ECO 317 – Economics of Uncertainty – Fall Term 2009
Notes for lectures
5. Portfolio Allocation with One Riskless, One Risky Asset

Characterization of the Optimum

Consider a risk-averse, expected-utility-maximizing investor with initial wealth W_0. Two assets are available:

[1] A riskless asset that pays a total or gross rate of return R_0. This comprises 1 plus the interest or dividend rate plus the capital gains (or minus capital losses). This is usually taken to be the interest rate on short-term government bonds, but even there the real rate of return may be uncertain because of uncertainty about inflation, and in some countries or through much of history even the nominal rate of return may be uncertain because of default risk. So don't take this example too literally; its main purpose is to develop the ideas and the methods.

[2] A risky asset whose total rate of return R is a random variable, with a cumulative distribution function $F(R)$ and the corresponding density function $f(R)$ defined over (having its support as) the interval (R_L, R_H). Note that R_L may be zero if there is limited liability. I assume that $0 < F(R_0) < 1$; otherwise all realizations of the random variable will either exceed R_0 or fall short of R_0, and in either case the portfolio choice problem will be trivial.

If the investor puts X in the risky asset, and therefore $(W_0 - X)$ in the riskless asset, his random final wealth will be

$$\begin{aligned} W &= R_0\,(W_0 - X) + R\,X \\ &= R_0\,W_0 + (R - R_0)\,X \end{aligned} \tag{1}$$

The second line of this can be interpreted as if the investor gets the safe return R_0 on his whole wealth W_0, and then a random excess return (whose realizations may be positive or negative both with positive probabilities) $(R - R_0)$ on just the part X invested in the risky asset.

In keeping with the traditional approach of economic theory, we suppose that the investor's utility-of-consequences function depends only on his final wealth, and not on the original situation or status quo. Writing this function as $u(W)$ (where $u'(W) > 0$ and $u''(W) < 0$ (risk aversion)), his expected utility can be written as a function of X:

$$EU(X) = \int_{R_L}^{R_H} u(W)\; f(R)\,dR = \int_{R_L}^{R_H} u(R_0\,W_0 + (R - R_0)\,X)\; f(R)\,dR \tag{2}$$

The investor wants to choose X to maximize $EU(X)$. There may be constraints, for example $X \geq 0$ unless "short sales" of the risky asset are allowed, and $X \leq W_0$ unless the investor can borrow at the safe rate to invest in the risky asset. Let us proceed assuming both of these constraints apply.

To find the derivative $EU'(X)$, we have to differentiate with respect to the parameter X occurring inside the integral. This gives

$$EU'(X) = \int_{R_L}^{R_H} (R - R_0)\ u'(R_0 W_0 + (R - R_0) X)\ f(R)\, dR \tag{3}$$

If you are unused to doing this, think of it just like differentiating a sum term by term. If the random variable R took on only a discrete set of values R_i with probabilities $f(R_i)$ for $i = 1, 2, \ldots n$, we would have

$$EU(X) = \sum_{i=1}^{n} u(R_0 W_0 + (R_i - R_0) X)\ f(R_i)$$

and then differentiaing each term would give, using the chain rule,

$$EU'(X) = \sum_{i=1}^{n} (R_i - R_0)\ u'(R_0 W_0 + (R_i - R_0) X)\ f(R_i)$$

The differentiation of the integral in (3) is basically exactly the same process.

Differentiating with respect to X again,

$$EU''(X) = \int_{R_L}^{R_H} (R - R_0)^2\ u''(R_0 W_0 + (R - R_0) X)\ f(R)\, dR$$

The integrand is always negative because u'' is. Therefore EU is a concave function of X, and is going to have a unique maximum over our interval $X \in [0, W_0]$. If the maximum occurs in the interior, it will be characterized by the usual first-order condition $EU'(X) = 0$. If the maximum occurs at one of the end-points, the first-order condition must be changed in a way that you should know from ECO 310:

A maximum at $X = 0$ is characterized by $EU'(0) \leq 0$;
a maximum at $X = W_0$ is characterized by $EU'(W_0) \geq 0$.

We now proceed to see which of these obtains under what conditions.

