0.04418)	(0.03548)	(0.03129)	(0.03448)	(0.01168)	(0.02792)	(0.04447)	(0.03033)	(0.01472)	(0.04868)
2. Footwear and	0.177639	0.018906	−0.030411	−0.016823	−0.004209	0.023468	−0.023264	−0.004787	0.012913	0.025128
clothing	(0.03335)	(0.02684)	(0.02366)	(0.02608)	(0.00883)	(0.02112)	(0.03363)	(0.02294)	(0.01113)	(0.03681)
3. Housing	0.084590	0.025374	−0.029457	−0.016667	−0.000010	−0.012011	0.037734	0.010112	−0.000796	−0.028316
	(0.01788)	(0.01439)	(0.01268)	(0.01398)	(0.00473)	(0.01132)	(0.01803)	(0.01230)	(0.00597)	(0.01973)
4. Fuel	0.089326	0.006438	−0.003114	−0.004935	−0.022042	0.001125	0.050402	−0.009050	−0.014028	0.003279
	(0.03003)	(0.02416)	(0.02131)	(0.02348)	(0.00795)	(0.01901)	(0.03028)	(0.02066)	(0.01002)	(0.03315)
5. Drink and	0.229275	0.009825	0.040699	−0.007015	0.009880	−0.042878	−0.005588	−0.003446	0.008205	0.012928
tobacco	(0.02549)	(0.02051)	(0.01808)	(0.01993)	(0.00675)	(0.01614)	(0.02570)	(0.01753)	(0.00851)	(0.02813)
6. Travel and	0.103688	0.041007	−0.000546	0.025342	−0.003091	0.019203	−0.058615	0.016429	0.007308	−0.013219
communication	(0.01489)	(0.01198)	(0.01056)	(0.01164)	(0.00394)	(0.00943)	(0.01501)	(0.01024)	(0.00497)	(0.01643)
7. Entertainment	0.023798	−0.011800	0.016214	0.000476	0.003617	0.007844	0.003895	−0.018019	0.001462	−0.002538
	(0.00920)	(0.00740)	(0.00653)	(0.00719)	(0.00244)	(0.00582)	(0.00928)	(0.00633)	(0.00307)	(0.01016)
8. Other goods	0.069496	0.003932	−0.005816	0.014819	0.001975	−0.005145	0.012303	−0.023728	−0.005415	0.009458
	(0.01556)	(0.01252)	(0.01104)	(0.01216)	(0.00412)	(0.00985)	(0.01569)	(0.01070)	(0.00519)	(0.01717)
9. Other services	0.087225	0.010089	0.003164	0.021156	0.012872	0.009702	0.017683	0.018405	−0.003024	−0.085420
	(0.01913)	(0.01539)	(0.01357)	(0.01495)	(0.00506)	(0.01211)	(0.01929)	(0.01316)	(0.00638)	(0.02111)
Intercepts		0.000477	−0.000955	0.001525	−0.000445	−0.002424	0.000648	0.000132	0.000096	0.000946
		(0.00081)	(0.00061)	(0.00033)	(0.00055)	(0.00047)	(0.00027)	(0.00017)	(0.00029)	(0.00035)

