ECO 305 – Fall 2003 – November 6

General Equilibrium and Efficiency – Mathematical Treatment

1 Framework and Assumptions

The themes are similar to those of partial equilibrium – market equilibrium and efficiency – but now we consider all markets simultaneously, and pay more explicit attention to their interactions. This makes the assumption of quasi-linear preferences inappropriate, and we do not impose it here.

We begin by listing the framework and assumptions that will be maintained throughout the analysis:

Technology and preferences: [1] Production functions are concave and preferences show diminishing MRS. [2] There are no public goods and no externalities in production or in consumption.

For market equilibrium: [1] There are markets for all economically relevant (valued by consumers and having a positive opportunity cost to produce) commodities (goods and services). [2] The commodities are unambiguously defined; there is no asymmetric information about the quality of goods or labor etc. [3] All consumers and firms are perfect competitors (act as price-takers) in all these markets.

For efficient planning: [1] There is complete information about all resource endowments, preferences, and production possibilities.

2 Notation:

Individuals (households) are labelled $h = 1, 2 \ldots H$, goods $g = 1, 2 \ldots G$, and factors $f = 1, 2 \ldots F$. Then

$$\begin{aligned}
x_{hg} &= \text{amount of good } g \text{ consumed by individual } h, \text{ (Vector } \mathbf{X}_h) \\
v_{hf} &= \text{amount of factor } f \text{ supplied by individual } h, \text{ (Vector } \mathbf{V}_h) \\
z_{fg} &= \text{amount of factor } f \text{ used in producing good } g, \text{ (Vector } \mathbf{Z}_g) \\
X_g &= \text{total amount of good } g \text{ produced} \\
U_h &= \text{utility of individual } h\,.
\end{aligned}$$

Production function for each good g

$$X_g = X_g(\mathbf{Z}_g) = X_g(z_{1g}, z_{2g} \ldots z_{Fg})\,. \tag{1}$$

Utility function for each household h

$$U_h = U_h(\mathbf{X}_h, \mathbf{V}_h) = U_h(x_{h1}, \ldots x_{hG}\,, v_{h1} \ldots v_{hF})\,. \tag{2}$$

Note that factors can be split up into those used in the production of one good and not in any other, and goods can be split up into amounts consumed by one consumer and not another, because of the assumption that there are no public goods and no externalities. If this assumption fails, the total amount of a public good can enter everyone's utility function, or the amount produced or consumed by one can affect the output or utility of another. We will consider these effects later in the course. The ambitious can try out now how the various conditions below will be changed if there are such effects.

Constraints: Material balance for each good g:

$$x_{1g} + x_{2g} + \ldots + x_{Hg} \leq X_g\,, \tag{3}$$

and for each factor f:

$$z_{f1} + z_{f2} + \ldots + z_{fG} \leq v_{1f} + v_{2f} + \ldots + v_{Hf}\,. \tag{4}$$

3 Optimization

Suppose a benevolent dictator plans all production and allocation to maximize social welfare

$$W = W(U_1, U_2 \ldots U_H) \tag{5}$$

subject to the constraints (1), (2), (3) and (4).

Substitute out the conditions (1) and (2) that define outputs and utilities. Keep the constraints of material balance, and define Lagrange multipliers α_g for the goods balance constraints and β_f for the factor balance constraints. Then, omitting arguments of functions for brevity, the Lagrangian is

$$\mathcal{L} = W + \sum_{g=1}^{G} \alpha_g \left[X_g - \sum_{h=1}^{H} x_{hg} \right] + \sum_{f=1}^{F} \beta_f \left[\sum_{h=1}^{H} v_{hf} - \sum_{g=1}^{G} z_{fg} \right] .$$

The first-order conditions are: For goods consumption quantities x_{hg}:

$$\frac{\partial W}{\partial U_h} \frac{\partial U_h}{\partial x_{hg}} - \alpha_g = 0 . \tag{6}$$

For factor supplies v_{hf}:

$$\frac{\partial W}{\partial U_h} \frac{\partial U_h}{\partial v_{hf}} + \beta_f = 0 . \tag{7}$$

For factor uses z_{fg}:

$$\alpha_g \frac{\partial X_g}{\partial z_{fg}} - \beta_f = 0 . \tag{8}$$

Note the difference of sign in the conditions for goods' consumption and factor supplies – this is because goods provide utility and labor supply gives disutility to consumers. If some factors (land) can be supplied without disutility, then their quantities will be set equal to the maximum each consumer possesses, and (7) will be replaced by inequality conditions for a boundary solution. This will not be discussed in any detail here, as it does not raise any important issues for our present purpose.

