MULTIVARIATE TRANSVARIATION THEORY AMONG SEVERAL DISTRIBUTIONS AND ITS ECONOMIC APPLICATIONS

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1. Introduction

The theory of transvariation was introduced by C. Gini [1916] and developed by this author and his school of statistics.[1]

This paper dealt with nonparametric and Gaussian transvariation theory between two or more than two multivariate distribution functions and its applications to economics.

The following probabilistic notation and abbreviations will be used.

By $\Lambda_i (i=1,\dots,s)$, we denote a random experiment or random observation (r.e.). In each realization of an r.e. we are interested in the probabilities that certain events will occur. Thus, we are interested in the measurement or observation of numerical quantities associated with each r.e.. For example, one may be interested in the observations of prices, wages, measurements of output and so on. Such numerical quantities are sample realizations of a random variable (r.v.) $\xi_i (i=1,\dots,s)$ that can assume values in a space of n dimensions $(n \geq 1)$. Hence, a random variable $\xi_i = \xi_i(\omega_i)$ is a real valued function defined for all sample point ω_i of a basic probability space

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[1] For an exhaustive up-to-date bibliography on transvariation see Camilo Dagum [1968].

(Ω_i, F_i, P_i), for $i=1,\dots,s$. Ω_i denotes a sample space, wherein a sample point (elementary event) $\omega_i \in \Omega_i$, and a set E_i of sample points is a subset in the sample space, such that $E_i \subset \Omega_i$. F_i is a σ-field of sets in Ω_i and P_i is a probability measure for sets in a σ-field F_i.

Each r.v. $\xi_i(\omega_i)$, for $i=1,\dots,s$, induces a new probability space (Ω_i', F_i', P_i') from a basic probability (Ω_i, F_i, P_i). In fact, the set of values that $\xi_i(\omega_i)$ can take for all $\omega_i \in \Omega_i$, defines Ω_i', that is the sample space (range space) of $\xi_i(\omega_i)$. The inverse function ξ_i^{-1} takes every interval $\xi_i \leq x_i$ into a measureable ω_i set. Therefore, if F_i' is a σ-field of sets in Ω_i', the r.v. $\xi_i(\omega_i)$ maps the sample points ω_i in Ω_i into sample points ξ_i in Ω_i' such that, for every Borel set $E_i' \in F_i'$, there is an event $E_i \in F_i$, $E_i \subset \Omega_i$, for which $\xi_i(\omega_i) \in E_i'$. Hence, the inverse image of the set E_i' is

$$E_i = \xi_i^{-1}(E_i')$$

and

$$P_i'(E_i') = P_i(\xi_i^{-1}(E_i')) .$$

By $F_i(x_{i1},\dots,x_{in})$ we denote a cumulative distribution function (c.d.f.), where

(1.1.a) $$F_i(x_{i1},\dots,x_{in}) = P_i(\xi_{i1} \leq x_{i1},\dots,\xi_{in} \leq x_{in}); \quad i=1,\dots,s;$$

or in a more compact notation

(1.1.b) $$F_i(x_i) = P_i(\xi_i \leq x_i) \qquad i=1,\dots,s.$$

The k-dimensional marginal c.d.f., for the first k ordinates, is

(1.2) $$F_i(x_{i1},\dots,x_{ik}) = F_i(x_{i1},\dots,x_{ik}, +\infty,\dots,+\infty)$$
$$= P_i(\xi_{i1} \leq x_{i1},\dots,\xi_{ik} \leq x_{ik}), \quad i=1,\dots,s; \quad k=1,\dots,n-1.$$

The sample spaces Ω_i of $\xi_i (i=1,\dots,s)$ are contained in the n-dimensional Euclidean space. If nothing is specified, it will be clear from the contest, whether we are working with the entire Euclidean space or its non-negative subspace. Many economic variables (prices, outputs, incomes, etc.) are non-negatives.

2. Multivariate Transvariation Theory

2.1 Let $\Lambda_i (i=1,\dots,s)$ be an r.e. and

(2.1) $$\xi_i = (\xi_{i1},\dots,\xi_{in})$$

a multivariate random variable associated to the r.e. Λ_i . The multivariate r.v. $\xi_i = \xi_i(\omega_i)$ induces the probability space

(2.2) $$(\Omega_i', F_i', P_i')$$

where

(2.3) $$P_i'(\xi_i \leq x_i) = F_i(x_i) = P_i'(\xi_{i1} \leq x_{i1},\dots,\xi_{in} \leq x_{in})$$

is the c.d.f. of (2.1).

Let

(2.4) $$\lambda_i = (\lambda_{i1},\dots,\lambda_{in}) \in R_n$$

be an arbitrary real parameter vector.

Let $\Lambda = (\Lambda_i,\Lambda_j)$, $i,j=1,\dots,s;\ i \neq j$, be a combined r.e. to which is associated the multivariate r.v.

(2.5) $$\xi = (\xi_i, \xi_j)$$

that induces the probability space

(2.6) $$(\Omega'_{ij}, F'_{ij}, P'_{ij})$$

where Ω'_{ij} is the Cartesian product of the sample spaces Ω'_i and Ω'_j and F'_{ij} is the product σ-field of F'_i and F'_j. With P'_{ij} is denoted the probability measure of the product measurable space (Ω'_{ij}, F'_{ij}).

The parameter vector

$$\lambda = (\lambda_i, \lambda_j) \tag{2.7}$$

is associated to the r.v. (2.5).