TABLE II

ROTTERDAM MODEL HOMOGENEOUS WITH INTERCEPTS

	B	*C*.1	*C*.2	*C*.3	*C*.4	*C*.5	*C*.6	*C*.7	*C*.8	C.9
1. Food	0.120611	−0.134691	0.048744	−0.000833	0.008813	0.028916	−0.084481	0.033611	−0.017958	0.117878
	(0.04773)	(0.03670)	(0.03032)	(0.03708)	(0.01235)	(0.02799)	(0.04428)	(0.03215)	(0.01541)	(0.05075)
2. Footwear and	0.177833	0.019384	−0.031022	−0.017063	−0.004330	0.023001	−0.022492	−0.005089	0.013089	0.024522
clothing	(0.03264)	(0.02510)	(0.02073)	(0.02536)	(0.00844)	(0.01914)	(0.03028)	(0.02199)	(0.01053)	(0.03470)
3. Housing	0.081181	0.018086	−0.020151	−0.013008	0.001829	−0.004887	0.025964	0.014715	−0.003467	−0.019080
	(0.01814)	(0.01395)	(0.01153)	(0.01409)	(0.00469)	(0.01064)	(0.01683)	(0.01222)	(0.00586)	(0.01929)
4. Fuel	0.091287	0.010631	−0.008467	−0.007040	−0.023101	−0.002974	0.057173	−0.011698	−0.012491	−0.002034
	(0.02952)	(0.02269)	(0.01875)	(0.02293)	(0.00764)	(0.01731)	(0.02738)	(0.01988)	(0.00953)	(0.03139)
5. Drink and	0.234766	0.021565	0.025710	−0.012908	0.006917	−0.054354	0.013370	−0.010860	0.012508	−0.001948
tobacco	(0.02611)	(0.02008)	(0.01659)	(0.02029)	(0.00676)	(0.01531)	(0.02422)	(0.01759)	(0.00843)	(0.02777)
6. Travel and	0.111821	0.058567	−0.022966	0.016527	−0.007523	0.002038	−0.030257	0.005339	0.013745	−0.035469
communication	(0.01861)	(0.01431)	(0.01182)	(0.01446)	(0.00482)	(0.01092)	(0.01726)	(0.01254)	(0.00601)	(0.01979)
7. Entertainment	0.024077	−0.011203	0.015451	0.000176	0.003467	0.007261	0.004859	−0.018397	0.001681	−0.003295
	(0.00901)	(0.00693)	(0.00572)	(0.00700)	(0.00233)	(0.00528)	(0.00836)	(0.00607)	(0.00291)	(0.00958)
8. Other goods	0.070075	0.005170	−0.007396	0.014198	0.001662	−0.006355	0.014301	−0.024510	−0.004962	0.007890
	(0.01525)	(0.01172)	(0.00969)	(0.01185)	(0.00394)	(0.00894)	(0.01414)	(0.01027)	(0.00492)	(0.01621)
9. Other services	0.088349	0.012492	0.000097	0.019950	0.012266	0.007353	0.021563	0.016888	−0.002144	−0.088464
	(0.01878)	(0.01444)	(0.01193)	(0.01459)	(0.00486)	(0.01101)	(0.01742)	(0.01265)	(0.00606)	(0.01997)
Intercepts		−0.000514	−0.000940	0.001292	−0.000311	−0.002048	0.001212	0.000151	0.000136	0.001023
		(0.00079)	(0.00054)	(0.00030)	(0.00049)	(0.00043)	(0.00031)	(0.00015)	(0.00025)	(0.00031)