We can rewrite the conditions to bring out their relation to the geometric treatment of November 4. We do this by picking conditions in suitable pairs, and dividing the two equations in each pair:

Consumption efficiency: For any two goods, say 1 and 2,

$$\frac{\partial U_h / \partial x_{h1}}{\partial U_h / \partial x_{h2}} = \frac{\alpha_1}{\alpha_2}, \text{ same for all } h .$$

The left hand side is the marginal rate of substitution between goods 1 and 2 along an indifference curve for household h; thus the MRS between any fixed pair of goods should be equal for all households. This is the tangency condition for goods allocation in the exchange Edgeworth box of Figure 1 of November 4. Then the right hand side, α_1/α_2 should play the role of the relative price of the two goods. We will soon see that such is indeed the case.

Production efficiency: For any two factors, say 1 and 2,

$$\frac{\partial X_g / \partial z_{1g}}{\partial X_g / \partial z_{2g}} = \frac{\beta_1}{\beta_2}, \text{ same for all } g .$$

The left hand side is the marginal rate of technical substitution between factors 1 and 2 along an isoquant for good g; thus the RTS should be equal for all goods. This is the tangency condition for factor allocation in the Edgeworth box of Figure 3 of November 4. Then the right hand side, β_1/β_2, should play the role of the relative price of the two factors, as indeed it will.

Joint efficiency: For any one good g and any one factor f, and for all h

$$\frac{-\partial U_h/\partial v_{hf}}{\partial U_h/\partial x_{hg}} = \frac{\beta_f}{\alpha_g} = \frac{\partial X_g}{\partial z_{fg}} \,.$$

This says that the marginal extra amount of good g that will induce any consumer to supply another marginal unit of factor f should equal the rate at which the production process can transform factor f into good g. This corresponds to the tangency in Figure 4 from November 4.

4 Equilibrium

Let $\mathbf{p} = (p_g)$ be the vector of prices of goods and $\mathbf{w} = (w_f)$ the vector of prices of factors. We specify the behavior of firms first; then households. We suppose that good g is made by just one firm, and define its profit function (this needs concave production functions)

$$\pi_g(p_g, \mathbf{w}) = \max\, p_g\, X_g(z_{1g}, \ldots z_{Fg}) - \sum_{f=1}^{F} w_f\, z_{fg} \,.$$

Then the output supply and factor demand functions are given by Hotelling's Lemma:

$$X_g = \partial\pi_g/\partial p_g \,, \qquad z_{fg} = -\partial\pi_g/\partial w_f \,. \tag{9}$$

In the present context of perfect competition (price-taking) and no externalities, the assumption of one firm per good is harmless: if several firms are making a good, their profit-maximization is carried out independently and simultaneously, and we can effectively regard them as one firm whose profit function is the sum of those of the actual separate ones (and whose output supply function and factor demand functions are likewise sums). Conversely, multi-product firms can be allowed by making the profit function depend on all the relevant output prices.

If I_h denotes the income of households other than that derived from their sales of factor services, their budget constraints are

$$\sum_{g=1}^{G} p_g\, x_{hg} \le \sum_{f=1}^{F} w_f\, v_{hf} + I_h \,.$$

Maximizing utility subject to this gives the demand functions

$$x_{hg} = x_{hg}(\mathbf{p}, \mathbf{w}, I_h) \,, \qquad v_{hf} = v_{hf}(\mathbf{p}, \mathbf{w}, I_h) \,. \tag{10}$$