Given a combined r.e. $\Lambda = (\Lambda_1, \Lambda_2)$ and using the r.v. (2.5) associated with it, as well as the parameter vector (2.7), we define[2]

$$\tau = \xi_2 - \xi_1 = (\xi_{21} - \xi_{11}, \ldots, \xi_{2n} - \xi_{1n}) = (\tau_1, \ldots, \tau_n) \tag{2.8}$$

and

$$\alpha = \lambda_2 - \lambda_1 = (\lambda_{21} - \lambda_{11}, \ldots, \lambda_{2n} - \lambda_{1n}) = (\alpha_1, \ldots, \alpha_n) \tag{2.9}$$

We assume, without a lost in generality, that α is a nonpositive vector, with at least one element strictly negative, i.e.

$$\alpha \leq 0, \quad \alpha \alpha' = \Sigma \alpha_i^2 > 0 \quad . \tag{2.10}$$

<u>Definition of k-dimensional marginal transvariation $(1 < k < n)$:</u> Given a combined r.e. $\Lambda = (\Lambda_1, \Lambda_2)$ to which is associated the 2n-dimensional r.v. (2.5), we define a k-dimensional marginal transvariation between the multivariate random variables ξ_1 and ξ_2 and the parameter vectors λ_1 and λ_2, when the differences $x_{2h} - x_{1h} = t_h$ are of opposite sign to the differences $\lambda_{2h} - \lambda_{1h} = \alpha_h$ for k values of h.

[2] For the sake of notational simplification, we will deal from now on, with the random variables ξ_1 and ξ_2 (i=1 and j=2) until we arrive to the development of multivariate transvariation theory among several distributions $(s > 2)$ in section 7.

Range of a marginal transvariation: The absolute value of $\Pi(x_{2h}-x_{1h}) = \Pi\, t_h$, for the k dimensions considered, defines the range of its corresponding transvariation.

Convention 1: Given a sample of independent replications of a combined r.e. Λ we then compute as transvariations one half of the number of null differences, i.e., one half of the times we observe $x_{2h} - x_{1h} = 0$, for at least one out of the k coordinates, in the realization of each combined r.e..

Convention 2: If $\alpha_h = \lambda_{2h} - \lambda_{1h} = 0$ for δ components out of the k's considered in (2.9), then we drop their corresponding components in (2.8) and we work with the (k-h)-dimensional marginal transvariation.

Convention 3: We will assume from now on that the k-dimensional marginal transvariation that we are dealing with are the first k coordinates of the r.v. (2.8).

Marginal transvariability: The k-dimensional marginal transvariability between a multivariate r.v. $\xi = (\xi_1,\xi_2)$, or its linear transformation $\tau = \xi_2-\xi_1$ and an arbitrary parameter $\lambda = (\lambda_1, \lambda_2)$, where $\alpha = \lambda_2 - \lambda_1$, is the probability that the r.v.

(2.11) $$\tau_{(k)} = (\tau_1,\ldots,\tau_k)$$

takes a value

(2.12) $$t_{(k)} = (t_1,\ldots,t_k)$$

of sign opposite to the parameter

(2.13) $$\alpha_{(k)} = (\alpha_1,\ldots,\alpha_k) .$$

Hence, by definition of marginal transvariability and taking account of Convention 3 and the sign of (2.10) we have

(2.14)
$$p_{n,k} = P(\tau_1 > 0,\ldots,\tau_k > 0 \mid \alpha_1 < 0,\ldots,\alpha_k < 0)$$
$$= \int_{t_{(k)} \geq 0} d\,G(t_{(k)})$$

where $G(t_{(k)})$ is the c.d.f. of the k-variate r.v. $\tau_{(k)}$.

LEMMA 1: Let

(2.15)
$$t_1 + t_2 + \ldots + t_k = 0$$

be a median hyperplane. If

(2.16)
$$P(\tau_1 \leq t_1,\ldots,\tau_k \leq t_k) = G(t_1,\ldots,t_k)$$

is symmetrically distributed with respect to the origin, then

(2.17)
$$p_{n,k} = 2^{-k} \quad .$$

The hyperplane (2.15) divide the k-dimensional space in two half spaces. The k-variate marginal transvariability is part of the amount of probability (the amount of the unit mass) lying on the half space that does not contain the vector (2.13).

Let

(2.18)
$$h = (h'' - h') = (h_1,\ldots,h_n)$$

be a variable vector such that

(2.19) $$\alpha \leq h \leq 0$$

where α was specified in (2.9) and (2.10). Now we introduce a new r.v.

(2.20) $$\tau - h = (\xi_2 - h'') - (\xi_1 - h')$$

and applying the k-variate marginal transvariability definition to (2.20) with respect to the parameter vector (2.13), we have

(2.21) $$p_{n,k}(h_{(k)}) = P(\tau_1 > h_1, \ldots, \tau_k > h_k \mid \alpha_1 < 0, \ldots, \alpha_k < 0; \alpha_{(k)} \leq h_{(k)} \leq 0)$$

The expression (2.21) is a non-decreasing and non-negative set function for decreasing values of $h_{(k)}$ within its domain specified in (2.19). The limit of (2.21) when $h_{(k)}$ tend to $\alpha_{(k)}$ defines the maximum of the k-variate marginal transvariability of the r.v. (2.11) with respect to the parameter vector (2.13). Therefore,

(2.22) $$p_{n,k;M} = \lim_{h_{(k)} \to \alpha_{(k)}} p_{n,k}(h_{(k)}) = P(\tau_1 > \alpha_1, \ldots, \tau_k > \alpha_k \mid \alpha_1 < 0, \ldots, \alpha_k < 0)$$

and

(2.23) $$P_{n,k} \leq p_{n,k;M} \leq 1 \quad .$$

Marginal probability of transvariation: The ratio between a k-variate marginal transvariability and its maximum defines the k-variate marginal probability of transvariation. Hence,

(2.24) $$P_{n,k} = \frac{p_{n,k}}{p_{n,k;M}} \quad .$$

The relevant characteristics of the marginal probability of transvariation are straightforward adaptation of those pointed out in Dagum (1968).

