ECO 317 – Economics of Uncertainty – Fall Term 2009
Notes for lectures
9. Demand for Insurance

The Basic Two-State Model

Insurance is a method for reducing (or in ideal circumstances even eliminating) *individual* risk by pooling with many others who have similar but not perfectly correlated (in ideal circumstances uncorrelated or even negatively correlated) risks, or trading it with someone who is much less averse to bearing the risk (or even risk-neutral).

Consider a person with initial wealth W_0, some of which will be lost if a bad event – fire or accident or theft, say – occurs. Here we treat a simple case where there just two outcomes of the uncertain prospect can arise: either zero loss, or loss of a given magnitude L. In technical terms, there are just two *states of the world,* or elementary events, or scenarios, 1: No loss, and 2: Loss. Let π denote the probability of loss (of state 2); for now we suppose this is exogenous and known to everyone.[1]

Suppose the individual is a risk-averse expected utility maximizer with an increasing and concave utility-of-consequences function u defined over wealth. Facing the prospect of the loss, his expected utility is

$$(1-\pi)\ u(W_0) + \pi\ u(W_0 - L)\,.$$

Insurance can increase this expected utility. This contract must be entered into *before* the outcome is known, that is, before the uncertainty is resolved. The contract stipulates actions to be performed by one party or the other [1] immediately, before the resolution of the uncertainty, and [2] after resolution of the uncertainty, in each conceivable outcome, that is, in each contingency.

In our simple two-state model, the contract is very simple: [1] The insured pays a *premium* P in advance to the insurance company. [2] After the uncertainty has resolved, the insurance company pays the insured an agreed *indemnity* I if the loss occurs (that is, if state 2 materializes), and nothing if the loss does not occur (that is, in state 1). Before proceeding to study what kinds of contracts will emerge, here are some general remarks.

Any such contingent contract, where the parties have unbalanced obligations to pay before and after the resolution of uncertainty, has to be enforceable; otherwise the party that was unilaterally required to pay the other after the resolution of the uncertainty would be tempted to renege. In most modern economies, this task belongs to the government's legal system; in other countries or in other contexts it could be a private enforcer, or a self-enforcing equilibrium of a repeated interaction between the two parties.

Enforceability requires that the enforcer can give a clear verdict as to which party owes what to the other. Therefore the states of the world have to be clearly delineated, and after the fact it must be equally clear to the enforcing authority which state has occurred. This

[1] Technically, we have to assume that it is *common knowledge* between this person and anyone who may enter into a contract with him: each knows π, each knows that the other knows, each knows that the other knows that he knows, and so on *ad infinitum.*

can be problematic in reality, as the dispute about wind damage versus water damage after the 2005 hurricanes illustrates. In the case of third-party enforcement, the contract can only distinguish those contingencies (states of the world) whose occurrence can be proved to that third party; in technical terms, the contract can only be based on information that is provable or *verifiable* to outsiders. For self-enforcement, it is enough if the contingency is *observable* to the two parties themselves.

For now we ignore all such issues; we assume that all states of the world are costlessly verifiable after the resolution of uncertainty, and a third party stands ready to enforce contracts perfectly and costlessly. We will return to issues of information and perhaps of enforcement later in the course.

First-Best Insurance

Under ideal circumstances, insurance could be statistically or actuarially fair, in the sense that the premium could equal the expected monetary value of the indemnity to be paid:

$$P = \pi\, I\,. \tag{1}$$

This can be done by pooling together a large number of similar independent risks. Suppose n identical people facing independent risks are in the insurance pool. It is sometimes said that the total risk is negligible because of the law of large numbers, but that is not correct. The total premium receipts equal $n\, P$. The total payout is a random variable with mean $n\, \pi\, I = n\, P$. The variance of the total payout is $n\, \pi\, (1 - \pi)\, I$, which does not get small as n increases; on the contrary, it grows proportionately with the size of the pool. But the *per capita* payout, being $(1/n)^{th}$ of the total, has variance $(1/n^2)$ times that of the total, that is, $\pi\, (1 - \pi)\, I\, /\, n$, which does become small as n increases. Thus the pool can provide an almost non-random indemnity to each of its members.