To examine the possibility that $X = 0$ is optimal, begin by observing that

$$EU'(0) = \int_{R_L}^{R_H} (R - R_0)\ u'(R_0 W_0)\ f(R)\, dR = u'(R_0 W_0)\ (\ E[R] - R_0\ )$$

Therefore

$$EU'(0) \leq 0 \qquad \text{if and only if} \qquad E[R] \leq R_0$$

Thus the optimal X is zero if $E[R] \leq R_0$, and positive if $E[R] > R_0$. This tells us that if the risky asset has an expected rate of return even slightly above the rate of return on the safe asset, then our investor will put at least a little of his initial wealth in the risky asset. That is, he will be willing to bear at least a little of risk so long as the expected return is positive.

The intuition is that if the investor puts a very small amount ϵ in the risky asset, the extra expected return he gets is proportional to ϵ, but the variance of his final wealth is proportional to ϵ^2. So long as his coefficient of absolute risk aversion at W_0 is finite, he will surely like the first-order extra return bearing only second-order extra risk. However, that may no longer be true if the utility-of-consequences function $u(W)$ has a kink at W_0, as in the loss-aversion or prospect theories; more on this later.

As for the possibility that $X = W_0$ is optimal,

$$EU'(W_0) = \int_{R_L}^{R_H} (R - R_0)\ u'(R\,W_0)\ f(R)\,dR$$

If $u'(W) = m$, constant for all W (risk-neutrality), this reduces to $m\,(\,E[R] - R_0\,)$, so $EU'(W_0) > 0$ if and only if $E[R] > R_0$. This is intuitively obvious. By continuity (this is easy to say; a rigorous proof is not so easy), an investor whose risk-aversion is very small will also invest everything in the risky asset if it pays a higher expected return than the safe asset. But we cannot get any clear and simple condition involving a bound on the risk-aversion. Each example must be examined separately.

Next let us assume that the optimum is interior. This X is defined implicitly as the solution to the equation

$$EU'(X) \equiv \int_{R_L}^{R_H} (R - R_0)\ u'(R_0\,W_0 + (R - R_0)\,X)\ f(R)\,dR = 0\,. \tag{4}$$

Comparative Statics - General Idea

If any of the exogenous entities (parameters W_0 and R_0, and the shape of the distribution $f(R)$) that go into this change, so does the optimal X. We often need to know how the optimal X will change in response to one of these changes - for example whether an investor with more initial wealth will invest a larger absolute amount, or a larger fraction of the initial wealth, in the risky asset. This is called comparative statics – comparative because we are comparing the decision-maker's choices under alternative conditions; statics because we merely ask how the choice under the new conditions compares with that under the original conditions, and not the dynamic process by which the consumer may recognize the change in conditions and gradually adapt to it, or anything of that kind. The expressions you found in ECO 310 for the price and income derivatives of demand functions were an instance of this – you were asking how the quantities chosen by the consumer differ for different exogenously specified prices and incomes.

Just about all comparative statics analyses follow the same method. It is worth knowing in a very general form, so you can then apply it mechanically in each context.

The general problem is as follows. Suppose x is being chosen to maximize a function $V(x, \theta)$ where θ is an exogenous parameter. For a given value of θ, the optimum x is defined as the solution to the implicit equation

$$V_x(x, \theta) = 0\,, \tag{5}$$

where the subscripts on V denote differentiation with respect to the indicated argument. Call the solution $x = H(\theta)$; in other words

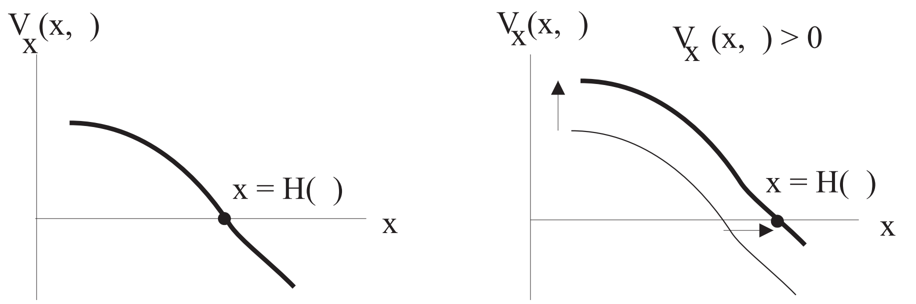
$$V_x(H(\theta), \theta) = 0\,, \tag{6}$$

for all θ. Then the aim of comparative statics is to find various properties of the function H, most importantly, whether it is increasing or decreasing in θ.