<u>Convention 4</u>: If the maximum of the marginal transvariability is null, <u>a fortiori</u>, the marginal transvariability will also be null. In that particular case we assign the value zero to the marginal probability of transvariation.

<u>Marginal moment of transvariation</u>: The expectation of the r power of the range of a k-variate marginal transvariation defines the r^{th} marginal moment of transvariation in a k-dimensional space. Hence

(2.25) $$m_{r;n,k} = E\left(\prod_{i=1}^{k} \tau_i^r \,\middle|\, \tau_i > 0, \alpha_i < 0;\ i=1,\dots,k \right).$$

The marginal moment of transvariation of the r.v. (2.20) with respect to the parameter vector (2.13) is a function of h. Thus

(2.26) $$m_{r;n,k}(h_{(k)}) = E\left(\prod_{i=1}^{k} (\tau_i - h_i)^r \,\middle|\, \tau_i > h_i, \alpha_i < 0;\ i=1,\dots,k \right).$$

The limit of (2.26) for $h_{(k)} \to \alpha_{(k)}$ defines the <u>maximum of the k-dimensional marginal moment of transvariation of the order r</u>. Therefore

(2.27) $$m_{r;n,k;M} = \lim_{h_{(k)} \to \alpha_{(k)}} m_{r;n,k}(h_{(k)})$$
$$= E\left(\prod_{i=1}^{k} (\tau_i - \alpha_i)^r \,\middle|\, \tau_i > \alpha_i, \alpha_i < 0;\ i=1,\dots,k \right).$$

The k-dimensional marginal moment of transvariation and its maximum contain, as a particular case, the k-dimensional marginal transvariability and its corresponding maximum. In fact, they are obtained when we give to r the value zero.

As in the bivariate case, the marginal moment of transvariation and its maximum satisfy the following inequalities, that determine their corresponding domain

(2.28) $$0 \leq m_{r;n,k} \leq m_{r;n,k;M} \leq \beta_{r;n,\alpha_{(k)}}$$

where $\beta_{r;n,\alpha_{(k)}}$ defines the kr^{th} absolute product moment of the deviations $\tau_{(k)} - \alpha_{(k)}$, with the proviso that the power of each component of the random vector $\tau_{(k)} - \alpha_{(k)}$ is equal to r. Therefore

(2.29) $$\beta_{r;n,\alpha_{(k)}} = E\left(\prod_{i=1}^{k} |\tau_i - \alpha_i|^r\right) .$$

The marginal moment of transvariation, its maximum and the absolute product moment of (2.28) are measures of the degree kr. Then, it is convenient, for practical and theoretical purposes, to introduce a zero degree measurement with range in the unit interval. This is achieved by the introduction of the intensity of transvariation concept.

Marginal intensity of transvariation: The r^{th} root of the ratio between the r^{th} marginal moment of transvariation (2.25) and its maximum (2.27) defines the r^{th} marginal intensity of transvariation in a k-dimensional space. Then

(2.30) $$I_{r;n,k} = \left(\frac{m_{r;n,k}}{m_{r;n,k;M}}\right)^{\frac{1}{r}} \quad ; \qquad r=1,2,\ldots \quad .$$

For $r=1$, we have

(2.31) $$I_{1;n,k} = \frac{m_{1;n,k}}{m_{1;n,k;M}} \quad .$$

The relevant characteristics of the marginal intensity of transvariation are similar to those of the marginal probability of transvariation. In particular, the range of $I_{r;n,k}$ is

(2.32) $$0 \leq I_{r,n,k} \leq 1 .$$

We must observe that the probability of transvariation deals with the sign of the differences meanwhile, the intensity of transvariation deals with weighted differences. That is why, in particular for small samples, both estimators give estimates that can show a quite large difference.

Convention 5: If, as in Convention 4

(2.33) $$P(\tau_{(k)} > 0 \mid \alpha_{(k)} < 0) = P(\tau_{(k)} > \alpha_{(k)} | \alpha_{(k)} < 0) = 0$$

then we assign the value zero to the marginal intensity of transvariation.

3. Case of Two Independent Multivariate Random Variables

3.1. If the two multivariate random variables ξ_1 and ξ_2 are independents, we still have the same results obtained in the preceding section for the marginal probability and intensity of transvariation. But, under this assumption of independence we can introduce the concepts of marginal space of transvariation and marginal discriminative value.

3.2. Marginal space of transvariation: Let ξ_1 and ξ_2 be two independent random vectors. Hence, for their marginal c.d.f., we have, from (1.2)

(3.1) $$F_i(x_{i(k)}) = F_i(x_{i1},\ldots,x_{ik}) \; ; \qquad i=1,2.$$

Let N_1 and N_2 represent the size of the populations corresponding to each of the sample spaces of ξ_1 and ξ_2. Then we introduce the function $g(x)$ of a new r.v. η defined as follows

$$g(x_{(k)}) = N_1 f_1(x_{(k)}) \quad \text{if} \quad N_1 f_1(x_{(k)}) \leq N_2 f_2(x_{(k)}) \tag{3.2.a}$$

$$g(x_{(k)}) = N_2 f_2(x_{(k)}) \quad \text{if} \quad N_1 f_1(x_{(k)}) > N_2 f_2(x_{(k)}) . \tag{3.2.b}$$