Alternatively, an ideal insurance company that has no administrative costs and is risk-neutral makes an expected profit equal to $P - \pi\, I$ from each of its customers. If competition among insurance companies can drive this expected profit down to zero in equilibrium, then the market will provide actuarially fair insurance. An insurance company can be risk-neutral because it is owned by investors for whom this risk is not correlated with the market as a whole, that is, the insurance company's stock has zero beta.

Suppose actuarially fair insurance is available, and the individual can choose any amount of coverage or indemnity I by paying the fair premium $P = \pi\, I$. His final wealth in the two states of the world will be

$$W = \begin{cases} W_1 = W_0 - P = W_0 - \pi\, I & \text{in state 1 (no loss)} \\ W_2 = W_0 - P - L + I = W_0 - L + (1 - \pi)\, I & \text{in state 2 (loss)} \end{cases} \tag{2}$$

His expected utility, expressed as a function of the choice variable I, is

$$\begin{aligned} EU(I) &= (1 - \pi)\; u(W_1) + \pi\; u(W_2) \\ &= (1 - \pi)\; u(W_0 - \pi\, I) + \pi\; u(W_0 - L + (1 - \pi)\, I) \end{aligned} \tag{3}$$

To maximize this, the first order condition is

$$EU'(I) = -\ (1-\pi)\ \pi\, u'(W_0 - \pi\, I) + \pi\, (1-\pi)\ u'(W_0 - L + (1-\pi)\, I) = 0 \tag{4}$$

As usual, risk aversion ensures that the second-order condition is satisfied.

Now (4) can be written as $u'(W_1) = u'(W_2)$, and therefore $W_1 = W_2$ at the optimum. The individual has equal wealth in the two states. Therefore he faces no risk. Alternatively, from the expressions for W_1 and W_2 in (2), we have $W_0 - \pi\, I = W_0 - L + (1-\pi)\, I$, or $I = L$; the optimum is characterized by full insurance. This is the best the individual can hope for, unless outside resources can be brought in to subsidize him.

Note well the associated condition, namely equalization of marginal utilities across the states of the world. This will prove very useful when interpreting future results where the ideal or first-best insurance is not attainable. Then the way in which marginal utilities differ across states gives us clues as to when and in what way and how far the solution falls short of the ideal.

With $I = L$, the expressions for final wealth in (2) become

$$W_1 = W_2 = W_0 - \pi\, L\,. \tag{5}$$

This is as if the individual simply bears his expected loss with certainty.

A Geometric Treatment

This analysis can be illustrated using the state-space diagram that was developed in the previous handout. We show on the two axes the amounts of final wealth of the individual in the two states. The indifference map consists of contours of equal expected utility; its properties were explained in the previous handout. Now, to develop the idea of choice using this diagram, we need a budget line.

We can get it by eliminating I from the two lines of (2). Multiply the first by $(1-\pi)$, the second by π, and add the two together. This yields

$$(1-\pi)\ W_1 + \pi\ W_2 = (1-\pi)\ W_0 + \pi\ (W_0 - L)\,. \tag{6}$$

This is a straight line, passing through the point $(W_0, W_0 - L)$, and having a negative slope equal to $(1-\pi)/\pi$ in absolute value. This is the steeper of the two straight lines in Figure 1.

We can interpret this as follows. In the absence of any trading in risk, the individual would have W_0 in state 1 and $(W_0 - L)$ in state 2. Before the resolution of uncertainty, he can make a contract whereby he promises to give up a dollar if state 1 occurs, in exchange for a promise that will get him say x dollars if state 2 occurs. What value of x will preserve a statistical or actuarial balance between the two trades? The probability of having to give up the dollar is $(1-\pi)$, so the expected monetary loss from the contract is $(1-\pi)$. The probability of receiving x dollars is π, so the expected monetary gain from the contract is $\pi\, x$. For balance, we equate the expected loss and the expected gain, so $\pi\, x = 1 - \pi$, or $x = (1-\pi)/\pi$. This is exactly the slope of the budget line (in absolute value). You can think of it as the *relative price*, expressed in units of "state-2 dollars," for which this person