We assume the second-order condition in its strict inequality (sufficient condition) form:

$$V_{xx}(x, \theta) < 0\,. \tag{7}$$

The left hand panel of Figure 1 shows this. The curve is the graph of V_x as a function of x holding θ fixed. It crosses zero at $x = H(\theta)$, and is decreasing from positive to negative values through this point because of the second-order condition (7).



First and second order conditions

Comparative statics of increase in

Figure 1: Comparative Statics

Now suppose θ increases. Consider the case where the second-order cross-partial derivative $V_{x\theta}$ is positive. This means that for any fixed x, V_x increases as θ increases. This raises the whole graph depicting V_x against x upward. This is shown in the right hand panel of Figure 1. The old position of the curve is shown by the thinner line, and the shift is indicated by the vertical arrow. We see that the point where the curve crosses the horizontal axis then shifts to the right, as shown by the right-pointing arrow along the axis. So the new position of the maximizer moves to the right: $x = H(\theta)$ increases.

Conversely, if $V_{x\theta} < 0$, then an increase in θ shifts the V_x curve vertically downward, and therefore lowers the optimal $x = H(\theta)$.

Note that the shift in the V_x curve does not have to be vertically parallel to itself, that is, V_x does not have to rise (or fall, as the case may be) by the same amount for all x. This is because $V_{x\theta}$ may itself depend on x. What matters for the direction of change in $H(\theta)$ is that the shift be everywhere upward (or downward, as the case may be).

We can also see this using calculus. Note that the equation (6) holds for all θ. Differentiate it totally with respect to θ using the chain rule:

$$V_{xx}(H(\theta), \theta)\; H'(\theta) + V_{x\theta}(H(\theta), \theta) = 0\,.$$

Therefore

$$H'(\theta) = -\,\frac{V_{x\theta}(H(\theta), \theta)}{V_{xx}(H(\theta), \theta)}\,. \tag{8}$$

By the second-order condition, $V_{xx} < 0$. Therefore the sign of $H'(\theta)$ is the same as the sign of $V_{x\theta}(H(\theta), \theta)$.

So if we want to know merely whether the optimal x increases or decreases as θ increases, all we have to do is to find the second-order cross-partial $V_{x\theta}$ of the objective function that is being maximized, and evaluate its sign at the optimum x. If we want the quantitative magnitude of the comparative static effect of θ on x, we have the exact formula (8).

Exercise: To fix these ideas in your mind, repeat the analysis for the case where the function V is to be minimized, not maximized.

Comparative Statics of the Two-Asset Portfolio Problem

Now we can apply this general idea to our portfolio allocation problem with one riskless asset and one risky asset. The objective function is the expected utility given by (2). There we expressed it solely as a function of X, but we can also show the dependence on any other exogenous parameter that enters the problem. Let us start with the initial wealth W_0 playing the role of θ. So now the question is: if the investor has a larger initial wealth, will he put more or less into the risky asset? Obviously more? Not quite so fast.

Write

$$V(X, W_0) = EU(X) = \int_{R_L}^{R_H} u(W)\ f(R)\, dR = \int_{R_L}^{R_H} u(R_0\, W_0 + (R - R_0)\, X)\ f(R)\, dR\,.$$

Differentiating with respect to X, we have

$$V_X(X, W_0) = EU'(X) = \int_{R_L}^{R_H} (R - R_0)\ u'(R_0\, W_0 + (R - R_0)\, X)\ f(R)\, dR\,.$$

Differentiating this with respect to W_0, the key second-order cross-partial derivative is

$$\begin{aligned} V_{X\,W_0} &= \int_{R_L}^{R_H} (R - R_0)\ u''(R_0\, W_0 + (R - R_0)\, X)\, R_0\ f(R)\, dR \\ &= R_0\ \int_{R_L}^{R_H} (R - R_0)\ u''(R_0\, W_0 + (R - R_0)\, X)\ f(R)\, dR\,. \end{aligned}$$

In the integrand, $u'' < 0$, but $(R - R_0)$ changes sign as R ranges from R_L to R_H. Therefore the sign of the integral is not immediately determinate. However, we have good reason to suspect it might be positive. So let us explore this, that is, try to find conditions under which it will be positive.

