ECO 199 – GAMES OF STRATEGY
Spring Term 2004
PROBLEM SET 4 – DRAFT ANSWER KEY

The distribution of grades was as follows.

Range	Numbers
100-	3
90-99	21
80-89	14
70-79	4
0-69	11

Question 1: 30 points

Games of Strategy, Chapter 11, Exercise 11, pp. 377-8.

COMMON ERRORS: [1] Some mistakes in computing the pay-offs in (c): the money that the players put into the escrow account comes out of their own pockets. [2] Also in (c), most students had trouble demonstrating why using the escrow account in order to sustain cooperation at date 2 was a best response at date 1 (-5).

(a) (4 points) If both choose Low, each gets his own low-card value, 2 to Row and 4 to Column. If both choose High, each gets the other's high-card value, 8 to Row and 7 to Column. If Row chooses Low and Column chooses High, Row gets his own low-card value 2 and Column's high-card value 8 for a total of 10; Column gets 0. If Row chooses High and Column chooses Low, Column gets his own low-card value 4 and Row's high-card value 7 for a total of 11; Row gets 0.

(b) (4 points) The Nash equilibrium in the one-stage game is (Low, Low). Low is a (strictly) dominant strategy for both players. The game is a prisoners' dilemma: for both players, the cheater payoff exceeds the cooperative payoff, which exceeds the defect payoff, which exceeds the loser payoff: For Row, 10 > 8 > 2 > 0; for Column, 11 > 7 > 4 > 0.

(c) Consider the second stage (8 points) of the two-stage game. Call the amount that Row puts in the escrow account (to reward Column for cooperative behavior) r, and the amount that Column puts in the escrow account (to reward Row for cooperative behavior) c. The second stage payoff table is thus:

		Column	
		Low	High
Row	Low	2, 4	10-r, 0+r
	High	0+c, 11-c	8-r+c, 7+r-c

For Row, High is a weakly dominant strategy when c = 2, and a strictly dominant strategy when c > 2.; for Column, High is a weakly dominant strategy when r = 4, and a strictly dominant strategy when r > 4. For these values of c and r, therefore, (High,High) is a Nash equilibrium.

For a tie-breaking rule, assume that a player always chooses High when both actions would give her the same payoff. This is the rule used here. (For the opposite rule, c will have to be slightly greater than 2 and r slightly greater than 4, in each case by the smallest monetary unit available. So stating this assumption, inferring that c = 3 and r = 5, and following through the subsequent analysis, is also correct.)

Now consider the first stage (10 points). Here each player chooses between making a payment into the escrow account to alter the other's incentive, or not. Given the tie-breaking rule stated and used here, if Column wants to alter Row's incentive, it should set c = 2 (setting it higher hurts Column and has no impact on Row's choice); otherwise, Column should set c = 0. Similarly, Row should set r equal to either 4 or 0.

Thus Column has two strategies in this first stage: set c = 0 or set c = 2; Row also has two strategies: set r = 0 or set r = 4. We can then show the first-stage payoff matrix:

		Column	
		c = 2	c = 0
Row	r = 4	6, 9	6, 4
	r = 0	2, 9	2, 4

The rollback equilibrium is then (r = 4, c = 2) in the first stage, and (High,High) in the second stage. The prisoners' dilemma is resolved in this game; players attain the joint-payoff- maximizing outcome of (High,High).

(4 points) The escrow accounts provide a way to credibly commit to sharing the benefits of cooperative behavior. In any prisoners' dilemma, Player A's decision to Cooperate always reduces his own payoff, but also increases the joint payoff received by the full group. In other words, Player B's gain exceeds the Player A's loss. If Player B has a credible way to share that gain, it would be in her best interest to do so in order to induce Player A to cooperate. By providing a side payment of appropriate size, Player B can create a situation in which A's cooperation helps both players. The escrow system provides just such a credible way to share the gains from cooperation, and thus to induce cooperative behavior.

(Additional information: The escrow method achieves the jointly optimal solution of the prisoners' dilemma, but the payoffs are somewhat redistributed because of the asymmetry between the escrow amounts needed from the two players. Row gets 6 instead of 8, and Column gets 9 instead of 7. When we played this game in class, to ensure each student the same potential total earning, each player alternated the roles of Row and Column for an even number of total plays against different opponents.)

Question 2: 20 points

Games of Strategy, Chapter 12, Exercise 4, pp. 421.

COMMON ERRORS: [1] In (a), failing to describe what type of collective action game was played (-3) or why it was a prisoner's dilemma (-2). [2] In (d), some students did not realize that the main problem was the unilateral incentives to deviate not the asymmetry of pay-offs.
NICE JOB: Some students gave nice and detailed discussions of enforcement mechanisms in (e) (e.g. social norms, information requirements) and got +1.