Finally, we tie together households' incomes and firms' profits. (This is like the "circular flow of income" concept of macroeconomics in ECO 101). Suppose household h owns a fraction θ_{hg} of firm g. Then

$$I_h = \sum_{g=1}^{G} \theta_{hg}\ \pi_g(p_g, \mathbf{w}) \,. \tag{11}$$

Equilibrium is defined by the market-clearing conditions for all goods g

$$\sum_{h=1}^{H} x_{hg}\left(\mathbf{p}, \mathbf{w}, \sum_{g=1}^{G} \theta_{hg}\, \pi_g(p_g, \mathbf{w})\right) = X_g(p_g, \mathbf{w}) \,, \tag{12}$$

and all factors f

$$\sum_{g=1}^{G} z_{fg}(p_g, \mathbf{w}) = \sum_{h=1}^{H} v_{hf}\left(\mathbf{p}, \mathbf{w}, \sum_{g=1}^{G} \theta_{hg}\, \pi_g(p_g, \mathbf{w})\right) . \tag{13}$$

Every household and firm makes its decisions accepting the ruling prices (p_g) and (w_f), and the prices are determined by the requirement of market-clearing. Thus we are to regard prices as the "unknowns" that are solved from the equilibrium conditions.

5 Existence etc.

Since the demand and supply functions can in general be very nonlinear, the question arises whether the system has a solution, and if so, whether the solution is unique. As a step toward answering this, let us count the number of independent equations and unknowns. It may seem that there should be $F + G$ unknown prices (the components of the vectors $\mathbf{p}$ and $\mathbf{w}$), and $F + G$ market-clearing conditions to determine them. But matters are a little more complicated.

First note that all the demand and supply functions are homogeneous of degree zero in the prices $(\mathbf{p}, \mathbf{w})$ jointly. That is, if μ is any positive scalar, then all demand and supply quantities are unchanged if we replace $(\mathbf{p}, \mathbf{w})$ by $(\mu\,\mathbf{p}\,,\ \mu\,\mathbf{w})$. (Exercise: use the homogeneity properties of the profit function etc. to prove this claim.) In other words, only price ratios (relative prices) matter. Thus, among the $G + F$ unknowns $(p_g \mid g = 1, \ldots G)$ and $(w_f \mid f = 1 \ldots F)$, there are only $G + F - 1$ that matter. We can arbitrarily designate any one of the prices to equal 1. This is choosing a good to be the unit in which all other prices are measured; it is then called the *numeraire* good. Or we can choose any combination such as a weighted sum $\Sigma_g\, a_g\, p_g + \Sigma_f\, b_f\, w_f$ to equal 1; this is choosing a composite commodity, or a basket of quantities a_g of the goods and b_f of the factors, as a numeraire. (For example, if we choose the quantities that go into constructing the consumer price index as the basket, the CPI will always equal 1, so inflation will be eliminated by definition. Fortunately, no administration has yet used this ploy.)

Correspondingly, among the $G + F$ equations in the system (12) and (13), there is one redundancy. If we multiply the equations by the corresponding prices and add, we get

$$\begin{aligned}
& \sum_{g=1}^{G} p_g \left[\sum_{h=1}^{H} x_{hg} - X_g \right] + \sum_{f=1}^{F} w_f \left[\sum_{g=1}^{G} z_{fg} - \sum_{h=1}^{H} v_{hf} \right] \\
= & \sum_{h=1}^{H} \left[\sum_{g=1}^{G} p_g\, x_{hg} - \sum_{f=1}^{F} w_f\, v_{hf} \right] - \sum_{g=1}^{G} \left[p_g\, X_g - \sum_{f=1}^{F} w_f\, z_{fg} \right] \\
= & \sum_{h=1}^{H} I_h - \sum_{g=1}^{G} \pi_g \equiv 0\,.
\end{aligned}$$

So given any $G + F - 1$ of the equations, the remaining one follows. The above identity, which says that the values of the excess demands for all goods and factors add up identically to zero, is called *Walras' Law.*

Note that in equilibrium, each of the terms in sums that appear in the first line of the above equation chain would be zero because the market for each good and each factor would clear separately. Then the sum would be zero. The additional point about Walras' Law is that the sum must be zero whether or not the prices are at their equilibrium values. All or indeed any of the markets don't have to clear, but an excess supply in any one market must be matched by an excess demand of equal value in the other markets, and an excess demand in any one market must be matched by an excess supply of equal value in the other markets. For example, if there is excess supply in the labor market, there must be excess demand in the goods market. How can this be? The households are making plans as to how much labor to supply and are planning to spend on goods the income they are planning to earn from this labor supply. But the firms are not planning to hire that much labor, and accordingly are not planning to produce that much output. Of course, if any actual transaction are attempted at these out-of-equilibrium prices, some of the plans will go unfulfilled, and that will have repercussions elsewhere. For example, households who do not actually succeed in selling as much of their labor services as they would like may have to cut back on their plans for consuming goods. But that would take us into macroeconomics.