Recall that the second-cross-partial derivative is to be evaluated at the optimum. But the optimum does not involve u''; it involves u'. We can bring in u' using the concept of the coefficient of (absolute) risk aversion:

$$A(W) = -\ u''(W)\, /\, u'(W)\,, \qquad \text{or} \qquad u''(W) = -\ A(W)\ u'(W)\,.$$

Therefore

$$V_{X\,W_0} = -\ R_0\ \int_{R_L}^{R_H} (R - R_0)\ A(W)\ u'(W)\ f(R)\, dR\,,$$

where I have abbreviated $R_0 W_0 + (R - R_0) X = W$.

Evaluation at the optimum must somehow involve using the first-order condition (3). That looks very similar to this expression except that it does not have $A(W)$ inside the integral. Can we get it out somehow? This involves a "trick" of finding just the right bounds.

We should expect that in the normal case $A(W)$ should decrease as W increases: the willingness to take on a risk of a given *absolute dollar size* should be higher when the investor has more wealth. Proceed on this assumption. Note that $W = R_0 W_0 + (R - R_0) X$ increases as R increases, and that when $R = R_0$, $W = R_0 W_0$. Therefore

$$\begin{aligned} &\text{When } R < R_0, \text{ or } (R - R_0) < 0, \text{ we have } A(W) > A(R_0 W_0) \\ &\text{When } R > R_0, \text{ or } (R - R_0) > 0, \text{ we have } A(W) < A(R_0 W_0) \end{aligned}$$

Multiplying, we have

$$(R - R_0)\ A(W) < (R - R_0)\ A(R_0 W_0)$$

regardless of the sign of $(R - R_0)$. This is straightforward when $(R - R_0) > 0$; when $(R - R_0) < 0$, multiplying by the negative number reverses the direction of the inequality $A(W) > A(R_0 W_0)$.

Substituting into our integral

$$\begin{aligned} V_{X W_0} &= -\, R_0 \int_{R_L}^{R_H} (R - R_0)\ A(W)\ u'(W)\ f(R)\, dR \\ &> -\, R_0 \int_{R_L}^{R_H} (R - R_0)\ A(R_0 W_0)\ u'(W)\ f(R)\, dR \end{aligned}$$

Again the negative sign outside the integral changes the direction of the inequality. Then

$$V_{X W_0} > -\, R_0\ A(R_0 W_0) \int_{R_L}^{R_H} (R - R_0)\ u'(W)\ f(R)\, dR$$

because $A(R_0 W_0)$ is the same at all points of integration and therefore can be taken outside the integral.

Now the integral on the right hand side is just the one in the first-order condition $V_X = 0$ or (3). Since we are evaluating things at the optimum, this is zero. So we finally have

$$V_{X W_0} > 0\,.$$

Our general comparative statics result then tells us that an increase in the initial wealth W_0 will increase the optimal amount X the investor optimally puts into the risky asset. This on the assumption (and quite a reasonable assumption) that the coefficient of absolute risk aversion $A(W)$ is a decreasing function of wealth W. It should now make intuitive sense why this condition is needed for the result.

Exercise: Repeat the same steps to show that if $A(W)$ is an increasing function of W, then an increase in W_0 will lower the optimal X. And repeat similar (easier) steps to show that if $A(W) = a$, a constant independent of W, then the optimal X is independent of W_0.