(a) (4 points) This game is a prisoners' dilemma since [1] $s(n)$ is greater than $p(n + 1)$ for all n. Any single player can always raise his payoff by switching from participating to shirking, and [2] $p(100) = 100 > 4 = s(0)$.. Everyone gets a higher payoff when everyone participates than when everyone shirks.

(b) (2 points) $T(n) = n^2 + (100 - n)(4 + 3n)$
(c) (8 points) Plugging the appropriate values into the text's Eq. (12.1) yields

$$\begin{aligned} T(n+1) - T(n) &= n + 1 - 4 - 3n + n(n + 1 - n) + (100 - n - 1)(4 + 3n + 3 - 4 - 3n] \\ &= -3 - 2n + n + 3(100 - n - 1) \\ &= 294 - 4n \end{aligned}$$

This formula is positive for all n up to $n = 73$, but turns negative for $n = 74$ and higher. Thus, the last time the change in total social payoff expression is positive is for $T(74) - T(73)$ T is thus maximized at $n = 74$.

Using calculus, you can find the same answer by differentiating T(n) from part (b). $T'(n) = 2n + 3(100 - n) - (4 + 3n) = 296 - 4n$. Set this derivative equal to zero to find that $T(n)$ is maximized at $n = 74$.

(d) (3 points) When $n = 74$, each participant receives a benefit $p(74) = 74$. If one participant were to switch to shirking, he would receive a benefit $s(73) = 4 + 3(73) = 223$. Each participant thus has a private incentive to become a shirker (as is always the case in a prisoners' dilemma).

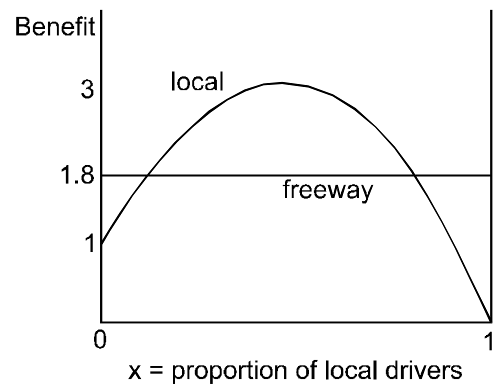
(e) (3 points) If the game is repeated, the players can take turns playing the role of shirker (and receiving the higher private benefit). Social norms or the threat of future punishment might keep each player in his assigned role for that game. A similar procedure might work even if the players' future interaction was in a different game.

Question 3: 25 points

Games of Strategy, Chapter 12, Exercise 6, pp. 422.

COMMON ERRORS: [1] In (b), most mistakes came from the endpoints: either ignoring x=0 (-2) or including x=1 (-2). [2] In (c), some students only looked at the benefits of using the local road or failed to compute the weighted average (-6).

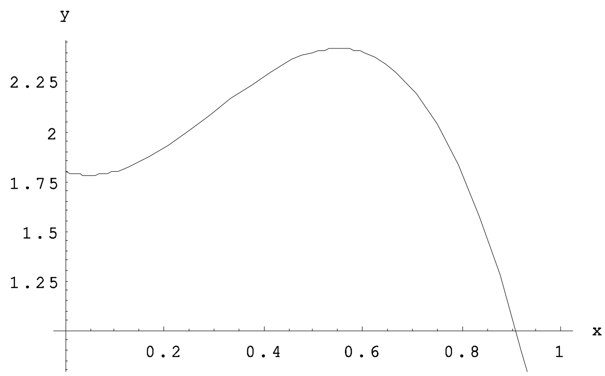
(a) (7 points) Diagram to the right.

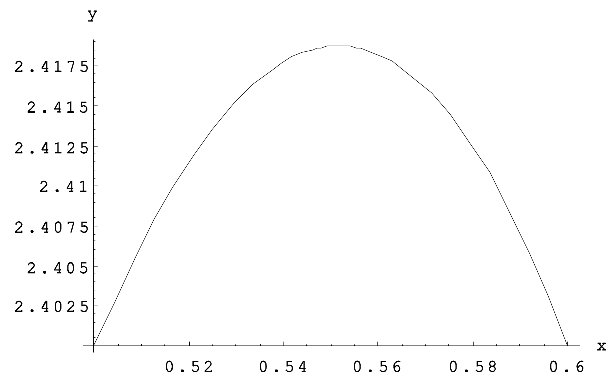


(b) (6 points) There are three possible Nash equilibria. At $x = 0$ (nobody on the local roads), the benefit on the highway exceeds the benefit on local roads; nobody will switch to a local road. At $x = 0.1$, the benefits are equal. This is, however, an unstable equilibrium, since if $x < 0.1$, the number of people driving the local road will fall, while if $x > 0.1$, local road usage will rise. At $x = 0.8$, benefits are again equal. This is a stable equilibrium: for $0.1 < x < 0.8$, local usage rises, for $x > 0.8$, local usage falls. Note that the other endpoint ($x = 1$) is not an equilibrium. When everybody is on the local roads, any driver could help himself by switching to the highway.