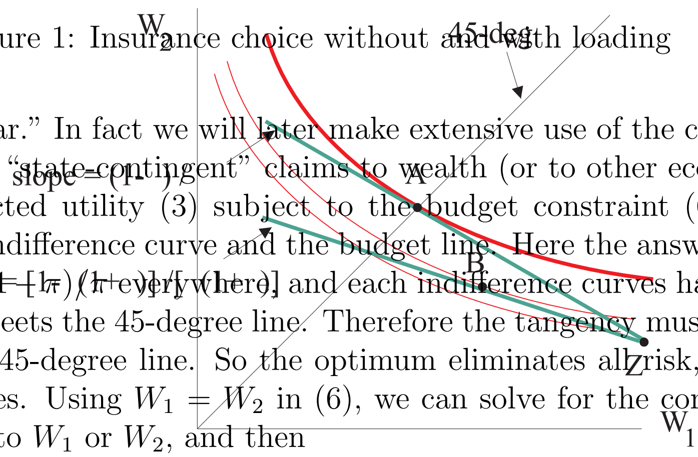

Figure 1: Insurance choice without and with loading

will sell a "state-1 dollar." In fact we will later make extensive use of the concept that trade in risk is trade in such "state-contingent" claims to wealth (or to other economic goods).

To maximize expected utility (3) subject to the budget constraint (6), we look for a tangency between an indifference curve and the budget line. Here the answer is obvious: the budget line has slope $(1-\pi)/\pi$ everywhere, and each indifference curves has slope $(1-\pi)/\pi$ at the point where it meets the 45-degree line. Therefore the tangency must occur where the budget line meets the 45-degree line. So the optimum eliminates all risk, and yields equal wealth in the two states. Using $W_1 = W_2$ in (6), we can solve for the common value. The left hand side reduces to W_1 or W_2, and then

$$W_1 = W_2 = (1-\pi)\, W_0 + \pi\, (W_0 - L) = W_0 - \pi\, L\,.$$

Thus the geometry confirms and illustrates the algebraic derivation above.

Loading

In practice, insurance is almost never available on actuarially fair terms. The insurance company has administrative costs it must cover, or competition in the insurance industry is imperfect so each company can charge a mark-up above its costs.[2] To keep the analysis simple, suppose the premium contains a constant "loading factor" λ, so (1) is replaced by

$$P = (1+\lambda)\,\pi\, I\,. \tag{7}$$

[2]In practice, the insurance companies collect the premiums some time before they have to pay out on claims. In the meantime they can invest the premiums. Therefore their profits come from two sources: any actuarial markups in their main business ("underwriting" profits), and gains (or losses) on their investments. Sometimes, if investments are producing large gains and competition is intense, companies that face fierce competition with other companies may even offer "superfair" insurance to attract business. And at other times when the financial markets are adverse, they may charge higher premiums for reasons that have nothing to do with the risks they are insuring.

This leads to corresponding changes in the expressions for the final wealth in the two states; (2) is replaced by

$$W = \begin{cases} W_1 = W_0 - P = W_0 - (1+\lambda)\,\pi\, I & \text{in state 1 (no loss)} \\ W_2 = W_0 - P - L + I = W_0 - L + [1-(1+\lambda)\,\pi]\, I & \text{in state 2 (loss)} \end{cases} \tag{8}$$

The expression for expected utility as a function of the choice variable I becomes

$$\begin{aligned} EU(I) &= (1-\pi)\; u(W_1) + \pi\; u(W_2) \\ &= (1-\pi)\; u(W_0 - (1+\lambda)\,\pi\, I) + \pi\; u(W_0 - L + [1-(1+\lambda)\,\pi]\, I) \end{aligned} \tag{9}$$

To maximize this, the first order condition is

$$EU'(I) = -\,(1-\pi)\;(1+\lambda)\,\pi\, u'(W_0 - (1+\lambda)\,\pi\, I) + \pi\,[1-(1+\lambda)\,\pi]\; u'(W_0 - L + [1-(1+\lambda)\,\pi]\, I) = 0 \tag{10}$$

Again, risk aversion ensures that the second-order condition is satisfied.