The second item in the exercise can be seen more directly. We saw that, within an increasing linear transformation, the utility-of-consequences function for the case of constant absolute risk aversion is

$$u(W) = -\, e^{-a\,W}\,.$$

For this, expected utility is

$$EU(X) = -\int_{R_L}^{R_H} \exp[-\,a\,(R_0\,W_0 + (R - R_0)\,X)\,]\; f(R)\,dR$$

and then the first-order condition is

$$\begin{aligned} EU'(X) &= a \int_{R_L}^{R_H} (R - R_0)\; \exp[-\,a\,(R_0\,W_0 + (R - R_0)\,X)\,]\; f(R)\,dR \\ &= a\; \exp[-\,a\,R_0\,W_0\,] \int_{R_L}^{R_H} (R - R_0)\; \exp[-\,a\,(R - R_0)\,X\,]\; f(R)\,dR = 0 \end{aligned}$$

or

$$\int_{R_L}^{R_H} (R - R_0)\; \exp[-\,a\,(R - R_0)\,X\,]\; f(R)\,dR = 0\,.$$

This does not involve W_0 at all, therefore the solution for the optimal X is independent of W_0.

Next consider the comparative statics effects of changing the distribution $F(R)$ (or equivalently, the density $f(R)$). We could take a particular distribution with parameters, e.g. a normal distribution with mean μ and standard deviation σ, and use these in the role of the parameter θ in the general theory above. But it proves better to work non-parametrically, and consider the kinds of shifts we studied in the previous unit: [1] The distribution becomes uniformly better, that is, improves in the sense of first order stochastic dominance. [2] The distribution becomes less risky, that is, improves in the sense of second order stochastic dominance.

Begin with the first-order improvement. Again you might think that an improvement in the distribution of return to the risky asset is sure to lead to an increase in the amount optimally invested in it, but again, not so fast. The key question is, will this shift V_X up or down? Since we are working non-parametrically, we cannot take a derivative like $V_{X\theta}$. But we can use properties of first order stochastic dominance to identify conditions under which V_X will increase for any given X.

We have

$$V_X(X, F(.)\,) = EU'(X) = \int_{R_L}^{R_H} (R - R_0)\; u'(R_0\,W_0 + (R - R_0)\,X)\; f(R)\,dR\,.$$

We know that a first-order stochastic dominant improvement in $F(R)$ raises the expected value of any increasing function of R, and lowers the expected value of any decreasing function of R. Fortunately the expression shows V_X as the expected value of something, namely of

$$(R - R_0)\; u'(R_0\,W_0 + (R - R_0)\,X)$$

This is to be regarded as a function of R, for any given X. Call it $\phi(R)$. Then the question is: under what conditions is $\phi(R)$ increasing, and under what conditions is it decreasing?

Its derivative is

$$\begin{aligned} \phi'(R) &= 1 * u'(R_0 W_0 + (R - R_0) X) + (R - R_0)\ X\ u''(R_0 W_0 + (R - R_0) X) \qquad (9) \\ &= u'(W) + (W - R_0 W_0)\ u''(W)\,. \end{aligned}$$

Write it as

$$\begin{aligned} u'(W) \left[1 + \frac{(W - R_0 W_0)\ u''(W)}{u'(W)} \right] &= u'(W) \left[1 + \frac{W - R_0 W_0}{W}\ \frac{W\ u''(W)}{u'(W)} \right] \\ &= u'(W) \left[1 - \frac{W - R_0 W_0}{W}\ \rho(W) \right] \end{aligned}$$

where $\rho(W)$ denotes the coefficient of relative risk aversion.

The fraction on the right hand side is less than 1; therefore if $\rho(W) < 1$, we can be sure that the right hand side is positive. Therefore $\rho(W) < 1$ is a sufficient (but not necessary) condition for $\phi'(R) > 0$, and therefore for V_X to be shifted up by a first-order stochastic dominant improvement in $F(R)$, and therefore for this improvement to lead to an increase in the optimal X.

Once we have the mathematical result, we can figure out an intuition for it. What can go wrong if $\rho(W) > 1$, or roughly speaking, if the investor is highly risk averse. Now an improvement in the distribution of the rate of return on the risky asset implies that the investor can reach the same expected wealth by investing a smaller amount in the risky asset, and in the process he will have reduced his risk. A highly risk-averse investor likes this a lot. Of course we do not expect him to go entirely that far – keep the expected return unchanged and merely reduce the risk. It will be optimal to go for some increase in expected wealth and some reduction in risk. But doing the latter means reducing his exposure to the risky asset to some extent.