TABLE III

ROTTERDAM MODEL SYMMETRIC WITH INTERCEPTS

	B	*C*.1	*C*.2	*C*.3	*C*.4	*C*.5	*C*.6	*C*.7	*C*.8	*C*.9
1. Food	0.151593	−0.072760	0.021231	0.010511	0.002436	0.010733	0.042088	−0.013969	−0.001375	0.001104
	(0.04085)	(0.02559)	(0.01548)	(0.00993)	(0.00921)	(0.01128)	(0.00960)	(0.00504)	(0.00770)	(0.01117)
2. Footwear and clothing	0.191381	0.021231	−0.016513	−0.027013	−0.003534	0.031628	−0.025480	0.014013	0.010209	−0.004541
	(0.02588)	(0.01548)	(0.01529)	(0.00829)	(0.00630)	(0.00870)	(0.00749)	(0.00434)	(0.00608)	(0.00826)
3. Housing	0.074695	0.010511	−0.027013	−0.016036	0.000133	−0.004457	0.019958	0.002228	−0.001190	0.015867
	(0.01659)	(0.00993)	(0.00829)	(0.01019)	(0.00403)	(0.00741)	(0.00765)	(0.00431)	(0.00449)	(0.00834)
4. Fuel	0.099717	0.002436	−0.003534	0.000133	−0.022519	0.010799	−0.005137	0.003282	0.001571	0.012969
	(0.02360)	(0.00921)	(0.00630)	(0.00403)	(0.00671)	(0.00499)	(0.00390)	(0.00192)	(0.00347)	(0.00422)
5. Drink and tobacco	0.214030	0.010733	0.031628	−0.004457	0.010799	−0.062191	−0.000752	0.005481	−0.001934	0.010692
	(0.02172)	(0.01128)	(0.00870)	(0.00741)	(0.00499)	(0.00992)	(0.00693)	(0.00377)	(0.00481)	(0.00750)
6. Travel and communication	0.092968	0.042088	−0.025480	0.019958	−0.005137	−0.000752	−0.046522	0.000775	0.005448	0.009621
	(0.01608)	(0.00960)	(0.00749)	(0.00765)	(0.00390)	(0.00693)	(0.01011)	(0.00457)	(0.00416)	(0.00826)
7. Entertainment	0.019279	−0.013969	0.014013	0.002228	0.003282	0.005481	0.000775	−0.019404	0.001275	0.006318
	(0.00763)	(0.00504)	(0.00434)	(0.00431)	(0.00192)	(0.00377)	(0.00457)	(0.00419)	(0.00221)	(0.00567)
8. Other goods	0.076850	−0.001375	0.010209	−0.001190	0.001571	−0.001934	0.005448	0.001275	−0.008633	−0.005370
	(0.01366)	(0.00770)	(0.00608)	(0.00449)	(0.00347)	(0.00481)	(0.00416)	(0.00221)	(0.00470)	(0.00462)
9. Other services	0.079567	0.001104	−0.004541	0.015867	0.012969	0.010692	0.009621	0.006318	−0.005370	−0.046659
	(0.01685)	(0.01117)	(0.00826)	(0.00834)	(0.00422)	(0.00750)	(0.00826)	(0.00567)	(0.00462)	(0.01274)
Intercepts		0.000371	−0.000806	0.001113	−0.000721	−0.002073	0.000977	0.000102	0.000244	0.000793
		(0.00068)	(0.00041)	(0.00025)	(0.00040)	(0.00034)	(0.00027)	(0.00012)	(0.00021)	(0.00026)

TABLE IV

INTERMEDIATE AND LINEAR EXPENDITURE SYSTEMS

	Intermediate					Linear Expenditure System				
	Intercepts			No intercepts		Intercepts			No intercepts	
	a	*b*	*c*	*b*	*c*	*a*	*b*	*c*	*b*	*c*
1. Food	.000530	.138484	.053729	.158535	.110380	−.000090	.174824	337.480	.168125	495.844
	(.000674)	(.038941)	(.051361)	(.027065)	(.047845)	(.049694)	(.033247)	(382.614)	(.023543)	(408.493)
2. Clothing and footwear	−.000781	.163657	.059816	.131905	.058833	−.000544	.128547	−413.996	.112502	−410.743
	(.000420)	(.024525)	(.033506)	(.018723)	(.031220)	(.049691)	(.020151)	(236.939)	(.015072)	(265.888)
3. Housing	.001063	.094835	.072285	.141458	.067913	.001009	.118373	11.458	.152647	33.554
	(.000251)	(.015105)	(.026041)	(.013477)	(.023830)	(.049690)	(.015734)	(592.954)	(.013723)	(643.173)
4. Fuel	−.000608	.095099	.088868	.073927	.060140	−.000301	.084510	−217.513	.072944	263.735
	(.000382)	(.022572)	(.018770)	(.016468)	(.015956)	(.049691)	(.014541)	(284.662)	(.012762)	(308.247)
5. Drink and tobacco	−.002233	.204766	.195931	.110405	.236785	−.002255	.215157	571.554	.133030	260.552
	(.000368)	(.021842)	(.037527)	(.020102)	(.044854)	(.049691)	(.017380)	(223.377)	(.017899)	(231.680)
6. Travel	.000819	.113302	.143209	.146716	.202631	.000844	.122475	192.443	.154542	342.221
	(.000299)	(.017212)	(.038622)	(.012967)	(.040235)	(.049690)	(.015162)	(371.626)	(.011580)	(382.825)
7. Entertainment	.000160	.013998	.064059	.022656	.060828	.000130	.018692	118.735	.024042	−110.098
	(.000124)	(.007362)	(.015218)	(.005384)	(.012705)	(.049690)	(.007007)	(336.650)	(.005528)	(312.015)
8. Other goods	.000161	.074296	.019529	.081806	.013981	.000575	.039508	−23.610	.061054	−182.681
	(.000203)	(.012153)	(.011760)	(.008794)	(.010200)	(.049690)	(.008414)	(328.829)	(.008336)	(314.398)
9. Other services	.000889	.101563	.302575	.132592	.188509	.000632	.097914	−405.899	.121114	−469.183
	(.000241)	(.014729)	(.055717)	(.013106)	(.038328)	(.049690)	(.017451)	(725.893)	(.013030)	(751.634)
		$\chi = -.353888$ (.039378)		$\chi = -.421969$ (.042751)						