The result of the combination of homogeneity of demand and supply functions and Walras' Law is that the number of independent equations and unknowns matches, but in a more subtle way: each number is $G + F - 1$, not $G + F$.

Equal numbers of equations and unknowns doesn't guarantee a solution, let alone a unique solution, especially in the nonlinear case. But we can prove rigorously that if the demand and supply functions are continuous, then an equilibrium exists. The general idea is as follows. Start with a trial set of prices $(\mathbf{p}, \mathbf{w})$.

At these prices, we can calculate all consumers' and firms' plans. Set up an adjustment rule that raises the prices of all goods and factors that are in excess demand and lowers the prices of those goods and factors that are in excess supply. This gives us a new vector of prices $(\mathbf{p}, \mathbf{w})$. In other words, the adjustment rule is a mapping from the set of price vectors into itself. If everything – the demand and supply functions (this needs diminishing MRS) and the adjustment rule itself – is continuous and we normalize things in such a way that the set of all possible prices is convex, then a general mathematical theorem (Brower's fixed point theorem or some extensions of it by Kakutani and others) provies that this mapping has a fixed point, that is, a vector of prices that stays unchanged. Given how the mapping was defined, at the fixed point price vector there must be zero excess demand or excess supply in all markets, and that must be a general equilibrium.

However, there can be many such price vectors; uniqueness of general equilibrium cannot be guaranteed in general, and in many realistic examples there may be multiple equilibria.

Also, the adjustment rule specified above looks like a dynamic process, but its dynamic stability cannot be guaranteed either. If all goods are Marshallian or gross substitutes, the process must converge to a unique equilibrium. The idea is as follows. Suppose there is an excess supply in the market for good 1. By Walras' Law, there must be an excess supply in all other markets on the average. Consider another good, say 2, that has such an excess supply. The price of good 1 rises in response to the excess demand. If it is a gross substitute for good 2, then the demand for good 2 increases, which tends to "cure" the excess supply in that market. But if good 1 is a complment to good 2, then the increase in the price of good 1 decreases the demand for good 2, which aggravates the excess supply there. Thus in the substitutes case the price adjustment in one market on the whole helps equilibrate other markets too; in the complements case it can hurt.

I have given only the conceptual basis for the studies of existence, uniqueness, and stability. That is sufficient for ECO 305. Those of you who are interested in pursuing the advanced mathematics can read the classic treatment by Kenneth Arrow and Frank Hahn, *General Competitive Analysis,* Holden-Day 1970.

6 Equivalence

In a competitive equilibrium, the consumers' FOCs are

$$\frac{\partial U_h}{\partial x_{hg}} - \lambda_h \; p_g = 0 = \frac{\partial U_h}{\partial v_{hf}} + \lambda_h \; w_f \,. \tag{14}$$

The firm's profit-maximization FOCs are

$$p_g \; \frac{\partial X_g}{\partial z_{fg}} = w_f \tag{15}$$

We see an immediate correspondence between these and the social optimality conditions (6), (7) and (8) above. If the social welfare function is chosen so that $\partial W/\partial U_h = 1/\lambda_h$, then the equilibrium satisfies the social optimality FOCs, the Lagrange multipliers being $\alpha_g = p_g$ for all g and $\beta_f = w_f$ for all f. Thus a competitive equilibrium is an optimum corresponding to some social welfare function. Then it is impossible to increase the utility of any one household without lowering that of some other – the equilibrium is *Pareto optimal* or *Pareto efficient*.

Conversely, suppose we have found the optimum allocation for a particular social welfare function. In the process we have found the Lagrange multipliers (α_g) and (β_f). Now let these be the prices for goods and factors respectively. Using lump-sum transfers, arrange the consumers' non-wage incomes I_h so that $\lambda_h = 1/(\partial W/\partial U_h)$ for all h. Then the social optimality FOCs are also the consumers' and the firms' FOCs – we have implemented the social optimum as a competitive market equilibrium, or decentralized it.