We define its corresponding relative frequency by

$$\frac{g(x_{(k)})}{N} = \frac{g(x_{(k)})}{N_1+N_2} . \tag{3.3}$$

The k-dimensional $(1 \leq k \leq n-1)$ marginal space of transvariation is, by definition, the common frequencies of the k-dimensional random vector $\xi_{1(k)}$ and $\xi_{2(k)}$ divided by the size of the combined populations (or samples). Therefore

$$C_{n,k} = \frac{1}{N} \int_{\Omega'_{1(k)} \cap \Omega'_{2(k)}} d\, G(x_{(k)}) = \frac{1}{N} \int_{\Omega_{(k)}} d\, G(\mathbf{x}_{(k)}) \tag{3.4}$$

and

$$\int_{\Omega^C_{(k)}} d\, G(x_{(k)}) = 0 \tag{3.5}$$

where

$$\Omega_{(k)} = \Omega'_{1(k)} \cap \Omega'_{2(k)} \subset R_k \tag{3.6}$$

and

$$\Omega^C_{(k)} = R_k \setminus \Omega'_{1(k)} \cap \Omega'_{2(k)} . \tag{3.7}$$

Because of the definition given in (3.2) for $g(x_{(k)})$, the space of transvariation has the domain

(3.8) $$0 \leq C \leq 0.50 \quad .$$

It takes the maximum value of 0.50 when $N_1 = N_2$ and $\xi_{1(k)}$ and $\xi_{2(k)}$ are equivalent k-dimensional random vectors.

Ratio of the space of transvariation: The ratio between the marginal space of transvariation (3.4) and its maximum, assuming $N_1 = N_2$ and equivalent random vectors, defines the ratio of the marginal space of transvariation. Therefore

(3.9) $$H_{n,k} = 2\,C_{n,k} = \frac{2}{N} \int_{\Omega_{(k)}} d\,G(x_{(k)}) \quad .$$

3.3 Marginal discriminative value: Let ξ_1, ξ_2, $f_1(x_{(k)})$, $f_2(x_{(k)})$, N_1 and N_2 stand as in section 3.2. Let $n_1(x_{(k)})$ be the error resultant from the assumption that ξ_{1i} $(i=1,\ldots,k)$ is greater (less) than x_i $(i=1,\ldots,k)$ and $n_2(x_{(k)})$ resultant from the assumption that ξ_{2i} $(i=1,\ldots,k)$ is less (greater) than x_i $(i=1,\ldots,k)$.[3]

Therefore, we define the error function by

(3.10) $$\epsilon(x_{(k)}) = \frac{n_1(x_{(k)}) + n_2(x_{(k)})}{N_1 + N_2} \quad .$$

The value $z_{(k)}$ of $x_{(k)}$ that minimizes the error function in (3.10) defines the marginal discriminative value between the random vectors $\xi_{1(k)}$ and $\xi_{2(k)}$.

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[3]This assumption does not imply that all components of the random vector $\xi_{1(k)}$ will be taken respectively as greater (or less) than the corresponding components of $x_{(k)}$ and the opposite behavior for $\xi_{2(k)}$, but rather that when one component of $\xi_{1(k)}$ is taken as greater than its corresponding components in $x_{(k)}$, the corresponding component in $\xi_{2(k)}$ is taken as less than, and vice versa.

4. Case of Independence Between the Components of Each Random Vector

If the n components of each random vector ξ_i $(i=1,2)$ in (2.1) are independents, we have, for the marginal transvariability, its maximum, and the marginal probability of transvariation, from (2.14), (2.22) and (2.24), respectively

(4.1)
$$p_{n,k} = \prod_{i=1}^{k} P(\tau_i > 0 | \alpha_i < 0) = \prod_{i=1}^{k} p_i$$

(4.2)
$$p_{n,k;M} = \prod_{i=1}^{k} P(\tau_i > \alpha_i | \alpha_i < 0) = \prod_{i=1}^{k} p_{i,M}$$

(4.3)
$$P_{n,k} = \prod_{i=1}^{k} \frac{P(\tau_i > 0|\alpha_i < 0)}{P(\tau_i > \alpha_i|\alpha_i < 0)} = \prod_{i=1}^{k} P_i$$

where P_i is the one-dimensional marginal probability of transvariation between the random component $\tau_i = \xi_{2i} - \xi_{1i}$ and its corresponding parameter $\alpha_i = \lambda_{2i}-\lambda_{1i}$.

For the marginal moment of transvariation, its maximum and the marginal intensity of transvariation, under the hypothesis of independence, we have, from (2.25), (2.27) and (2.30), respectively.

(4.4)
$$m_{r;n,k} = \prod_{i=1}^{k} E(\tau_i^r | \tau_i > 0, \alpha_i < 0) = \prod_{i=1}^{k} m_{r,i}$$

(4.5)
$$m_{r;n,k;M} = \prod_{i=1}^{k} E[(\tau_i - \alpha_i)^r | \tau_i > \alpha_i, \alpha_i < 0] = \prod_{i=1}^{k} m_{r,M,i}$$

(4.6)
$$I_{r;n,k} = \prod_{i=1}^{k} \left(\frac{m_{r,i}}{m_{r,M,i}}\right)^{\frac{1}{r}} = \prod_{i=1}^{k} I_{r,i} .$$

Then, the k-dimensional marginal intensity of transvariation is equal to the product of the one-dimensional marginal intensity of transvariation of each of the k components.