TABLE V

ADDITIVE ROTTERDAM, DIRECT ADDILOG, AND NO SUBSTITUTION SYSTEMS

	Additive Rotterdam			Direct Addilog			No Substitution		
	Intercepts		No Intercepts	Intercepts		No Intercepts	Intercepts		No Intercepts
	a	b	b	a	γ	γ	a	b	b
1. Food	−.000097	.181283	.169167	−.000088	.232138	.252501	.000654	.136515	.163248
	(.000697)	(.034027)	(.024044)	(.000602)	(.051976)	(.041684)	(.000708)	(.041497)	(.029681)
2. Clothing and footwear	−.000327	.119813	.108771	−.000489	.494100	.453247	−.000681	.167827	.139968
	(.000504)	(.020651)	(.015384)	(.000413)	(.091773)	(.073752)	(.000430)	(.025218)	(.018375)
3. Housing	.001052	.108853	.149475	.001132	.242911	.322940	.001028	.085039	.127071
	(.000371)	(.015219)	(.013729)	(.000242)	(.042364)	(.047271)	(.000253)	(.014854)	(.012371)
4. Fuel	−.000390	.081949	.073213	−.000353	.669404	.651908	−.000739	.111338	.081133
	(.000430)	(.014512)	(.012216)	(.000331)	(.123568)	(.112885)	(.000447)	(.026202)	(.019138)
5. Drink and tobacco	−.002208	.214316	.136484	−.002161	.661383	.478304	−.002538	.238433	.134643
	(.000423)	(.017027)	(.017943)	(.000323)	(.068368)	(.070092)	(.000482)	(.028221)	(.025647)
6. Travel	.000778	.126153	.155801	.000659	.657646	.830241	.001077	.110671	.154703
	(.000399)	(.015565)	(.011608)	(.000247)	(.092628)	(.088692)	(.000330)	(.019312)	(.015283)
7. Entertainment	.000073	.022542	.025346	.000115	.245636	.388880	.000243	.006915	.016832
	(.000306)	(.007254)	(.005519)	(.00124)	(.092679)	(.082520)	(.000148)	(.008659)	(.006322)
8. Other goods	.000474	.046430	.061908	.000401	.467823	.602024	.000145	.078719	.084652
	(.000341)	(.008658)	(.007589)	(.000190)	(.078387)	(.078653)	(.000207)	(.012107)	(.008624)
9. Other services	.000646	.098660	.119834	.000785	.351319	.528337	.000812	.064543	.097751
	(.000381)	(.016358)	(.012906)	(.000256)	(.073153)	(.078236)	(.000319)	(.018699)	(.014207)
$\phi =$	−.358052		−.349673						
	(.037803)		(.036094)						

not strictly allowable within the theory, are included to try to remove the bias which is liable to result from the possible omission of important variables. It is undoubtedly true that factors other than prices and income affect demand and while, within the Rotterdam framework, one might reasonably expect their combined effects about their means to be adequately represented by the stochastic structure, it would not be reasonable to expect the means themselves to be zero. Thus the intercepts should be interpreted as indicating those commodity groups where variables not discussed here were important over the period. As we shall see, they are significant for all goods; thus, for reasons of space, only the intercept cases are tabulated for models 1 to 3.