Another way to think of this is by analogy with income and substitution effects. An increase in the expected rate of return on an asset is similar to a reduction in the price of the asset – you can now get the same return with less investment in it, that is, get it more cheaply. Like any reduction in price, this has a substitution effect – it makes you want to buy more of it. But the same price decrease also has an income effect – you are better off, and want to buy more of all normal goods. If the investor is highly risk-averse, this takes the form of wanting to buy a lot more of the riskless asset. That income effect can then overpower the substitution effect, causing on balance a reduction in the amount of initial wealth he invests in the risky asset.

Note all this is vague and qualitative, and the precise condition on $\rho(W)$ can come only by doing the math, but it is good to have the economic intuition to back up the mathematical result.

Next consider a change in $F(R)$ that makes it less risky, in the sense of second order stochastic dominance. This will raise the expected value of the function $\phi(R)$ if it is concave.

(And will lower its expected value if it is convex.) Differentiating the above expression (9), we have

$$\begin{aligned} \phi''(R) &= 2\ X\ u''(R_0\, W_0 + (R - R_0)\, X) + (R - R_0)\ X^2\ u'''(R_0\, W_0 + (R - R_0)\, X) \\ &= X\ \left[\ 2\ u''(W) + (W - R_0\, W_0)\ u'''(W)\ \right]. \end{aligned}$$

We want to identify sufficient conditions to ensure

$$2\ u''(W) + (W - R_0\, W_0)\ u'''(W) < 0\,.$$

That will make $\phi(R)$ concave. Then a reduction in risk, that is, a second order stochastic dominant improvement in $F(R)$, will increase the expected value of $\phi(R)$, therefore raise the V_X curve, and therefore increase the amount X invested in the risky asset.

In the textbook (p. 17), they define a measure of *absolute prudence*: $P(W) = -\, u'''(W)/u''(W)$. In its terms, the condition can be written as

$$2 - (W - R_0\, W_0)\ P(W) > 0\,.$$

But this does not carry much intuition. It would be better to interpret the result in terms of the more familiar measures of absolute and/or relative risk aversion. But those involve u'', and here we have u'''. So we are going to need something to do with how the risk aversion measures *change* as wealth increases.

Let $\alpha(W) = -\, u''(W)/u'(W)$, the coefficient of absolute risk aversion. The coefficient of relative risk aversion is $\rho(W)$ as above. Write these as

$$u''(W) = -\alpha(W)\ u'(W) \qquad W\ u''(W) = -\rho(W)\ u'(W)\,.$$

Differentiating,

$$\begin{aligned} u'''(W) &= -\ \alpha'(W)\ u'(W) - \alpha(W)\ u''(W) \\ u''(W) + W\ u'''(W) &= -\ \rho'(W)\ u'(W) - \rho(W)\ u''(W)\,. \end{aligned}$$

Therefore

$$\begin{aligned} & 2\ u''(W) + (W - R_0\, W_0)\ u'''(W) \\ =\ & 2\ u''(W) + W\ u'''(W) - R_0\, W_0\ u'''(W) \\ =\ & 2\ u''(W) - [\ \rho'(W)\ u'(W) + \rho(W)\ u''(W) + u''(W)\ ] + R_0\, W_0\ [\ \alpha'(W)\ u'(W) + \alpha(W)\ u''(W)\ ] \\ =\ & R_0\, W_0\ u'(W)\ \alpha'(W) + \{\ [1 - \rho(W)]\ + W_0\, R_0\ \alpha(W)\ \}\ u''(W) - \rho'(W)\ u'(W) \end{aligned}$$

Therefore the conditions [1] $\alpha(W)$ is decreasing, [2] $\rho(W)$ is increasing, and $\rho(W) < 1$ are jointly sufficient for our purpose. It would not be worth doing all this algebra but for the fact that at least some of these conditions are empirically reasonable so worth stating.

How does one think of arranging and grouping the terms like this? Basically by trial and error, although after some experience one needs fewer trials and commits fewer errors.