If $\alpha_i (i=1,\ldots,k)$ is the median of τ_i, we have for (4.2)

(4.7) $$p_{n,k;M} = 2^{-k}$$

and (4.3) becomes

(4.8) $$P_{n,k} = 2^k \prod_{i=1}^{k} p_i \quad .$$

If $\alpha_i (i=1,\ldots,k)$ is the mean of τ_i, then we have, for (4.5)

(4.9) $$m_{r;n,k;M} = 2^{-k} \prod_{i=1}^{k} \beta_{r,\alpha_i} \quad .$$

If we suppose further, that the random components ξ_{1i} and ξ_{2i}, for $i=1,\ldots,k$, are independent random variables and the size of their corresponding populations are N_{1i} and N_{2i} respectively, for $i=1,\ldots,k$, the marginal space of transvariation defined in (3.4) becomes

(4.10) $$C_{n,k} = \prod_{i=1}^{k} \frac{1}{N_{1i} + N_{2i}} \int_{-\infty}^{\infty} d\, G_i(x_i) = \prod_{i=1}^{k} C_i$$

and, for (3.9)

(4.11) $$H_{n,k} = \prod_{i=1}^{k} H_i \quad .$$

5. Hypothesis of Normal Distributions

5.1 The n-variate normal distribution. The p.d.f. of the n-variate normal distribution is

(5.1) $$f(x_1,\ldots,x_n) = \frac{\sqrt{|\sigma^{ij}|}}{(2\pi)^{\frac{1}{2}}} \exp[-\tfrac{1}{2} Q(x_1,\ldots,x_n)]$$

where

(5.2) $$Q(x_1,\ldots,x_n) = (x-\lambda)' (\sigma^{ij}) (x-\lambda) = \sum_{i,j} \sigma^{ij}(x_i-\lambda_i)(x_j-\lambda_j)$$

(5.3) $$\lambda_i = E(x_i) ; \quad i=1,\ldots,n .$$

(5.4) $$\sigma_{ij} = \text{cov}(x_i,x_j) = E(x_i-\lambda_i)(x_j-\lambda_j) ; \quad i,j=1,\ldots,n .$$

(5.5) $$C = (\sigma_{ij}) = \text{covariance matrix}$$

(5.6) $$C^{-1} = (\sigma^{ij}) = \text{the inverse of the covariance matrix}$$

(5.7) R_n = the sample space of the n-dimensional random variable, i.e., the entire n-dimensional space.

The p.d.f. (5.1) will be denoted by

(5.8) $$N[(\lambda_i), (\sigma_{ij})] ; \quad i,j=1,\ldots,n .$$

The characteristic function of the n-variate normal distribution is

(5.9) $$\varphi(u_1,\ldots,u_n) = E(\exp i \sum_{j=1}^{n} u_j x_j) = \exp (i \sum_{j=1}^{n} \lambda_j u_j - \tfrac{1}{2} \sum_{j,h} \sigma_{jh} u_j u_h) .$$

LEMMA 2: Let $x = (x_1, \ldots, x_n)$ be a vector random variable having the n-variate normal distribution (5.1). Then, the marginal distribution of $x_{(k)} = (x_1, \ldots, x_k)$, $(k < n)$, is the k-variate normal distribution $N[(\lambda_i), (\sigma_{ij})]$; $i,j=1,\ldots,k$.

If in (5.9) we put $t_{k+1} = \ldots = t_n = 0$, we have

$$(5.10) \quad \varphi(u_1, \ldots, u_k, 0, \ldots, 0) = \exp\left(i \sum_{j=1}^{k} \lambda_j u_j - \frac{1}{2} \sum_{j,h=1}^{k} \sigma_{jh} u_j u_h \right) .$$

Hence, $x_{(k)} = (x_1, \ldots, x_k)$ has the distribution

$$(5.11) \qquad N[(\lambda_i), (\sigma_{ij})] ; \quad i,j=1,\ldots,k; \quad 1 \leq k < n .$$

5.2. Distribution of linear functions of normally correlated random variables

LEMMA 3: Let $x = (x_1, \ldots, x_n)$ be an n-dimensional random variable having the n-variate normal distribution $N[(\lambda_i, (\sigma_{ij})]$; $i,j=1,\ldots,n$. That is, its p.d.f. is given by (5.1) and its characteristic function by (5.9). Then, the linear function

$$(5.12) \qquad L = \sum_{j=1}^{n} u_j x_j$$

where the u_j are not all zero, has the distribution

$$(5.13) \qquad N[\sum_{j=1}^{n} u_j \lambda_j , \sum_{j,h} \sigma_{jh} u_j u_h] .$$

The characteristic function of the random variable (5.12) is

$$(5.14) \qquad \varphi_L(v) = E(\exp i L v) = E(\exp i v \sum_{j=1}^{n} u_j x_j) = \varphi_x(vu_1, \ldots, vu_n)$$

$$= \exp(i v \sum_{j=1}^{n} u_j \lambda_j - \frac{1}{2} v^2 \sum \sigma_{jh} u_j u_h) .$$

Therefore, the random variable (5.12) has the distribution (5.13).

LEMMA 4: Let the n-dimensional random variables

$$x_s = (x_{s1},\dots,x_{sn}) \; ; \quad s = 1,2 \tag{5.15}$$

have the n-variate normal distribution

$$N[(\lambda_{si}), (\sigma_{sij})] \; ; \quad i,j=1,\dots,n; \quad s=1,2. \tag{5.16}$$

Then, the k-dimensional random variable

$$t_{(k)} = (x_{21}-x_{11},\dots, x_{2k}-x_{1k}), \quad 1 \leq k \leq n \tag{5.17}$$

has the k variate distribution

$$N[\alpha_i), (\sigma_{ij})] \; ; \quad i,j=1,\dots,k \tag{5.18}$$

where

$$\alpha_i = \lambda_{2i} - \lambda_{1i} = E(x_{2i} - x_{1i}) \tag{5.19}$$

$$\begin{aligned} \sigma_{ij} &= E(t_i-\alpha_i)(t_j-\alpha_j) = \operatorname{cov}(t_i,t_j) \\ &= \operatorname{cov}(x_{2i}-x_{1i}, x_{2j}-x_{ij}) = \\ &= \operatorname{cov}(x_{1i},x_{1j}) + \operatorname{cov}(x_{2i},x_{2j}) - \operatorname{cov}(x_{1i},x_{2j}) - \operatorname{cov}(x_{1j},x_{2i}) \end{aligned} \tag{5.20}$$

Applying Lemmas 2, and 3, and the method of mathematical induction, we prove Lemma 4.