This does not yield a simple solution like that in the ideal or first-best case. But we can infer which way the solution here differs from the ideal. Write (10) as

$$\begin{aligned} \frac{u'(W_1)}{u'(W_2)} &= \frac{\pi\,[1-(1+\lambda)\,\pi]}{(1-\pi)\;(1+\lambda)\,\pi} \\ &= \frac{[1-(1+\lambda)\,\pi]/[(1+\lambda)\,\pi]}{(1-\pi)/\pi} \end{aligned} \tag{11}$$

Since $(1+\lambda)\,\pi > \pi$,

$$\frac{1-(1+\lambda)\,\pi}{(1+\lambda)\,\pi} < \frac{1-\pi}{\pi}\,,$$

and therefore (11) gives $u'(W_1) < u'(W_2)$, or $W_1 > W_2$. The individual chooses to have less wealth in the loss state than in the no-loss state, that is, he chooses partial insurance coverage and bears some of the risk himself.

This is illustrated in the state-space figure (1). Now the budget line, found by eliminating I between the two lines of (8), is

$$[1-(1+\lambda)\,\pi]\; W_1 + (1+\lambda)\,\pi\; W_2 = [1-(1+\lambda)\,\pi]\; W_0 + (1+\lambda)\,\pi\;(W_0 - L)\,.$$

This again passes through the point $(W_0, W_0 - L)$, but has slope $[1-(1+\lambda)\,\pi]/[(1+\lambda)\,\pi]$, which is smaller than the slope $(1-\pi)/\pi$ in the case of actuarially fair insurance. Intuitively, when the insurance is actuarially unfair, the individual is able to get less in the loss state for each dollar he gives up in the no-loss state.

The indifference curves of expected utility all have slope $(1-\pi)/\pi$ on the 45-degree line, and are flatter below that line. So now, with a flatter budget line, the tangency must occur below the 45-degree line, that is, in the region where $W_1 > W_2$.

So long as the loading factor is not too high, the individual will choose to have some insurance. But suppose the loading factor get so high that the slope of the budget line equals the slope of the indifference curve at the initial point $(W_0, W_0 - L)$. Then the individual will optimally stay at this point, that is, buy no insurance. If the loading factor gets even larger, he may want to move south-east along the budget line, that is, take a position even riskier than his original risk, if he can benefit from the same loading factor as the insurance company. But this is usually not possible, so we have a corner solution where he buys no insurance.

Deductibles

Deductibles and coinsurance are common features of actual insurance contracts. If a policy has deductible D, and only a fraction β of the loss in excess of the deductible is covered, then the indemnity I is related to the loss L by

$$I = \beta\,(L - D) \text{ if } L > D, \text{ and } I = 0 \text{ if } L \leq D.$$

However, in the two-state example we have considered up to now, these added features make no substantive difference; all that matters is the I and the premium $P = (1 + \lambda)\,\pi\, I$, irrespective of the details by which these come about. So we need a more general setting for a meaningful analysis of deductibles and coinsurance. In this section we consider deductibles. Specifically, we ask when and how they emerge as outcomes in the market or as optima.

Initial wealth is W_0. There are n states with probabilities $p_1, p_2, \ldots p_n$, and loss amounts $0 \leq L_1 < L_2 < \ldots < L_n$. Write $\overline{L} = \sum_i p_i\, L_i$ for the expected loss.

The most general insurance contract can take the form: the individual pays the company P in advance, and the company pays him specified indemnities $I_1, I_2, \ldots I_n$ in the various states. There is a given loading factor λ, so

$$P = (1 + \lambda) \sum_{i=1}^{n} p_i\; I_i\,. \tag{12}$$

The insured individual's final wealth is given by

$$W_i = W_0 - P - L_i + I_i \qquad i = 1, 2 \ldots n\,. \tag{13}$$

His expected utility is

$$EU = \sum_{i=1}^{n}\; p_i\; u(W_i)\,. \tag{14}$$

Suppose competition among insurance companies ensures that the contract is the best that can be offered to the insured subject to the requirement (12) for covering the company's expected total cost. Alternatively, you can think of this as a social optimum, constrained by the need to break even and cover unavoidable administrative costs. Writing μ for the Lagrange multiplier on the constraint, the conditions for the optimal choice of the indemnity levels I_i are

$$p_i\; u'(W_i) - \mu\; (1 + \lambda)\, p_i = 0, \quad \text{or} \quad u'(W_i) = \mu\; (1 + \lambda)$$

for all i. Thus marginal utilities are equalized across all states. This also implies that all the W_i are equal. So the individual bears no risk. Of course the common value of all the W_i is lower because of the premium.