(c) (12 points) Total benefit = $x(1 + 9x - 10x^2) + 1.8(1 - x) = 1.8 - 0.8x + 9x^2 - 10x^3$. The derivative of this is $-0.8 + 18x - 30x^2$. Set this equal to zero and solve the resulting quadratic equation for x. This yields x = 0.05 and 0.55 . The second derivative $18 - 60x$ is positive when x = 0.05 and negative when x = 0.55, so the latter yields the maximum. (Additional information for the mathematically advanced: The total benefit function has a local maximum at x = 0. So we should make a discrete comparison between the two local maxima to find the global maximum, and it is easy to verify that x = 0.55 yields higher total benefit.)

Numerical calculations can also be used, with a more refined search between x = 0.5 and 0.6. Or a graph with a scale large enough to enable inspection. A graph over the full domain of x from 0 to 1, and one over the smaller relevant range from 0.5 to 0.6, are shown below. Note the slight dip near x = 0, yielding a local maximum at 0 and the local minimum at x =0.05.





Question 4: 25 points

Games of Strategy, Chapter 13, Exercise 3, pp. 464.

COMMON ERRORS: [1] In (e), including the polymorphic equilibrium among the ESS (-2). [2] In (d), failing to recognize that the cut-off p=(n-2)/(n-1) was increasing with n (-3).

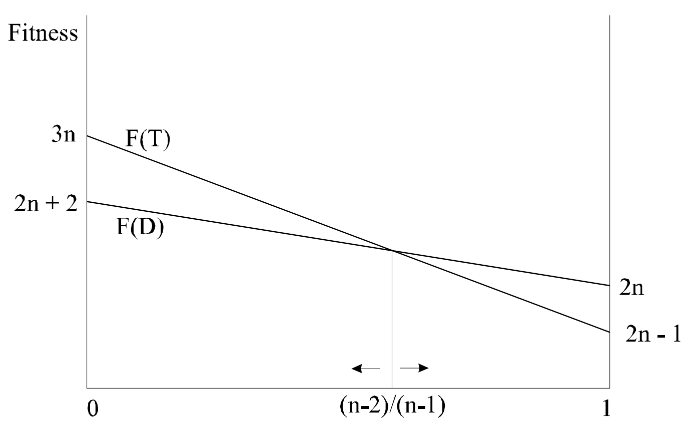
(a) (8 points) Let D represent an Always Defector, and T a Tit-for-Tat player.

		Player 2	
		D	T
	D	2n , 2n	2n+2 , 2n-1
Player 1	T	2n-1 , 2n+2	3n , 3n

(b,c) (2 points) F(D) = p 2n + (1-p) (2n+2) = 2 (n+1-p); F(T) = p (2n-1) + (1-p) 3n = 3n - p(n+1)

(d) (6 points) F(D) > F(T) when 2n+2 - 2p > 3n - n p - p, or 2 - p > n - n p. This yields
n p - p > n - 2 OR p (n - 1) > n - 2, OR p > (n-2) / (n-1).
Similarly, F(T) > F(D) when 3n - n p - p > 2n + 2 - 2p or when n - n p > 2 - p. This yields
n - 2 > n p - p, OR n - 2 > p (n - 1), OR p < (n - 2) / (n - 1) .

(e) (5 points)



x = proportion of defectors (D-types)

Thus there are two pure ESS: all-A, and all-T. (At the point $p = (n-2)/(n-1)$ the two types have equal fitness, but this polymorphism is not an ESS because if a few more of either pure type arrive, that type will do better than the population and the proportion will move toward one of the extremes.)

(f) (4 points) The population will move towards the all-tit-for-tat equilibrium (in which cooperation always occurs) whenever the original $p < (n-2)/(n-1)$. As n rises, $(n-2)/(n-1)$ also rises (approaching 1), which increases the likelihood that a given p will meet this condition.

Intuitively, the possible advantage of being hard-wired to play tit-for-tat is that it gives a player the chance to establish long-lasting (and beneficial) cooperation if it meets another tit-for-tat player. The disadvantage is that a tit-for-tat player is left open for one-time exploitation when it meets an always defector. As the number of rounds rises, the possible benefits of (long-lasting) cooperation rise relative to the possible one-time cost. Thus, a larger n shifts the relative benefit-cost ratio in favor of tit-for-tat players.