Tables I to III, which show the first three forms of the Rotterdam system, list the b vectors and C matrices for each of the systems; the first column is the vector of b values; those subsequent are the columns of the C matrix. Thus, for example, the entry in the row "fuel" under column heading C.5 gives the compensated response of fuel demand to a change in the price of drink and tobacco. These numbers may be converted to elasticities by division by the corresponding value share; i.e., the income elasticity e_i is given by $e_i = b_i/w_i$ and the compensated price elasticity e_{ij}^* by $e_{ij}^* = c_{ij}/w_i$. To give an idea of the magnitudes involved, the average shares in 1963 were as follows: food, 28.4 per cent; clothing and footwear, 10.0 per cent; housing, 15.3 per cent; fuel, 5.3 per cent; drink and tobacco, 13.3 per cent; travel and communication, 9.7 per cent; entertainment, 3.3 per cent; other goods, 5.1 per cent; and other services, 9.6 per cent. Thus to take examples from Table I, we calculate the income elasticity for demand for fuel as 1.67, the compensated own price elasticity as -0.41, and the compensated cross price elasticity with respect to say the price of housing services as $-.09$. The results of the other models are presented in Tables IV and V.

The final concentrated likelihood values for each of the models are presented in Table VI. The numbers in each column are twice the logarithm of the concentrated likelihood; the number in brackets is the number of free parameters in each of the respective models. For the nested models, i.e., 1, 2, 3, 4, 5, and 8, the difference between any two of the numbers in the table is asymptotically distributed as χ^2 with q degrees of freedom where q is the number of restrictions imposed, i.e., the difference between the numbers in brackets. For models 6 and 7, which are not nested either within one of themselves or within the most general model 1, no

TABLE VI

	Intercepts	No intercepts
1. Free Rotterdam	4353.75 (88)	4306.50 (80)
2. Homogeneous Rotterdam	4322.70 (80)	4267.35 (72)
3. Symmetric Rotterdam	4279.50 (52)	4222.80 (44)
4. Intermediate	4209.30 (25)	4161.60 (17)
5. Additive Rotterdam	4173.75 (17)	4119.75 (9)
6. Linear Expenditure System	4183.20 (25)	4125.15 (17)
7. Direct Addilog	4187.70 (17)	4127.40 (9)
8. No Substitution	4137.75 (16)	4054.95 (8)

such test is possible. Nevertheless, it could well be argued that the likelihood values are the best criteria of discrimination that we have in the present state of knowledge. Even so we cannot say whether the difference between the values for the linear expenditure system and the direct addilog system is in any sense significant.

However, difficulties are not confined to the non-nested models. The use of a testing distribution which is only asymptotically correct involves considerable danger, especially in small sample work, of rejecting valid hypotheses. Since the true distribution will always have fatter tails than its limiting counterpart, a test based on the latter which leads to acceptance could never be reversed by appeal to the true distribution but this does not hold for rejections. Indeed, for small numbers of observations, the horizontal distance between the two distributions may be quite large at the confidence levels which are of interest. In the cases where the restrictions on the parameters fall within equations, i.e., the imposition of zero intercepts, homogeneity, and zero price substitution, the correcting factors to the likelihood ratio test are known[4] (see, e.g., Anderson [**1**, pp. 207–210]). There is thus no problem in testing these three types of constraint. Also for models where the number of parameters being estimated or restricted is small relative to the number of observations (in our case the tests between intermediate, additive and zero price substitution models), the correction is likely to be small and to have little effect on the test outcome. This only leaves symmetry as an awkward case. One possibility is to make a correction of the same order as is made in the within equation models; this procedure seems to give sensible results in practice. Nevertheless, it must be recognized that this problem does not as yet have a general solution.

Looking at the table of likelihood values we see that in no case is the hypothesis of zero intercepts acceptable. This rejection seems due to the necessity for constant terms for two categories in particular. These are housing and drink and tobacco. The former contains a large element of imputed rent which one would not expect to be closely related to current income and prices and the latter, due to the considerable increases in indirect taxation since the turn of the century, is also somewhat of a special case. These results then should be taken as not so much contrary to the theory as indicative of its incompleteness.

Before passing on to the comparison of the principle models we must first establish that the substitution effects of price changes are indeed of importance. This can be done by comparing the likelihood values of models 1 and 8 from the table. Applying the correction to the likelihood ratio we find that the probability that such a decrease could be random given the truth of the null hypothesis to be less than one thousandth of one per cent. This would seem to establish a firm base for the other experiments; compensated price effects are important in demand analysis and the way in which we allow for them is thus a matter of more than trivial importance.

