ECO 317 – Economics of Uncertainty – Fall Term 2009
Tuesday October 6
Portfolio Allocation – Mean-Variance Approach

Validity of the Mean-Variance Approach

Constant absolute risk aversion (CARA):

$$u(W) = - \exp(- \alpha W)$$

Final wealth W will be a random variable, whose distribution is affected by the allocation choices

Assume normal distribution: mean $\mathsf{E}[W]$, Variance $\mathsf{V}[W]$

These are functions of the allocation choices

$$EU = - \mathsf{E}[\exp(- \alpha W)] = - \exp\{ - \alpha \mathsf{E}[W] + \tfrac{1}{2} \alpha^2 \mathsf{V}[W] \}$$

So maximizing EU is equivalent to

$$\text{minimizing} \qquad - \alpha \mathsf{E}[W] + \tfrac{1}{2} \alpha^2 \mathsf{V}[W]$$

or

$$\text{maximizing} \qquad \mathsf{E}[W] - \tfrac{1}{2} \alpha \mathsf{V}[W]$$

One Riskless, One Risky Asset

Safe asset: gross return rate R (1 plus interest rate)

Risky asset: random gross return rate r

Mean $\mu = \mathsf{E}[r] > R$, Variance $\sigma^2 = \mathsf{V}[r]$

Initial wealth W_0. If x in risky asset,

final wealth $W = (W_0 - x)\,R + x\,r = R\,W_0 + (r - R)\,x$

$$\begin{aligned} \mathsf{E}[W] &= W_0\,R + x\,(\mu - R) \\ \mathsf{V}[W] &= x^2\,\sigma^2; \quad \text{Std. Dev.} = x\,\sigma \end{aligned}$$

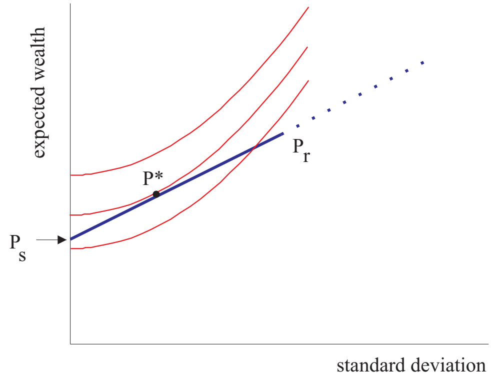
Choose x to maximize $W_0\,R + x\,(\mu - R) - \frac{1}{2}\,\alpha\,x^2\,\sigma^2$

FOC $\mu - R - a\,x\,\sigma^2 = 0$, therefore optimum

$$x = \frac{\mu - R}{\alpha\,\sigma^2}$$

Observe x independent of W_0. CARA-Normal model under uncertainty is like quasi-linear utility in ordinary demand theory.

As x varies, straight line in (Mean,Std.Dev.) figure.



$P_s = (0, W_0\, R)$ safe; $P_r = (W_0\, \sigma, W_0\, \mu)$ risky;

Beyond P_r possible if leveraged borrowing OK

(In dotted line as shown if borrowing rate $=$ safe rate R;

with kink if borrowing rate $>$ safe rate.)

P^* is optimal portfolio

Two Risky Assets

$W_0 = 1$; Random gross return rates r_1, r_2

Means $\mu_1 > \mu_2$; Std. Devs. σ_1, σ_2, Correl. Coefft. ρ

Portfolio $(x, 1-x)$. Final $W = x\, r_1 + (1-x)\, r_2$

$$\mathsf{E}[W] = x\,\mu_1 + (1-x)\,\mu_2 = \mu_2 + x\,(\mu_1 - \mu_2)$$

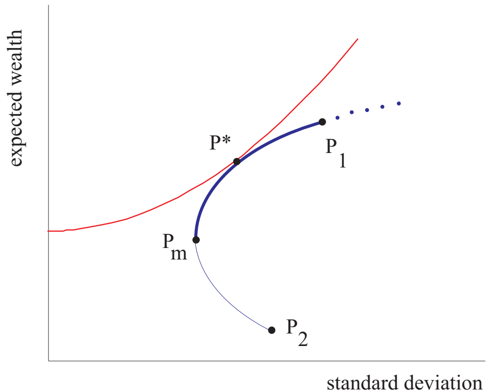
$$\begin{aligned} \mathsf{V}[W] &= x^2\,(\sigma_1)^2 + (1-x)^2\,(\sigma_2)^2 + 2\,x\,(1-x)\,\rho\,\sigma_1\,\sigma_2 \\ &= (\sigma_2)^2 - 2\,x\,[\,(\sigma_2)^2 - \rho\,\sigma_1\,\sigma_2\,] + x^2\,[(\sigma_1)^2 - 2\,\rho\,\sigma_1\,\sigma_2 + (\sigma_2)^2] \end{aligned}$$