If $\lambda = 0$, then full coverage $I_i = L_i$ for all i, leading to $P = \overline{L}$ from (12), and then $W_i = W_0 - \overline{L}$ from (13), satisfies all the conditions, and is therefore optimal. It is essentially unique.[3] In other words, if there is no loading, it is not optimal to have any deductibles.

What if $\lambda > 0$? Denote the common value of all the W_i by W^*, Then, from (13) we have

$$I_i = W^* - W_0 + P + L_i \,. \tag{15}$$

This tells us that $I_i - L_i$ should be the same for all states. Indemnities now fall short of covering the full loss because of the loading factor. But in the constrained optimum, the absolute amount by which they fall short should be the same for all states.

This raises a potential problem: there may be some states with small losses for which the indemnities should become negative, that is, the insured should pay the insurance company something extra if one of those states occurs. This is a logical implication, but it is impractical. If negative indemnities are ruled out, then the maximization of EU must be carried out with additional constraints $I_i \geq 0$ for all i.

To maximize expected utility given by (14) subject to the loading condition for the premium (12) and the non-negativity conditions on all the indemnity amounts, we have the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{n} p_i \; u(W_0 - P - L_i + I_i) + \mu \; \left[P - (1+\lambda) \; \sum_{i=1}^{n} p_i \; I_i \right] + \sum_{i=1}^{n} \nu_i \; I_i \,,$$

where ν_i are the Lagrange-Kuhn-Tucker multipliers on the non-negativity constraints. The first order conditions with respect to the I_i are

$$p_i \; u'(W_0 - P - L_i + I_i) - \mu \; (1+\lambda) \, p_i + \nu_i = 0 \,.$$

And we have the *complementary slackness* conditions:

$$\text{If } I_i > 0, \text{ then } \nu_i = 0 \,. \qquad \text{If } \nu_i > 0, \text{ then } I_i = 0 \,.$$

Suppose there is a particular pair of states j, k such that $I_j = 0$ and $I_k > 0$. Then $\nu_j \geq 0$ (the equality could arise but that would be an exceptional or razor's-edge case), and $\nu_k = 0$. Therefore the first order conditions for the indemnity amounts in these two states give us

$$\begin{aligned} p_j \; u'(W_0 - P - L_j) - \mu \; (1+\lambda) \, p_j + \nu_j &= 0 \,, \\ p_k \; u'(W_0 - P - L_k + I_k) - \mu \; (1+\lambda) \, p_k &= 0 \,, \end{aligned}$$

or

$$\begin{aligned} u'(W_0 - P - L_j) - \mu \; (1+\lambda) + \nu_j / p_j &= 0 \,, \\ u'(W_0 - P - L_k + I_k) - \mu \; (1+\lambda) &= 0 \,. \end{aligned}$$

[3] Increasing P and all the I_i by equal amounts will also be a solution, but it gives the same outcomes in all states as the solution proposed, so the two are equivalent.

Subtract the second from the first:

$$u'(W_0 - P - L_j) - u'(W_0 - P - L_k + I_k) + \nu_j/p_j = 0\,.$$

Therefore

$$u'(W_0 - P - L_j) - u'(W_0 - P - L_k + I_k) \le 0\,,$$

or

$$u'(W_0 - P - L_j) \le u'(W_0 - P - L_k + I_k)\,,$$

or

$$W_0 - P - L_j \ge W_0 - P - L_k + I_k\,,$$

or

$$L_j \le L_k - I_k < L_k\,.$$

Therefore any state in which no indemnity is paid must have a lower loss than any state in which a positive indemnity is paid (and the difference is not wholly made up by the indemnity in the latter state). So the states where the constraint $I_i \ge 0$ is binding, that is, the no-indemnity states, must be the least-loss states.

Now consider two other states, say k and h, in both of which positive indemnity is paid, so $I_h > 0$, $I_k > 0$ and therefore by complementary slackness, $\nu_h = 0$, $\nu_k = 0$. Then tracing steps similar to those above (Useful practice: Do this.) we have

$$u'(W_0 - P - L_k + I_k) = u'(W_0 - P - L_h + I_h)\,,$$

so $W_k = W_h$ and the insured ends up with the same amount of final wealth in these two states.