5.3. Transvariation theory assuming normal distribution.

5.3.1. If the n-dimensional random variables (2.1) (i=1,2), are normally distributed, then the k-dimensional random variable (2.11) has a normal distribution (by application of Lemma 4). Using (5.18) in (2.14), (2.22), (2.24), (2.25), (2.27) and (2.30), we then obtain the parameters of the transvariation. In particular, because of (5.19) and Lemma 1, we have for (2.22)

(5.21) $$p_{n,k;M} = \frac{1}{2^k} .$$

If x_1 and x_2 in (5.15) are independent random variables, (5.20) becomes

(5.22) $$\sigma_{ij} = \mathrm{cov}(x_{1i}, x_{1j}) + \mathrm{cov}(x_{2i}, x_{2j})$$

Assuming normality and independence between two n-dimensional random variables, we obtain the marginal space of transvariation and the marginal discriminative value, replacing (5.16) in the corresponding formulas of section 3.

5.3.2. If we assume further that the components of the n-dimensional random variable τ in (2.8) are independent (and normally distributed), we have

(5.23) $$\sigma_{ij} = \mathrm{cov}(\tau_i, \tau_j) = \mathrm{cov}(\xi_{2i}-\xi_{1i}, \xi_{2j}-\xi_{1i}) = 0 \quad \text{if } i \neq j$$

(5.24) $$\sigma_{ii} = \mathrm{var}(\tau_i) = \mathrm{var}(\xi_{1i}) + \mathrm{var}(\xi_{2i})$$

(5.25) $$g(t_1, \dots, t_n) = \frac{1}{(2\pi)^{\frac{1}{2}n}} \prod_{i=1}^{n} \frac{1}{\sqrt{\sigma_{ii}}} \exp\left(-\frac{(t_i-\alpha_i)^2}{2\sigma_{ii}}\right)$$

where

(5.26) $$\alpha_i = E(\tau_i) = E(\xi_{2i}-\xi_{1i}) = \lambda_{2i}-\lambda_{1i} < 0 \; ; \quad i=1,\dots,n ,$$

and

(5.27) $$\Phi(u_1, \dots, u_n) = \prod_{i=1}^{n} \Phi(u_i) = (2\pi)^{-\frac{1}{2}n} \prod_{i=1}^{n} \int_{-\infty}^{u_i} \exp(-\tfrac{1}{2}t_i^2)\, dt_i .$$

Therefore, the k-variate marginal transvariability (4.1) becomes

(5.28) $$p_{n,k} = \prod_{i=1}^{k} P(\tau_i > 0 \mid \alpha_i < 0) = \prod_{i=1}^{k} [1-\Phi(-\frac{\alpha_i}{\sqrt{\sigma_{ii}}})]$$
$$= \prod_{i=1}^{k} \Phi(\frac{\alpha_i}{\sqrt{\sigma_{ii}}}) = \prod_{i=1}^{k} p_i .$$

Its maximum (4.2) is

(5.29) $$p_{n,k;M} = 2^{-k}$$

For the k-dimensional moment of transvariation and its maximum we have, from (4.4) (4.5) and (4.9) in conjunction with (5.24), (5.26) and (5.27)

(5.30) $$m_{r;n,k} = \prod_{i=1}^{k} E(\tau_i^r | \tau_i > 0, \alpha_i < 0) = \prod_{i=1}^{k} m_{r,i}$$

$$= \prod_{i=1}^{k} \alpha_i^r \Phi\left(\frac{\alpha_i}{\sqrt{\sigma_{ii}}}\right) + \frac{1}{\sqrt{2\pi}} \prod_{i=1}^{k} \sum_{s=1}^{r} \binom{r}{s} \alpha_i^{r-s} \sigma_{ii}^{\frac{1}{2}s} G\left(s; -\frac{\alpha_i}{\sqrt{\sigma_{ii}}}\right)$$

(5.31) $$m_{r;n,k;M} = 2^{-k} \prod_{i=1}^{k} \beta_{r,\alpha_i} = \left[\frac{2^{\frac{r}{2}-1}}{\sqrt{\pi}} \Gamma\left(\frac{r+1}{2}\right)^k\right] \prod_{i=1}^{k} \sigma_{ii}^{\frac{r}{2}}$$

where $G(s;-b)$, for $s = 2k$ [Dagum 1960.a and 1968]

(5.32) $$G(2k;-b) = \frac{(2k)!}{2^k k!} \sqrt{2\pi}\ \Phi(b) - e^{-\frac{1}{2}b^2} \sum_{i=0}^{k-1} \frac{(2i)! \binom{2k}{2i}}{2^i i! \binom{k}{i}} b^{2k-2i-1}$$

and for $s = 2k+1$,

(5.33) $$G(2k+1;-b) = e^{-\frac{1}{2}b^2} \sum_{i=0}^{k} 2^i\ i! \binom{k}{i} b^{2k-2i} .$$

In (5.31), we have [Dagum 1960a and 1968]