$$\frac{\partial \mathsf{V}[W]}{\partial x} = \begin{cases} -\,2\,[(\sigma_2)^2 - \rho\,\sigma_1\,\sigma_2] & \text{at } x = 0 \\ 2\,[(\sigma_1)^2 - \rho\,\sigma_1\,\sigma_2] & \text{at } x = 1 \end{cases}$$

So diversification can reduce variance if $\rho < \min\,[\sigma_1/\sigma_2, \sigma_2/\sigma_1]$

$$\text{To minimize variance,} \quad x = \frac{(\sigma_2)^2 - \rho\,\sigma_1\,\sigma_2}{(\sigma_1)^2 - 2\,\rho\,\sigma_1\,\sigma_2 + (\sigma_2)^2}$$

Optimum: $$x = \frac{\dfrac{\mu_1 - \mu_2}{\alpha} + (\sigma_2)^2 - \rho\,\sigma_1\,\sigma_2}{(\sigma_1)^2 - 2\,\rho\,\sigma_1\,\sigma_2 + (\sigma_2)^2}$$



P_1, P_2 points for each asset; P_m minimum-variance portfolio, P^* optimum
Portion $P_2\,P_m$ dominated; $P_m\,P_1$ efficient frontier
Continuation past P_1 if short sales of 2 OK

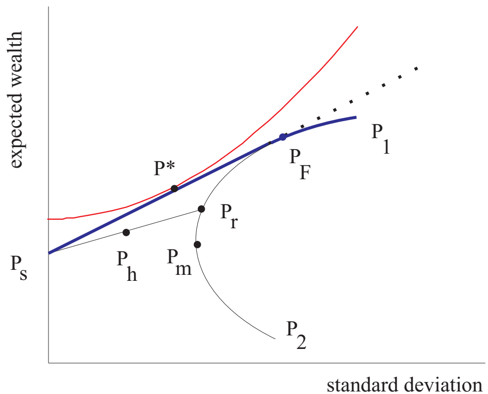
One Riskless, Two Risky Assets

First combine two riskies; this gets

  all points like P_h on all lines like $P_s\, P_r$

Then mix with riskless; this gets

  Efficient frontier $P_s\, P_F$ tangential to risky combination curve



Then along curve segment $P_F\, P_1$ if no leveraged borrowing;
continue straight line $P_s\, P_F$ if leveraged borrowing OK

With preferences as shown, optimum P^*
mixes safe asset with particular risky combination P_F
"Mutual fund" P_F is the same for all investors
regardless of risk-aversion (so long as optimum in $P_s\,P_F$)
Investors who are even less risk-averse may go beyond P_F
including corner solution at P_1
or tangency past P_1 if can sell 2 short to buy more 1

Capital Asset Pricing Model

Individual investors take the rates of return as given
but these must be determined in equilibrium

Suppose one safe and two risky assets

Investor h with initial wealth W_h
Invests x_1^h dollars in the shares of firm 1,
x_2^h dollars in the shares of firm 2,
and $(W_h - x_1^h - x_2^h)$ in the safe asset.

Expression for random final wealth $W =$

$$(W_h - x_1^h - x_2^h)\, R + x_1^h\, r_1 + x_2^h\, r_2 = W_h\, R + x_1^h\, (r_1 - R) + x_2^h\, (r_2 - R)\,,$$

Maximizes

$$\mathsf{E}[W] - \tfrac{1}{2}\, \alpha_h\, \mathsf{V}[W]$$

where

$$\begin{aligned} \mathsf{E}[W] &= W_h\, R + x_1^h\, (\mathsf{E}[r_1] - R) + x_2^h\, (\mathsf{E}[r_2] - R) \\ \mathsf{V}[W] &= (x_1^h)^2\ \mathsf{V}[r_1] + 2\, x_1^h\, x_2^h\, \mathsf{Cov}[r_1, r_2] + (x_2^h)^2\ \mathsf{V}[r_2] \end{aligned}$$

