ECO 317 – Economics of Uncertainty – Fall Term 2009
Notes for lectures
13. MARKETS AND EFFICIENT RISK-BEARING: EXAMPLES AND EXTENSIONS

1. Allocation of Risk in Mean-Variance Framework

Suppose there are S states of the world, and a single physical good (often labelled "corn") whose aggregate quantitites in the different states are given by

$$Y_s = \overline{Y} + y_s \qquad \text{for } s = 1, 2, \ldots S$$

The probabilities of the states are π_s for $s = 1, 2, \ldots S$, and

$$\sum_{s=1}^{S} \pi_s \, y_s = 0$$

This choice of notation (use of the mean $\overline{Y}$ and zero-expected-value deviations y_s from it) is merely to simplify the algebra.

There are two people, A and B, with mean-variance objective functions, and coefficients of risk-aversion α_A and α_B respectively. Consider an allocation where

$$\text{A gets} \quad X_s = \overline{X} + x_s, \qquad \text{B gets} \quad Y_s - X_s = (\overline{Y} - \overline{X}) + (y_s - x_s)$$

where

$$\sum_{s=1}^{S} \pi_s \, x_s = 0$$

We want to choose this allocation to be Pareto efficient, that is, to maximize A's mean-variance objective, written MV_A for short:

$$MV_A = \overline{X} - \tfrac{1}{2} \, \alpha_A \sum_{s=1}^{S} \pi_s \, (x_s)^2$$

subject to giving B a specified minimum of his mean-variance objective MV_B given by

$$MV_B = (\, \overline{Y} - \overline{X} \,) - \tfrac{1}{2} \, \alpha_B \sum_{s=1}^{S} \pi_s \, (y_s - x_s)^2 \geq k \, ,$$

where k is some given constant that helps determine which point on the Pareto frontier is optimal. By varying k parametrically, we can trace out the entire Pareto frontier. The higher is k, the more favorable to B is the distribution.

Formulate the Lagrangian

$$\mathcal{L} = \left[\, \overline{X} - \tfrac{1}{2} \, \alpha_A \sum_{s=1}^{S} \pi_s \, (x_s)^2 \, \right] + \lambda \left[\, (\, \overline{Y} - \overline{X} \,) - \tfrac{1}{2} \, \alpha_B \sum_{s=1}^{S} \pi_s \, (y_s - x_s)^2 - k \, \right] + \mu \sum_{s=1}^{S} \pi_s \, x_s$$

where λ and μ are Lagrange multipliers.

If the maximization occurs in the interior of the range of $\overline{X}$, we have the first-order condition for $\overline{X}$:

$$\frac{\partial \mathcal{L}}{\partial \overline{X}} = 1 - \lambda = 0\,.$$

I will assume this to be so. There will be upper and lower limits on $\overline{X}$ implied by the requirement that the consumption quantities of both A and B must be non-negative in every state s, but I will not pursue this here.

With $\lambda = 1$, the Lagrangian simplifies to

$$\mathcal{L} = \overline{Y} - \tfrac{1}{2}\,\alpha_A \sum_{s=1}^{S} \pi_s\,(x_s)^2 - \tfrac{1}{2}\,\alpha_B \sum_{s=1}^{S} \pi_s\,(y_s - x_s)^2 - k + \mu \sum_{s=1}^{S} \pi_s\,x_s\,.$$

Note that $\overline{X}$ cancels, so its exact choice is actually indeterminate. Varying k over its permissible range changes $\overline{X}$ over its permissible range and traces out the entire set of Pareto efficient allocations.

The first-order condition with respect to any one of the x_s, say x_σ, is

$$\frac{\partial \mathcal{L}}{\partial x_\sigma} = -\alpha_A\;\pi_\sigma\,x_\sigma + \alpha_B\;\pi_\sigma\,(y_\sigma - x_\sigma) + \mu\;\pi_\sigma = 0$$

Summing these over σ and using the conditions that the expected deviations are zero and that the probabilities sum to 1, we have

$$-\alpha_A\ *\ 0 + \alpha_B\ *\ 0 + \mu\ *\ 1 = 0$$

or $\mu = 0$. (A Lagrange multiplier being zero is unusual but not impossible; it means that a slight shift of the constraint has no first-order effect, but only a second-order effect, on the maximized value of the objective function.)