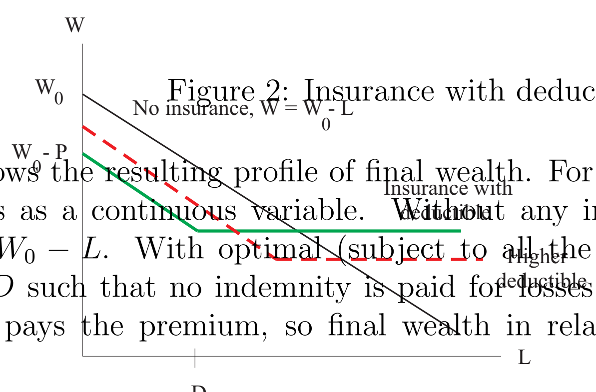

Figure 2: Insurance with deductible

Figure 2 shows the resulting profile of final wealth. For convenience in graphing, I have treated the loss as a continuous variable. Without any insurance, final wealth is simply the line $W = W_0 - L$. With optimal (subject to all the constraints) insurance, there is a level of loss D such that no indemnity is paid for losses below this level. Of course the individual still pays the premium, so final wealth in relation to loss is the straight line

$W = W_0 - P - L$. For losses exceeding D, the final wealth is kept constant independent of the level of loss, so $W = W_0 - P - D$. In the region $L > D$, the indemnity I must be satisfy $W_0 - P - L + I = W = W_0 - P - D$, so $I = L - D$. Therefore the contract takes the form of a pure deductible: the insured bears all losses up to the deductible, but there is full coverage of losses in excess of the deductible.

The graph of the final wealth with such insurance is piecewise linear, with a kink at $L = D$. This is the thick kinked line in the figure. Comparing this with the line for the case of no insurance, we see that the insured gives up some wealth in the low-loss states, and in return gets more final wealth in the states with larger losses. This is very intuitive when we think of the risk-averse individual as wishing to reduce exposure to risk. But the precise form of the optimum – a pure deductible – requires the math to figure out.

The book (p. 57) develops in a different way the idea that the pure-deductible contract optimally reduces risk. Its intuition is this: Start with the kinked line of the pure deductible case, and consider any other combination of indemnity payments with the same expected value. You cannot reduce the indemnity payments at any point in the region $L < D$ because they are already zero. If you increase them by making some offsetting reduction at some point in the region $L > D$, you will be increasing the final wealth in one state where it is already higher than in some other state where it is already lower. This can only lower expected utility. Or you could make offsetting changes at two or more points in the region $L > D$. But that can't be good either because that would bring variability to what was a constant level of final wealth there.

What is the optimal deductible? A higher deductible implies lower indemnities $I_i = L_i - D$, paid out in fewer states; therefore it implies a lower premium P from (12). This raises the line $W_0 - P - L$ in figure 2. But with a higher D, the line carries on farther to the right. In general, it crosses the horizontal part of the kinked final wealth curve for the older smaller D, before itself becoming horizontal. This is the dashed line in the figure. So the final wealth is higher in the lower-loss states and lower in the higher-loss states, that is, it becomes riskier. Choice between the two final wealth profiles depends on the risk aversion. An individual with a smaller risk aversion will prefer the higher deductible. This makes intuitive sense.

If the loading factor λ increases, it becomes optimal to have a higher deductible D, and eventually if λ gets too high, the individual may choose not to buy any insurance at all.

These results can be derived mathematically but that gets a little difficult so I will just rely on the intuition.

Finally some insurance policies have a ceiling on the coverage. This is hard to understand purely on the basis of optimal allocation of risk between the insured and the insurance company. Very large losses can threaten the company's solvency if they hit many of its customers simultaneously, but that is an extremely rare event unless the risks are highly correlated across people. And insurance companies can shift their own very large risks to other "reinsurance" companies. Ceilings may better be explained as responses to the possibility of asymmetric information: in insurance industry jargon, these are situations of moral hazard (extreme carelessness on part of the insured, or even deliberate fraud such as arson if the indemnity exceeds the value of the item insured) or adverse selection (the policy selectively attracts customers who know themselves to have high probabilities of huge risks, that is, FOSD rightward-shifted loss distributions). We will consider asymmetric information later in the course.