FOCs for optimal portfolio choice (allowing short sales etc. if necessary)

$$\begin{aligned} \mathsf{E}[r_1] - R &= \alpha_h \; \{ \; x_1^h \, \mathsf{V}[r_1] + x_2^h \, \mathsf{Cov}[r_1, r_2] \; \} \\ \mathsf{E}[r_2] - R &= \alpha_h \; \{ \; x_1^h \, \mathsf{Cov}[r_1, r_2] + x_2^h \, \mathsf{V}[r_2] \; \} \end{aligned}$$

like "inverse demand functions".

Rewrite these equations as

$$\begin{aligned} \tau_h \; \{ \, \mathsf{E}[r_1] - R \, \} &= x_1^h \, \mathsf{V}[r_1] + x_2^h \, \mathsf{Cov}[r_1, r_2] \\ \tau_h \; \{ \, \mathsf{E}[r_2] - R \, \} &= x_1^h \, \mathsf{Cov}[r_1, r_2] + x_2^h \, \mathsf{V}[r_2] \end{aligned}$$

where $\tau_h = 1 \, / \, \alpha_h$ is the investor's *risk-tolerance*.
Sum these across all investors. Impose equilibrium condition:
Total dollars invested $=$ total values of the firms F_1, F_2.
Take F_1, F_2 as given here; related to firms' profits in Note 6.

$$T\{ \, \mathsf{E}[r_1] - R \, \} = F_1 \, \mathsf{V}[r_1] + F_2 \, \mathsf{Cov}[r_1, r_2] \tag{1}$$

$$T\{ \, \mathsf{E}[r_2] - R \, \} = F_1 \, \mathsf{Cov}[r_1, r_2] + F_2 \, \mathsf{V}[r_2] \tag{2}$$

where $T =$ sum of τ_hs is the *market's risk tolerance*.

The market rate of return r_m is weighted average

$$r_m = (\ r_1 \ F_1 + r_2 \ F_2 \ ) \, / \, (F_1 + F_2)$$

Then multiply (1) by F_1, (2) by F_2 and add:

$$\begin{aligned} & T \ (F_1 + F_2)\{ \, \mathsf{E}[r_m] - R \, \} \\ = \ & (F_1)^2 \, \mathsf{V}[r_1] + 2 \, F_1 \, F_2 \, \mathsf{Cov}[r_1, r_2] + (F_2)^2 \, \mathsf{V}[r_2] \\ = \ & \mathsf{V}[r_1 \, F_1 + r_2 \, F_2] = (F_1 + F_2)^2 \ \mathsf{V}[r_m] \end{aligned}$$

or

$$\mathsf{E}[r_m] - R = \frac{F_1 + F_2}{T} \ \mathsf{V}[r_m]$$

Risk premium on the market as a whole is
 $\sim$ variance of the market rate of return, and
 $\sim$ 1 / market's risk tolerance
Factor $(F_1 + F_2)/T$ is the *market price of risk*
 It is endogenous in the whole equilibrium.

Similar work with FOC for asset 1 yields:

$$\begin{aligned} \mathsf{E}[r_1] - R &= \frac{F_1 + F_2}{T} \ \mathsf{Cov}[r_1, r_m] \\ &= \frac{\mathsf{Cov}[r_1, r_m]}{\mathsf{V}[r_m]} \ \{ \ \mathsf{E}[r_m] - R \ \} \end{aligned}$$

This gives two important conclusions

$$\mathsf{E}[r_1] - R = \frac{\mathsf{Cov}[r_1, r_m]}{\mathsf{V}[r_m]} \ \{ \ \mathsf{E}[r_m] - R \ \}$$

Risk premium on firm-1 stock depends on its
systematic risk (correlation with whole market) only,
not *idiosyncratic* risk (part uncorrelated with market)
Coefficient is *beta* of firm-1 stock

The risk premium in the market on any one stock depends on
the covariance of returns between the stock and the market
not on variance of the stock itself.

The "idiosyncratic" risk in one stock
(the part that is not correlated with the market)
can be diversified away, so investor not paid for bearing it

The risk in the whole market must be borne by the
collectivity of investors, so this earns a risk premium
proportional to their collective risk aversion $1/T$.