Then the first-order conditions become

$$-\alpha_A\;\pi_\sigma\,x_\sigma + \alpha_B\;\pi_\sigma\,(y_\sigma - x_\sigma) = 0$$

They yield the solution

$$x_\sigma = \frac{\alpha_B}{\alpha_A + \alpha_b}\;y_\sigma$$

and therefore

$$y_\sigma - x_\sigma = \frac{\alpha_A}{\alpha_A + \alpha_b}\;y_\sigma$$

for all states of the world $\sigma = 1,\ 2,\ \ldots\ S$. In words, the shares of the two people in the deviations of aggregate output from its mean should be inversely proportional to their coefficients of risk-aversion (or equivalently, proportional to the coefficients of risk-tolerance; see the textbook, pp.153-5, also a more general formulation on pp.161-3). This makes good intuitive sense. In particular, note that if one of the two people is risk-neutral, he should bear all the risk, giving the other a constant amount across all states.

The simplicity of the formula comes from the mean-variance assumption. Recall that under this assumption the absolute amount of risk a person wants to take is independent of his initial wealth (similar to the case of zero income effects under quasilinear utility functions in the theory of consumer choice without uncertainty). That is why the risk-bearing optimum depends only on the risk aversion coefficients and not on the two people's wealth levels (or on the parameter k that governs where on the Pareto frontier the two will be located). In particular, the allocation of risk is determinate, and the same at all Pareto efficient allocations, independently of whether A or B is overall better off. Again this is because the willingness of each to bear risk is independent of the extent of his well-being.

To justify the mean-variance structure rigorously, we should assume that the traders have CARA utility-of-consequences functions and that the random output has a normal distribution. To do this mathematically rigorously would require a continuum of states, and therefore a constrained optimization problem where one chooses a whole function $x(s)$ using the calculus of variations (isoperimetric problem). This can be done, but the discrete formulation here is simpler to explain; think of it as an approximation to the normal.

In later topics we will come across situations where unobservable actions of one person can affect the overall outcome (moral hazard). Then to give this person a stronger incentive to take the socially efficient action, we will have to inflict more of risk on him than is indicated by the above formula. In other words, the optimum will have to strike a balance between the considerations of risk-bearing and incentive. Keep in mind the above formula as a reference point or standard with which to compare these later optima under moral hazard.

In this section we did not consider markets where people could trade voluntarily. Instead the focus was on the Pareto frontier, where a government or some such agency reallocated quantities. But we know from the Arrow-Debreu theory of general equilibrium under uncertainty that any Pareto efficient allocation can be achieved as an equilibrium of price-taking trading in a complete set of Arrow-Debreu securities (or equivalently, any set of securities whose payoff vectors span the full state space) provided the initial purchasing power is suitably distributed.

2. Incomplete Markets – Example

What happens if a complete set of Arrow-Debreu securities (or a set of fully spanning securities) is not available? Here we develop the ideas using an example, and focusing on efficient allocations. In the next section we will consider the relation between efficient allocations and equilibria of price-taking trade in markets in the available limited set of securities.

Suppose there is one physical good, corn. There are two farmers, A and B. Each is subject to risk, and with equal probability may get a high output or a low output. Their risks are independent of each other. A's high output is 30 and low output is 10 (mean 20 with equiprobable deviations of 10 on each side of the mean), whereas B's high output is 40 and low output is 0 (mean 20 with equiprobable deviations of 20 on each side). Thus they have equal expected levels of output ($\mu_A = \mu_B = 20$) but B faces more risk (variance $\sigma_B^2 = 400$ as against A's $\sigma_A^2 = 100$).