If no constraints are placed upon the C matrix, the maximum likelihood parameter estimates are those given in Table I. The intercepts conform to the general

[4] I am grateful to A. P. Barten for indicating this reference to me.

pattern, i.e., groups 3 and 5 are significantly different from zero at the one per cent level; otherwise there is little difference between the estimates with and without intercepts. Given the general importance of the constants, the models which include them are probably the more reliable. Thus from Table I, taking the one per cent level as our criterion of significance, we see that all the marginal budget shares are significantly different from zero. However, this is only true of seven out of eighty-one price responses, six of these lying along the diagonal. Note that all the own price values are negative in accordance with the theory; the full sign conditions are difficult to test on a non-symmetric matrix. These overall results are not out of line with expectations; the data have a great deal of income information but relatively little on prices. By calculating the R^2 statistics[5] we see that factors other than prices and incomes are important.

One might expect, from these somewhat attenuated price effects, that the imposition of homogeneity would follow without difficulty and this expectation would receive superficial support from the apparent similarity between Table I and Table II. However, inspection of the likelihood values indicates that this is not the case; with intercepts the probability of these likelihoods arising under the null hypothesis is 0.87 per cent; without intercepts, it is 0.08 per cent. Once again the null hypothesis is firmly rejected. This rejection may be traced further by inspecting F ratios for each of the commodity groups. This shows that homogeneity is rejected at the 0.1 per cent level by category 6, travel and communication, and at the 5 per cent level by category 1, food. In other words, a proportional increase in all prices and income will cause an increase in expenditure on travel and communication and a decrease in expenditure on food. There is no problem with other commodities.

The fact that this rejection was also suffered by Byron [**6** and **7**] as well as by Barten in his later experiments [**4**, though not **3**], does not make it any the more palatable. Though it is of course possible to think of reasons why it might occur, none of these are really convincing. Homogeneity is a very weak condition. It is essentially a function of the budget constraint rather than the utility theory and it is difficult to imagine *any* demand theory which would not involve this assumption. Indeed to the extent that the idea of rationality has any place in demand analysis, it would seem to be contradicted by non-homogeneity. It is of course possible that changes in the distribution of income have systematically favored transport users rather than food consumers, but this too seems implausible.

Now we may accept this rejection, implying our acceptance of the framework within which the experiment was carried out, or we may refuse to do so, claiming that the experiment was wrongly performed and that a correct experiment would have led to the opposite result. The first implies the acceptance of non-homogeneous behavior and would seem to require some hypothesis of "irrational" behavior; this is not an attractive alternative. We are thus left to excuse our failure but without further information it is difficult to do this in a convincing fashion. Obvious

[5] Drink and tobacco and travel and communication are over 0.8; the others cluster around 0.6, except for entertainment which is less than 0.5

possible causes are measurement errors or inappropriate aggregation of commodities or of consumers. In favor of the first of these, it might be argued that the high level of significance of the positive substitution elasticity of travel with respect to the price of food is itself anomalous and that it is this term which causes much of the difficulty over the restriction. If this coefficient is due to errors in the data (which are very likely in a series of this length), it is quite appropriate to *enforce* homogeneity; indeed, we should expect better estimates of the other parameters by doing so. Whether or not we are justified, this is what we shall do here. The formal rejection must go on record but it would seem that to continue with further tests having imposed homogeneity is more acceptable than turning away altogether.

The symmetric estimates are given in Table III. The likelihood tests give acceptance as compared with homogeneity but rejection as compared with the unrestricted model. Clearly this latter is due to the unacceptability of homogeneity; the additional restrictions of symmetry do not make the situation any worse. This is perhaps surprising; symmetry imposes twenty-eight constraints over and above the eight of homogeneity. Nevertheless, this fact would be quite consistent with our interpretation of the rejection of homogeneity and could perhaps be interpreted as justifying our enforcement of the constraint. Thus, if we accept the previous result as anomalous, this much more powerful test yields considerable evidence in favor of the acceptability of the utility theory. For symmetry, unlike homogeneity, derives from assumptions about the utility function rather than the budget constraint; evidence in this regard is thus more valuable for the testing of this theory. Indeed in order to test whether or not it is possible to imagine the data as having been generated within the theory, it is now only necessary to examine the further postulate, that of negativity.