Coinsurance

If deductibles are the optimal way to cope with a requirement of loading, why use coinsurance? What coinsurance does is to require the individual to bear a portion of risk *at the margin*. This can serve a useful purpose if the person has some control over the magnitude of the risk, for example by exercising some precautionary effort or care. Then coinsurance gives him some incentive to make such effort. In other words, coinsurance can be a useful response to moral hazard. We will analyze this issue later. For now we consider coinsurance assuming that it is the only instrument available for coping with loading.

The underlying situation is the same as in the case of deductibles. Initial wealth is W_0. There are n states with probabilities p_i and loss amounts L_i for $i = 1, 2, \ldots n$. The expected loss is $\overline{L}$. A fraction β of any loss is covered by the indemnity, so $I_i = \beta\, L_i$. The loading factor is λ, so the premium is given by

$$\begin{aligned} P &= (1+\lambda) \sum_{i=1}^{n} p_i\, I_i = (1+\lambda)\, \beta \sum_{i=1}^{n} p_i\, L_i \\ &= (1+\lambda)\, \beta\, \overline{L} = \beta\, P_f\,, \end{aligned} \tag{16}$$

where I have defined $P_f = (1+\lambda)\, \overline{L}$ to be the premium for full coverage.

The insured individual's final wealth in state i is given by

$$\begin{aligned} W_i &= W_0 - P - L_i + I_i \\ &= W_0 - \beta\, P_f - (1-\beta)\, L_i \end{aligned} \tag{17}$$

Expected utility is $EU = \sum_{i=1}^{n} p_i\, u(W_i)$ as usual.

To find the optimal choice of β, begin with

$$\frac{d\,EU}{d\beta} = \sum_{i=1}^{n} p_i\, u'(W_i)\, (L_i - P_f)\,. \tag{18}$$

Also

$$\frac{d^2 EU}{d\beta^2} = \sum_{i=1}^{n} p_i\, u''(W_i)\, (L_i - P_f)^2\,,$$

which is everywhere negative, so the first-order condition (which, however, may be an inequality at an end-point) yields a global optimum.

Begin by asking when full insurance may be optimal. At $\beta = 1$, we have $I_i = L_i$ and therefore $W_i = W_0 - P_f$ for all i. Then

$$\begin{aligned} \frac{d\,EU}{d\beta} &= \sum_{i=1}^{n} p_i\, u'(W_0 - P_f)\, (L_i - P_f) = u'(W_0 - P_f) \sum_{i=1}^{n} p_i\, (L_i - P_f) \\ &= u'(W_0 - P_f)\, [\, \overline{L} - (1+\lambda)\, \overline{L}\,] = -\,\lambda\, u'(W_0 - P_f)\, \overline{L}\,. \end{aligned}$$

If $\lambda = 0$, this is zero, so $\beta = 1$ is optimal. (EU as a function of β is increasing and concave throughout the range from 0 to 1, and just reaches the point of becoming flat as β hits 1.) So once again, if actuarially fair insurance (without any loading) is available, it is optimal for a risk-averse individual to choose full insurance.

If $\lambda > 0$, then $d\,EU/d\beta < 0$ at $\beta = 1$, so some $\beta < 1$ must be optimal. If there is loading, the individual chooses to bear some fraction of the risk. That reduces the premium enough to raise the expected wealth, so bearing a little risk is acceptable, as in the portfolio choice problem.