Suppose each has a mean-variance objective function, with

$$\begin{aligned} MV_A &= \mu_A - \tfrac{1}{5}\,\sigma_A^2 \\ MV_B &= \mu_B - \tfrac{1}{20}\,\sigma_B^2 \end{aligned}$$

Thus A has a coefficient of absolute risk aversion $\alpha_A = \frac{2}{5}$, whereas B has $\alpha_B = \frac{1}{10}$.

Without any trade in risk, the values of the objective functions for the two people are

$$\begin{aligned} MV_A &= 20 - \tfrac{1}{5}\, 100 = 0 \\ MV_B &= 20 - \tfrac{1}{20}\, 400 = 0 \end{aligned}$$

There is nothing special about these zeroes; we could have added any constant to each objective function, or multiplied it by any positive constant (taken any affine transformation of each) without affecting anything. But the zeroes are easier to remember and compare against alternatives we will consider.

Specifically, we will ask how the two can trade risks for mutual benefit. We know from the general theory that price-taking trading with complete markets in Arrow-Debreu securities achieves a Pareto efficient allocation of risk. Let us consider fully efficient allocations in detail for this example.

In Arrow-Debreu terms, there are four states of the world, counting all possible combinations of high and low outputs for the two people. Label these by two-letter combinations, the first showing A's high or low outcome and the second showing B's high or low. The aggregate output quantities are

HH: 70; LH :50; HL: 30; LL :10

The probability of each state is $\frac{1}{4}$. Using the terminology of the previous section, the mean aggregate output is $\overline{Y} = 40$, and the deviations from the mean in the four states are respectively 30, 10, -10 and -30.

We know how to characterize the Pareto frontier from the general theory of the previous section. The deviations should be allocated between A and B in inverse proportions of their coefficients of absolute risk aversion, that is, 1:4. So A's deviations from his mean should be 6, 2, -2 and -6, and B's should be 24, 8, -8 and -24. The distributional considerations are optimally handled by giving A the average $\overline{X}$ and giving B the average $40 - \overline{X}$. The Pareto frontier is traced out by varying $\overline{X}$.

With these deviations, the variance of A's final consumption of corn is

$$\tfrac{1}{4}\,[\,6^2 + 2^2 + (-2)^2 + (-6)^2\,] = \tfrac{1}{4}\,[36 + 4 + 4 + 36] = 20$$

and therefore A's mean-variance objective is

$$MV_A = \overline{X} - \tfrac{1}{5}\, 20 = \overline{X} - 4\,.$$

The variance of B's final consumption of corn is

$$\tfrac{1}{4}\,[\,24^2 + 8^2 + (-8)^2 + (-24)^2\,] = \tfrac{1}{4}\,[576 + 64 + 64 + 576] = 320$$

and therefore B's mean-variance objective is

$$MV_B = 40 - \overline{X} - \tfrac{1}{20}\, 320 = 40 - \overline{X} - 16 = 24 - \overline{X}\,.$$

To find the Pareto frontier, eliminate $\overline{X}$ between the two equations:

$$MV_A + MV_B = 20\,.$$

As an example, suppose we want to achieve the point (10, 10) on the frontier (this divides the potential gains equally between the two, and will also be the Nash bargaining outcome). That requires setting $\overline{X} = 14$. Then A gets respectively 20, 16, 12, and 8 units of corn in the four states, whereas B gets 50, 34, 18, and 2. This requires a lot of freedom to reallocate quantities separately in each state of the world, and such freedom is exactly what the four separate Arrow-Debreu securities create.[1]

What if we do not have a full set of Arrow-Debreu securities, or alternatively four or more composite securities whose payoff vectors span the whole four-dimensional state space? A more realistic case is one where it is possible to allocate shares in each person some share in the other's farm. For example, if A is given a 50% share in B's farm, then A gets 50% of B's output in each state of the world. But the fraction cannot be varied from one state to another. Specifically, suppose we give B a fraction ϕ of A's farm and give A a fraction $(1-\psi)$ of B's farm. Thus we have only two degrees of freedom in allocating risk profiles, not the full four that exist with allocation of Arrow-Debreu securities. Let us see what values of the mean-variance objective functions can be attained using shares. The quantities of corn the two get to consume in the various states are given in Table 1.