(5.34) $$\beta_{r,\alpha_i} = E(|\tau_i - \alpha_i|^r .$$

From (5.28) and (5.29) we have, for the marginal probability of transvariation (4.3)

(5.35) $$P_{n,k} = 2^k \prod_{i=1}^{k} p_i = 2^k \prod_{i=1}^{k} \Phi\left(\frac{\alpha_i}{\sqrt{\sigma_{ii}}}\right)$$

and from (5.30) and (5.31) we have, for the marginal intensity of transvariation (4.6)

(5.36) $$I_{r;n,k} = 2^{\frac{k}{r}} \prod_{i=1}^{k} \left(\frac{m_{r,i}}{\beta_{r,\alpha_i}}\right)^{\frac{1}{r}} = \prod_{i=1}^{k} I_{r,i} \quad .$$

For $r=1$ we have

(5.37) $$I_{1;n,k} = (2\pi)^{\frac{1}{2}k} \prod_{i=1}^{k} \left[\Phi'\left(\frac{\alpha_i}{\sqrt{\sigma_{ii}}}\right) + \frac{\alpha_i}{\sqrt{\sigma_{ii}}}\Phi\left(\frac{\alpha_i}{\sqrt{\sigma_{ii}}}\right)\right] \quad .$$

6. Linear Transformation

The k-variate $(1 \leq k \leq n)$ parameters of transvariation give us a complete set of measurements that will allow a comprehensive analysis of the relevant characteristics of both n-variate distributions.

In the preceding sections we introduced and mathematically formalized the marginal parameters of transvariation. However, their operational complications increase tremendously with the number of dimensions, except for the case of independence developed in section 4 and, for the normal distribution, in section 5.3.2. For this reason we need some transformation that will reduce the computations required to estimate the parameters of transvariation. With this purpose we introduce a linear transformation that will result in some information loss (cf. preceding section where we worked with full information).

Given the n-variate random variables of (2.1)

(6.1) $$\xi_s = (\xi_{s1}, \ldots, \xi_{sn}), \qquad s=1,2$$

we define the n^{th} reduced random variables

(6.2) $$\eta_1 = \sum_{i=1}^{n} \eta_{1i} = \sum_{i=1}^{n} \frac{\xi_{1i}}{\Delta_{ii}}$$

(6.3) $$\eta_2 = \sum_{i=1}^{n} \eta_{2i} = \sum_{i=1}^{n} \frac{\xi_{2i}}{\Delta_{ii}} \quad .$$

The sample spaces of the reduced random variables (6.2) and (6.3) are the real line R_1 .

$\triangle_{ii}$ denotes Gini's mean difference between the random variables ξ_{1i} and ξ_{2i} . Hence,

(6.4) $$\triangle_{ii} = E(|(\xi_{1i}-\lambda_{1i}) - (\xi_{2i}-\lambda_{2i})|) = E(|\tau_i-\alpha_i|) = \beta_{1,\alpha_i}$$

where $\lambda_{1i}, \lambda_{2i}$ and α_i are the means of their corresponding random variables as in (5.26) and τ_i as defined in (2.8)

The k^{th} marginal reduced random variables corresponding to (6.2) and (6.3) are, respectively

(6.5) $$\eta_{(k)1} = \sum_{i=1}^{k} \eta_{1i}$$

(6.6) $$\eta_{(k)2} = \sum_{i=1}^{k} \eta_{2i} \quad .$$

By virtue of the linear transformation (6.2) and (6.3), including the marginal cases (6.5) and (6.6), we now have unidimensional random variables. Then we apply to these new random variables the theory developed in Dagum (1960a) and (1968), in order to estimate the parameters of transvariation.

Given two n-variate random variables we can apply the bivariate transvariation theory to the n^{th} reduced random variables (6.2) and (6.3) and to the k^{th} marginal reduced random variables (6.5) and (6.6), for $k=1,\ldots,n-1$. Therefore, we have a total of $\binom{n}{k}$ possible cases of k^{th} reduced random variables. For each of these cases we can estimate the parameters of transvariation, making a total of 2^n-1 estimates for each parameter of transvariation. For $k=1$ we have the same results, working with either the reduced or the original random variables.

7. Transvariation Among Several Distributions

7.1 Introduction. In this section we will deal with more than two multivariate distributions and we will work with the marginal reduced random variables introduced in section 6.

Let $\Lambda_i (i=1,\ldots,s)$ be a random experiment to which is associated an n-variate r.v.

(7.1) $$\xi_i = (\xi_{i1},\ldots,\xi_{in})$$

and an arbitrary real parameter vector

(7.2) $$\lambda_i = (\lambda_{i1},\ldots,\lambda_{in}) \ .$$

Let $N_i (i=1,\ldots,s)$ be the size of the populations (or the samples), and

(7.3) $$N = \sum_{i=1}^{s} N_i \ .$$

Convention 6: The N elements of (7.3) are a complete set of independent and equally possible events.

Let

(7.4) $$\Lambda_{ij} = (\Lambda_i,\Lambda_j) \ ; \quad i,j=1,\ldots,s \ ; \quad i\neq j$$

be a combined r.e. with which is associated a combined vector random variable (ξ_i,ξ_j).