At $\beta = 0$ (no insurance), we have $W_i = W_0 - L_i$, so

$$\begin{aligned}\frac{d\,EU}{d\beta} &= \sum_{i=1}^{n} p_i\ u'(W_0 - L_i)\ [\,L_i - (1+\lambda)\,\overline{L}\,] \\ &= \sum_{i=1}^{n} p_i\ u'(W_0 - L_i)\ [\,L_i - \overline{L}\,] - \lambda\,\overline{L}\ \sum_{i=1}^{n} p_i\ u'(W_0 - L_i) \\ &= \mathrm{Cov}[u'(W_0 - L_i),\ L_i] - \lambda\ \mathrm{E}[L_i]\ \mathrm{E}[u'(W_0 - L_i)]\end{aligned}$$

A higher L_i corresponds to a lower $W_0 - L_i$ and therefore to a higher $u'(W_0 - L_i)$, so the covariance is positive. Therefore the sign of the expression is ambiguous. If λ is small, it will be positive, so EU will be an increasing function of β at 0, and the individual will insure a positive fraction of his losses. But if λ exceeds the threshold

$$\frac{\mathrm{Cov}[u'(W_0 - L_i),\ L_i]}{\mathrm{E}[L_i]\ \mathrm{E}[u'(W_0 - L_i)]}\,,$$

then $d\,EU/d\beta$ at $\beta = 0$ becomes negative, so $\beta = 0$ is optimum (Kuhn-Tucker condition for an optimum at the left end-point), and the individual buys no insurance.

Suppose the solution is in the interior, given by the first-order condition

$$\frac{d\,EU}{d\beta} = \sum_{i=1}^{n} p_i\ u'(W_i)\ (L_i - P_f) = 0\,. \tag{19}$$

Next we examine its comparative statics:

[1] As W_0 increases, the whole function $EU(\beta)$ shifts downward if the coefficient of absolute risk aversion is a decreasing function of wealth. The argument is similar to that in the portfolio choice problem, and it will be a useful exercise for you to work it out. The steps are also similar to those in [2] below. See Proposition 3.3 pp. 53-4 in the textbook for a different way to do this.

[2] What happens if the individual gets more risk-averse? The book gives a general proof (Proposition 3.2 on p. 53). Here is a special case. Suppose the individual has constant absolute risk-aversion α, so $u'(W) \sim \exp(-\alpha\,W)$. Then the first-order condition is

$$\sum_{i=1}^{n} p_i\ \exp[-\,\alpha\,W_i\,]\ (L_i - P_f) = 0\,. \tag{20}$$

As α changes, the derivative of the left hand side with respect to α is

$$- \sum_{i=1}^{n} p_i \; W_i \; \exp[- \alpha W_i] \; (L_i - P_f) \, .$$

We have labeled the states in order of increasing losses, and therefore in order of decreasing W_i. And $L_i - P_f$ must change sign as i goes from 1 to n, otherwise the sum could not be zero. Suppose

$$\begin{array}{l} L_1 - P_f, \; L_2 - P_f, \; \ldots \; L_k - P_f \text{ are negative} \\ L_{k+1} - P_f, \; \ldots \; L_n - P_f \text{ are positive} \end{array}$$

We also have

$$W_1 > W_2 > \ldots > W_k \, ,$$

therefore for $i = 1, 2, \ldots k$,

$$p_i \; W_i \; \exp[- \alpha W_i] \; (L_i - P_f) < p_i \; W_k \; \exp[- \alpha W_i] \; (L_i - P_f) \, .$$

(We are multiplying the inequality $W_i > W_k$ by a negative number so its direction is reversed.) And

$$W_k > W_{k+1} > \ldots > W_n \, ,$$

therefore for $i = k + 1, \ldots n$,

$$p_i \; W_i \; \exp[- \alpha W_i] \; (L_i - P_f) < p_i \; W_k \; \exp[- \alpha W_i] \; (L_i - P_f) \, .$$

(We are multiplying the inequality $W_i < W_k$ by a positive number so its direction is unchanged.)

So the same inequality holds for all i, and we can sum over i to get

$$\sum_{i=1}^{n} p_i \; W_i \; \exp[- \alpha W_i] \; (L_i - P_f) < W_k \; \sum_{i=1}^{n} p_i \; \exp[- \alpha W_i] \; (L_i - P_f) \, .$$

But the sum on the right hand side is zero by the first order condition. Therefore

$$- \sum_{i=1}^{n} p_i \; W_i \; \exp[- \alpha W_i] \; (L_i - P_f) > 0 \, .$$

That is, as α increases, the whole function in the first order condition (20) shifts up. By our standard comparative statics result, this raises the optimal β. So the more risk-averse the individual, the higher the fraction of risk he chooses to insure. This is intuitive, but the formal proof is not easy.