Of course this is only a comparison of the FOCs, but the assumptions above that all consumers' preferences have diminishing MRS and all production functions are concave (nonincreasing returns to scale and diminishing returns to each factor individually) ensure that the FOCs yield true optima.

7 Uses of General Equilibrium Analysis

Most improtantly, general equilibrium emphasizes that the economy is an interconnected system, and any disturbance to one part of it propagates to a greater or smaller extent to all the other parts. In reality it takes time for a new equilibrium to get established (and in the meantime other disturbances may hit). So if you can think through the indirect effects and take some action before everyone else figures out, you can profit from this.

Next, the analysis clarifies the conditions that are needed for markets to yield an optimal outcome, something that is taken as unqualified faith by some and denied equally unthinkingly by others. Specifically, we need perfect competition (price-taking behavior), no externalities or public goods, complete markets, and no information asymmetries. In the absence of any of these, there is potential for beneficial government intervention. And markets produce a distribution of income that has no necessary ethical merit. But the analysis also clarifies the daunting computational task facing a social planner, and also clarifies that such a planner may face problems in getting individuals to reveal the necessary information.

However, subject to these provisos, the method has very a broad range of application. It readily adapts to analyze equilibrium over time. A good available at different times is economically two distinct commodities, since its manifestations at the different times are not perfect substitutes for either producer or consumers. So we can introduce a relative price between the two. For example, with two periods labeled 1 and 2, consider the budget constraint of a consumer who expects incomes I_1 and I_2 dollars in the two periods and plans to consume C_1 and C_2 dollars. Thinking of "a dollar of consumption in period t" as a good in itself, we can denote the prices of these two by p_1 and p_2 and write the budget constraint as

$$p_1\ C_1 + p_2\ C_2 = p_1\ I_1 + p_2\ I_2\,.$$

Subject to this, the consumer will maximize a utility $U(C_1, C_2)$. Now think of everything from the perspective of period 1. Take "a dollar in period 1" as the unit of account in which all other prices are measured, so $p_1 \equiv 1$. Then p_2 is the value in period-1 dollars of having a dollar in period 2, that is, the discounted present value of a period-2 dollar. We usually write this as $1/(1+r)$, where r is the dollar rate of interest between the two periods. Then the budget constraint expresses the equality of the discounted present values of consumption and income:

$$C_1 + \frac{1}{1+r}\ C_2 = I_1 + \frac{1}{1+r}\ I_2\,.$$

The right hand side of this is called the "present-value wealth". Thus the usual formulation of the saving problem in macroeconomics (ECO 304) becomes a special case of microeconomic general equilibrium analysis.

In the same way, general equilibrium handles uncertainty. We distinguish different "scenarios" or "states of nature" that affect economic outcomes – the farmer gets rain or shine, your car gets stolen or it doesn't, and so on. Various goods, or purchasing power more generally, is not equally valuable in different scenarios, and you may want to trade off some money in one scenario for that in another. Thus there arises the need for markets that allocate or trade risks. General equilibrium methods allow us to price these risks. Thus the method becomes the foundation for the analysis of insurance markets, and financial markets more generally. Note that we can have lots of uncertainty without having any asymmetric information – everyone has to be equally uninformed about which scenario will actually transpire.

Of all the qualifications to general equilibrium analysis pointed out above, the underlying assumptions of perfect competition and symmetric information give rise to the most drastic modifications of the approach. When firms (and sometimes consumers too) are not price-takers, they can engage in all kinds of strategic behavior whose analysis requires *game theory.* When firms or consumers have asymmetric information advantages or disadvantages, the strategies of signaling to convey information credibly, screening to elicit information truthfully, and incentive design to elicit appropriate effort, come into play. In fact these two modifications interact, because someone with special information *ipso facto* has some monopoly power in trades involving this information, and can engage in strategic behavior of information manipulation. This combination of game theory and the theory of asymmetric information has revolutionized economics over the last quarter-century and taken it far beyond what is covered in ECO 102. These new ideas will form the core of the rest of this course.