Table 1: Corn Allocations Using Shares Only

State	HH	LH	HL	LL
A	$30(1-\phi) + 40(1-\psi)$	$10(1-\phi) + 40(1-\psi)$	$30(1-\phi)$	$10(1-\phi)$
B	$30\,\phi + 40\,\psi$	$10\,\phi + 40\,\psi$	$30\,\phi$	$10\,\phi$

We can then calculate the means, variances, and the objectives MV_A and MV_B as functions of (ϕ, ψ). It is hard to solve the constrained efficiency problem of choosing (ϕ, ψ) to maximize MV_A subject to $MV_B \geq k$. But by finding (MV_A, MV_B) for a large grid of values of (ϕ, ψ) and plotting these points (MV_A, MV_B) in a graph, we can get a good idea of the shape of the feasible frontier using only shares to allocate risk. Figure 1 shows this. The Pareto frontier with complete markets is also shown for comparison.

We see that the point (10,10) is not going to lie on the frontier using only shares. We calculated above the quantities of corn in various states that should be given to the two people to achieve these values of the mean-variance objective functions. To attain these quantities using shares we need

$$30\,\phi + 40\,\psi = 50, \qquad 10\,\phi + 40\,\psi = 34, \qquad 30\,\phi = 18, \qquad 10\,\phi = 2.$$

[1] Note that if we wanted to give A more than 12, that would require giving B a negative quantity in the LL state. Likewise, if we wanted to give B more than 18, then A's quantity in the LL state would have to be negative. If this is infeasible, the Pareto frontier will have to be truncated at these two extremes. But that is not relevant to the comparison with incomplete markets I turn to next; therefore I will ignore this problem.

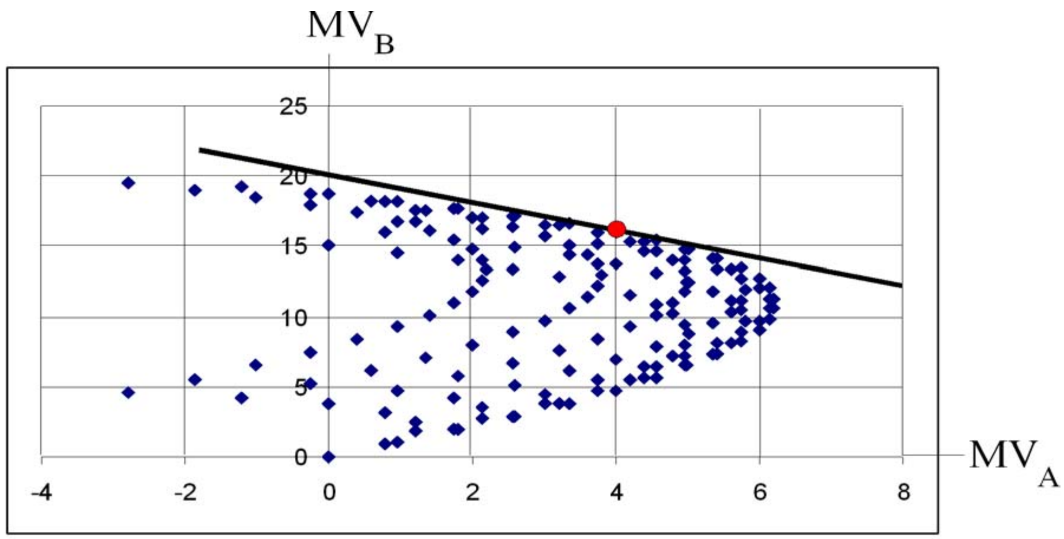

Figure 1: Frontiers With and Without Complete Markets

This is obviously impossible, as the third equation requires $\phi = 0.6$ and the fourth requires $\phi = 0.2$. The share allocation method has too few degrees of freedom (2) for the dimensionality of the state space (4).