To do this the eigenvalues of the estimated C matrices with and without intercepts were calculated; in both cases one of the eight non-zero values proved to be positive. Thus neither of the estimates satisfy the negative semi-definiteness of the theory. In order to test the significance of this shortcoming it would be best to estimate the symmetry model subject to the negativity constraint and compare the resulting likelihood values with those of model 3. Though this presents no difficulty in principle, the programming difficulties have so far prevented a satisfactory outcome to this test. A second-best solution is to calculate an asymptotic standard error[6] for the offending eigenvalue. This is an expensive operation, and since both sets of symmetric estimates are very similar, it was carried out only for the model without intercepts. This gave a standard error of 0.2738 corresponding to the eigenvalue of 0.0246 and would suggest that the violation is not serious.

It may be noted at once that all the diagonal elements of both symmetric substitution matrices are negative; thus, if only one price alters, the consumers' response has the appropriate maximization characteristics. We may then go on

[6] The calculation uses the matrix of derivatives of the eigenvalue λ with respect to the elements of the matrix $V = C - \lambda I$. This derivative matrix is the adjoint of V scaled by its trace and is used to map the four-dimensional covariance tensor of C into the scalar variance of λ.

to investigate responses to all possible changes in two prices, three prices, and so on until the maximization conditions cease to hold. That they will do so we are assured in advance by the presence of the positive eigenvalue. It has been suggested by Professor John Wise of Southampton University that the maximization conditions are more likely to break down in the higher rather than the lower orders. The basic hypothesis is that consumers can respond correctly to simple price changes but tend to become confused, if they become confused at all, only when large numbers of prices change together. This hypothesis is completely consistent with the present evidence. When the diagonal determinants of each of the C matrices were calculated, it was found that up to the seventh order all had the appropriate sign; only at the eighth order are the signs incorrect[7]. Thus every price in the model must change simultaneously before non-maximizing behavior can be observed. With respect to all other stimuli, the behavior patterns are appropriate to the utility maximizing consumer.

Whether this result is in fact the outcome of Wise-type behavior or whether it is due to the insignificance of the deviation from negativity cannot be decided at present. However, taking all the evidence together it would seem that the utility maximizing consumer is a paradigm which can take us a considerable way in the interpretation of the United Kingdom experience. Given this, attention shifts to the problem of whether or not it is possible to restrict behavior even further. Even with symmetry there are $(\frac{1}{2}n + 1)(n - 1)$ independent parameters, and for large n this is too many to be estimated in many empirical situations. We thus turn to the other models, each of which restricts behavior considerably more than does the theory alone. The parameter estimates for these models are given in Tables IV and V, and we discuss the most important aspects of these below.

Looking at additivity first, the likelihood values make it clear that this is not an acceptable restriction. The additive Rotterdam model which is a subcase of the symmetric model is rejected at a very high confidence level, and the likelihoods for the direct addilog and linear expenditure systems suggest that they, too, suffer from the same inadequacies. Now while it may be true that a different aggregation of commodities might reverse this result, it seems unlikely given the strength of the rejection. Rather we must accept that there exist specific substitution effects even between fairly broad categories of goods. Note, too, that the choice of functional form seems less important than additivity itself; the likelihood values for all three systems are quite close together. However, if we compare these models with model 8, we see that to allow even the limited substitution effects of additivity is better than ignoring such terms altogether. This result, though a negative argument in favor of additivity, is still of practical importance. For it provides some justification of the use of this assumption where no other model can be used. It would indicate that, at least for the United Kingdom, the results of the Frisch method of calculating price elasticities or of the linear expenditure system would be of more practical use than those derived from a system which allowed only for the income effects of price changes.

As to the choice between the additive models tested here, the direct addilog system does best. The linear expenditure system yields very little for the extra

[7] The ninth order determinant is of course zero.

parameters it absorbs; indeed, it has a likelihood function less than that of the direct addilog system which has fewer independent parameters. Each of these systems does better than the additive Rotterdam system. Evidently there is some return to allowing ϕ to vary over time though not enough to compensate for the nine degrees of freedom permitted to it in the linear expenditure system. The reason for the superiority of the addilog system presumably rests in the formulation of the income coefficients; this model allows each of the income elasticities to drift slowly downwards over time. This situation is clearly preferable to the constant tending towards unity of the other models. This result is not followed up here; yet there is clearly useful work to be done on the correct formulation of income elasticities and their dependence on the level of income.

We can also derive from these models evidence on ϕ and thus on Frisch's $\breve{\omega}$, the flexibility of the marginal utility of money. The Rotterdam additive model holds ϕ fixed, and our estimate of -0.3581 is quite close to the value -0.4888 estimated by Barten for Holland. The United Kingdom estimate gives a value for $\breve{\omega}$ of -2.79, which lies between the values given by Frisch for the median and poorer parts of the population and is consistent with the results of other studies; see, e.g., Section IV.3 of Brown and Deaton [**5**]. If we move to the addilog system, ϕ now increases very slowly in absolute value, from -0.3815 in 1900 to $-.4117$ in 1970, most of the increase taking place after the Second World War. This change is in the direction envisaged by Frisch, yet is much smaller than one might expect. For the linear expenditure system ϕ varies quite dramatically; it is in order to allow this that the values of the c parameters in Table IV deviate so violently from those which result from the normal estimation of the system.[8] Here ϕ follows the trade cycle quite closely from 1900 to 1938, decreasing from just below zero to -0.6; in the post-war period there is relative constancy around -0.4, the trade cycle relationship seemingly having disappeared. This variation is however unlikely to be significant and though we have some evidence for $\breve{\omega}$ increasing with income, this would not be strong enough to contradict earlier negative results, e.g., Theil and Brooks [**19**]. It is notable, however, that no values of ϕ numerically greater than unity or even close to it have appeared in any of these models, though Frisch indicates values of $\breve{\omega}$ of -0.7 and -0.1 for the better off and rich. However, the assumption that utility is bounded above, that bliss is finite, implies that $\breve{\omega}$ should be bounded above by a number strictly less than -1.[9] This seems to me more attractive than the alternative possibility and would provide a rationale for a much slower increase in the flexibility than originally imagined.

Having rejected additivity we turn finally to the intermediate system; this though much weaker in implication is still unacceptable. Equally, however, the additive model is an unacceptable restriction of the intermediate model; the latter thus lies truly between symmetry and additivity. Note from Table IV that the

[8] These extraordinary estimates require further comment. Since the system is not solved for predicted values of q, it is not obvious that the convexity requirements $q_i > c_i$ are violated. But if not, the model must fit very badly. Either way the model is being stretched beyond its limits.

[9] This excludes $\breve{\omega} \rightarrow -1$ as $\mu \rightarrow \infty$, as is the case for the linear expenditure system; the direct addilog utility (8) is bounded for $0 < \beta_i < 1$ and the estimated values satisfy this. I am particularly grateful to David Champernowne for help on these points. He assures me from personal experience that bliss *is* finite.

ordering of the elements of the b and c vectors is quite different; other services and entertainment are much more price-sensitive than they are income-sensitive; the opposite is true of food and clothing and footwear. This extra freedom over the enforced equality of the vectors under additivity does give a considerable improvement. Thus though we may not take these restrictions as valid, the system offers us a way of using extra price information where it is available. For example, we often have information on income and own price elasticities only; this model allows us to construct a complete system of demand equations from that alone. This requires more knowledge than the Frisch method, but for the United Kingdom, at least, it will give more accurate results.

This concludes the comparison of the alternative models. Though our consistent use of the Rotterdam format has enabled us to make direct comparisons in a way not so far possible, it must not be thought that this provides the final word on these systems. The selection of an optimal model depends on the use to which it is to be put and on the circumstances surrounding its estimation. Furthermore, our results are valid only for the United Kingdom, and only to the extent that the error structures we have assumed are appropriate. It is, of course, possible that the selection of similar error structures for all the models discriminates unfairly against one or the other of them. These questions can only be settled by further work, not only with different stochastic assumptions but with data from different countries.

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