Implementing a point on the full-efficiency frontier using only shares is possible in exceptional cases at most. Here we see that the two frontiers do have one point in common, namely (4,16). To achieve this point, we need $\overline{X} = 8$, and then A gets 14, 10, 6, 2 whereas B gets 56, 50, 24, 8. This is done by giving A a 20% share and giving B 80% share in each person's farm. But that is a special feature of the mean-variance setup. With more general utility functions, efficient allocation may require a pattern of consumption across states of the world that cannot be reproduced using shares (that is, patterns of production across states of nature). Then the share-constrained efficient frontier may not have any fully efficient points at all.

3. Incomplete Markets – Some General Theory

The following material is algebraically somewhat heavy. We will go through it in class for the broad ideas, but you are not expected to be able to reproduce the detailed calculations.

With complete markets, we have the usual equivalence between Pareto efficiency and competitive equilibria: a competitive equilibrium is Pareto efficient, and any Pareto efficient allocation can be achieved as a competitive equilibrium starting from suitably chosen initial endowments (or lump sums of purchasing power). Is there a similar correspondence between efficiency and equilibrium with incomplete markets? Does a competitive equilibrium in a limited set of markets for contingent claims have a property of constrained Pareto efficiency, in the sense that a government that must reallocate wealth in different state of the world obeying the same set of restrictions as the ones on market contracts cannot make one person better off without making someone else worse off? Here we will find that to be the case in a simple model of trading in shares. In more general contexts the equivalence breaks down; that must be left for more advanced courses.

Suppose there are two people (allowing more people just complicates the algebra without affecting any substantive economic conclusions). There are several states $s = 1, 2, \ldots S$. The endowments of the two people in the different states will be $(X^0_{i1}, X^0_{i2}, \ldots X^0_{iS})$ for $i = A$, B. Their final consumption quantities will be denoted by $(X_{i1}, X_{i2}, \ldots X_{iS})$.

Begin by setting up an ideal or standard of full or unrestricted Pareto efficiency, where consumption in each state of the world can be reallocated independently across states. So we maximize

$$U_A(X_{A1}, X_{A2}, \ldots X_{AS})$$

subject to

$$U_B(X_{B1}, X_{B2}, \ldots X_{BS}) \geq k$$

and

$$X_{As} + X_{Bs} \leq X^0_{As} + X^0_{Bs} \quad \text{for } s = 1, 2, \ldots S$$

Note that the utility functions don't have to have the expected utility form; they can be perfectly general so long as they have convex indifference curves in state space.

Form the Lagrangian

$$\mathcal{L} = U_A(X_{A1}, X_{A2}, \ldots X_{AS}) + \theta\ [U_B(X_{B1}, X_{B2}, \ldots X_{BS}) - k] + \sum_{s=1}^{S} \lambda_s \left[X^0_{As} + X^0_{Bs} - X_{As} - X_{Bs} \right]$$

Observe that the Lagrangian is just as if we were maximizing the weighted sum of utilities: $U_A + \theta\ U_B$. The textbook takes this approach, compare for example equation (10.2) on p. 157).

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X_{As}} &= \frac{\partial U_A}{\partial X_{As}} - \lambda_s = 0 \\
\frac{\partial \mathcal{L}}{\partial X_{Bs}} &= \theta\ \frac{\partial U_B}{\partial X_{Bs}} - \lambda_s = 0
\end{aligned}$$

for $s = 1, 2, \ldots S$. These imply

$$\frac{\partial U_A / \partial X_{As}}{\partial U_B / \partial X_{Bs}} = \theta \quad \text{for all } s = 1, 2, \ldots S\,. \tag{1}$$

If we shift one unit of consumption in state s from person B to person A, the ratio of the changes in utilities is given by the left hand side of this equation. The optimality condition then requires that the ratio should be the same in all states, and it is the freedom to arrange such shifts of consumption on a state-by-state basis that enables us to achieve this optimum. Remember this important property; we will use this to contrast what is possible with restricted markets and the equivalent restricted methods for allocation.

Next suppose we can carry out allocations using only shares in the two people's enterprises. Thus, if B gets the share ϕ in A's enterprise, that by itself will give B consumption quantities $\phi\, X^0_{As}$ in state s. Suppose B also gets a share ψ in his own enterprise. Then B's total consumption quantities will be

$$X_{Bs} = \phi\ X^0_{As} + \psi\ X^0_{Bs}\,.$$

And then A's consumption quantities will be

$$X_{As} = (1-\phi)\ X^0_{As} + (1-\psi)\ X^0_{Bs}\,.$$

Now we want to choose ϕ and ψ to maximize

$$U_A(X_{A1}, X_{A2},\ \ldots\ X_{AS})$$

subject to

$$U_B(X_{B1}, X_{B2},\ \ldots\ X_{BS}) \geq k\,.$$

Form the Lagrangian

$$\mathcal{L} = U_A(X_{A1}, X_{A2},\ \ldots\ X_{AS}) + \nu\ \left[\, U_B(X_{B1}, X_{B2},\ \ldots\ X_{BS}) - k\,\right]$$

where ν is a new Lagrange multiplier. The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= \sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}}\ (-\,X^0_{As}) + \nu\ \sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}}\ X^0_{As} = 0 \\
\frac{\partial \mathcal{L}}{\partial \psi} &= \sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}}\ (-\,X^0_{Bs}) + \nu\ \sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}}\ X^0_{Bs} = 0
\end{aligned}$$

Write these as

$$\frac{\displaystyle\sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}}\ X^0_{As}}{\displaystyle\sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}}\ X^0_{As}} = \frac{\displaystyle\sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}}\ X^0_{Bs}}{\displaystyle\sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}}\ X^0_{Bs}} = \nu \tag{2}$$

If (1) holds, then substituting into (2) we see that it holds as well, with $\nu = \theta$. However, the converse is not true in general. There are too few equations in (2) to satisfy (1). Another way to see this is to write the two equations as

$$\begin{aligned}
\sum_{s=1}^{S} \left[\, \frac{\partial U_A}{\partial X_{As}}\ - \nu\ \frac{\partial U_B}{\partial X_{Bs}}\,\right]\ X^0_{As} &= 0 \\
\sum_{s=1}^{S} \left[\, \frac{\partial U_A}{\partial X_{As}}\ - \nu\ \frac{\partial U_B}{\partial X_{Bs}}\,\right]\ X^0_{Bs} &= 0
\end{aligned}$$

For these to yield

$$\frac{\partial U_A}{\partial X_{As}}\ - \nu\ \frac{\partial U_B}{\partial X_{Bs}} = 0 \qquad \text{for all } s$$

we should have $S = 2$, and the matrix formed by the (X^0_{As}) and (X^0_{Bs}) coefficients should be non-singular, that is, the endowment vectors of the two people across the states should not be perfectly correlated.

Now let us examine the connection between this constrained Pareto optimal allocation and the equilibrium of competitive (price-taking) trading between the two people of shares in their enterprises.

Let p denote the price of A's enterprise measured in units of B's enterprise. Therefore, if A sells a fraction ϕ of his enterprise to B, he will get the fraction $p\,\phi$ of B's enterprise. Then his consumption quantities will be

$$X_{As} = (1 - \phi)\ X^0_{As} + p\,\phi\ X^0_{Bs}\,.$$

He chooses ϕ to maximize utility. The first-order condition for that is

$$\sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}}\ [\,-X^0_{As} + p\ X^0_{Bs}\,] = 0\,,$$

or

$$\sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}}\ X^0_{As} = p\ \sum_{s=1}^{S} \frac{\partial U_A}{\partial X_{As}}\ X^0_{Bs}\,.$$

A similar optimization for B yields his condition

$$\sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}}\ X^0_{As} = p\ \sum_{s=1}^{S} \frac{\partial U_B}{\partial X_{Bs}}\ X^0_{Bs}\,.$$

(You should actually derive this; that will be a good exercise for improving your understanding of the material.)

Dividing A's equation by the corresponding sides of B's immediately yields (2), the condition for Pareto optimality of allocation using shares. Therefore the "constrained equilibrium" is "constrained Pareto efficient", the constraint in both cases being the need to use shares instead of complete Arrow-Debreu securities.

An alternative approach to the same question uses the "slick proof" of Pareto efficiency of competitive equilibrium on p. 16 of Note 11. Think of shares just like goods. A is endowed with one unit of the good "A-share", and similarly B is endowed with one unit of the good "B-share". Suppose that as a result of trade, A ends up with ("consumes") ϕ_A and ψ_A units of the respective shares, and B ends up with ϕ_B and ψ_B units. Then A's consumption quantities of the state-contingent actual goods are

$$X_{As} = \phi_A\ X^0_{As} + \psi_A\ X^0_{Bs}\,,$$

and he gets utility which we can express as a function of ϕ_A and ψ_A:

$$V_A(\phi_A, \psi_A) \equiv U_A(\phi_A\,X^0_{A1} + \psi_A\,X^0_{B1},\ \phi_A\,X^0_{A2} + \psi_A\ X^0_{B2},\ \ldots\ \phi_A\ X^0_{AS} + \psi_A\ X^0_{BS})\,.$$

This is obviously an increasing function of (ϕ_A, ψ_A). Denote the prices of the shares by q_A and q_B respectively. Then A's budget constraint is that what he spends on buying B-shares should not exceed what he gets by selling A-shares, that is,

$$q_B \, \psi_A \leq q_A \, (1 - \phi_A) \,,$$

or

$$q_A \ \phi_A + q_B \ \psi_A \leq q_A \ * \ 1 + q_B \ * \ 0 \,,$$

where I have added the (zero) value of A's (zero) endowment of B-shares on the right hand side to make the constraint look exactly like the familiar budget constraint in an exchange economy with two goods. In a competitive market, with given share prices (q_A, q_B), the optimal choice (ϕ_A, ψ_A) will maximize $V_A(\phi_A, \psi_A)$ subject to this budget constraint.

Similarly, B will choose (ϕ_B, ψ_B) to maximize

$$V_B(\phi_B, \psi_B) \equiv U_A(\phi_B \, X^0_{A1} + \psi_B \, X^0_{B1}, \, \phi_B \, X^0_{A2} + \psi_B \ X^0_{B2}, \, \ldots \, \phi_B \ X^0_{AS} + \psi_B \ X^0_{BS}) \,.$$

Denote the prices of the shares by q_A and q_B respectively. Then A's budget constraint is that what he spends on buying B-shares should not exceed what he gets by selling A-shares, that is,

$$q_A \, \phi_B \leq q_B \, (1 - \psi_B) \,,$$

or

$$q_A \ \phi_B + q_B \ \psi_B \leq q_A \ * \ 0 + q_B \ * \ 1 \,,$$

A central planner seeking a Pareto efficient allocation will allocate the shares between the two to maximize $V_A(\phi_A, \psi_A)$ subject to $V_B(\phi_B, \psi_B) \geq k$, and feasibility constraints

$$\phi_A + \phi_B \leq 1, \qquad \psi_A + \psi_B \leq 1 \,.$$

The formal structure is identical to that of the calculus-based analysis of Pareto efficiency in Note 11; therefore the result applies and need not be proved afresh.

However, this result does not survive much generalization to several periods etc. I must leave these refinements for more advanced mathematical economics or finance courses. A couple of references: The equivalence result just proved is from Diamond's paper which is an optional reading in topic 4. You don't have to read the Diamond paper, but you are expected to know what is covered in these notes. For the more advanced materials alluded to just above (truly optional), you can read Mas-Colell, Whinston, and Green, *Microeconomic Theory,* chapter 19 and the references cited at the end of that chapter.