(7.4) is the r.e. under consideration, where the two populations are chosen at random and thereafter the combined r.e. is performed upon them. Hence, we have $s(s-1)$ possible random experiments and the probability of having one observation belonging to the i^{th} population and another to the j^{th} population, in this order, in the realization of a combined r.e., is

(7.5) $$A_{ij} = \frac{N_i N_j}{N(N-N_i)} \ .$$

The marginal reduced random variables associated to the combined r.e. (7.4) are, according with (6.5) and (6.6)

$$(7.6) \qquad \eta_{(k)i} = \sum_{h=1}^{k} \eta_{ih} = \sum_{h=1}^{k} \frac{\xi_{ih}}{\Delta_{hh}}$$

$$(i,j=1,\dots,s;\ k=1,\dots,n)$$

$$(7.7) \qquad \eta_{(k)j} = \sum_{h=1}^{k} \eta_{jh} = \sum_{h=1}^{k} \frac{\xi_{jh}}{\Delta_{hh}} .$$

Their corresponding parameters are

$$(7.8) \qquad \mu_{(k)i} = \sum_{h=1}^{k} \mu_{ih} = \sum_{h=1}^{k} \frac{\lambda_{ih}}{\Delta_{hh}}$$

$$(i,j=1,\dots,s;\ k=1,\dots,n)$$

$$(7.9) \qquad \mu_{(k)j} = \sum_{h=1}^{k} \mu_{jh} = \sum_{h=1}^{k} \frac{\lambda_{jh}}{\Delta_{hh}} .$$

7.2 Probability of transvariation.

7.2.1. Denoting by p_{ij} the bivariate transvariability between the reduced random variables (7.6) and (7.7) and the parameters (7.8) and (7.9), we have, for the realization of the combined r.e. (7.4); its transvariability

$$(7.10) \qquad {}_{s}p_{(k)} = \sum_{i \neq j} \frac{N_i N_j\, p_{ij}}{N(N-N_i)}$$

its maximum

$$(7.11) \qquad {}_{s}p_{M(k)} = \sum_{i \neq j} \frac{N_i N_j p_{ij,M}}{N(N-N_i)}$$

and, the probability of transvariation

$$(7.12) \qquad {}_{s}P_{(k)} = \frac{{}_{s}p_{(k)}}{{}_{s}p_{M(k)}} .$$

If $N_1 = N_2 = \ldots = N_s$, we have for (7.10, (7.11) and (7.12), respectively

$$(7.13) \qquad {}_sP_{(k)} = \frac{2}{s(s-1)} \sum_{i<j} p_{ij}$$

$$(7.14) \qquad {}_sP_{M(k)} = \frac{2}{s(s-1)} \sum_{i<j} p_{ij,M}$$

$$(7.15) \qquad {}_sP_{(k)} = \frac{\sum_{i<j} p_{ij}}{\sum_{i<j} p_{ij,M}} .$$

If the parameter (7.8) and (7.9) are the median of their corresponding reduced random variables in (7.6) and (7.7), we have

$$(7.16) \qquad {}_sP_{(k)} = 2\,{}_sp_{(k)} = 2 \sum_{i \neq j} \frac{N_i N_j \, p_{ij}}{N(N-N_i)}$$

$$(7.17) \qquad {}_sP_{(k)} = 2\,{}_sp_{(k)} = \frac{4}{s(s-1)} \sum_{i \neq j} p_{ij} .$$

<u>7.2.2. Gini's Aggregative method.</u> The results obtained in (7.16) and (7.17) imply the computations of $\frac{1}{2} s(s-1)$ transvariability. It is possible to reduce them to $2(s-1)$ in (7.16), when $s > 4$ and to $s-1$ in (7.17), when $s > 2$. With that end we order the s reduced random variables $\eta_{(k)i}$ $(i=1,\ldots,s)$ according to the increasing or decreasing order of their corresponding medians. Therefore, we have for (7.16)

$$(7.18) \qquad {}_sP_{(k)} = \frac{2}{N}\Big(\sum_{i=1}^{s-1} \frac{N_i}{N-N_i} \sum_{j=i+1}^{s} N_j p_{ij} + \sum_{i=2}^{s} \frac{N_i}{N-N_i} \sum_{j=1}^{i-1} N_j p_{ij} \Big) .$$

If we denote by Q_{ij} the number of transvariabilities between the reduced random variables $\eta_{(k)i}$ and $\eta_{(k)j}$, we have

$$(7.19) \qquad Q_{ij} = N_i N_j \, p_{ij} .$$

Hence, because of the requirement set above, i.e., ordering the s reduced random variables $\eta_{(k)i}(i=1,\ldots,s)$ according to the increasing or decreasing values of their corresponding medians, we have for the number of transvariabilities between $\eta_{(k)i}$ and the remaining random variables with greater and less medians, respectively

$$Q_{i(i+1,\ldots,s)} = \sum_{j=i+1}^{s} Q_{ij} = N_i \sum_{j=i+1}^{s} N_j p_{ij} = \tag{7.20}$$

$$= N_i \left(\sum_{j=i+1}^{s} N_j \right) p_{i(i+1,\ldots,s)}$$

$$Q_{i(1,\ldots,i-1)} = \sum_{j=1}^{i-1} Q_{ij} = N_i \sum_{j=1}^{i-1} N_j \, p_{ij} \tag{7.21}$$

$$= N_i \left(\sum_{j=1}^{i-1} N_j \right) p_{i(1,\ldots,i-1)} \; ,$$

where we denote by $p_{i(1,\ldots,i-1)}$ the transvariability between the reduced r.v. $\eta_{(k)i}$ and the $(i-1)$ reduced random variables with medians less than (or greater than) the median of $\eta_{(k)i}$ and, therefore $p_{i(i+1,\ldots,s)}$ denotes the transvariability between the i^{th} reduced r.v. and the $(s-i)$ reduced random variables with medians greater than (or less than) the median of the i^{th} reduced random variable.

After replacing (7.19), (7.20) and (7.21) in (7.18) we have the following simplified result

$$sP_{(k)} = \frac{2}{N} \left[\sum_{i=1}^{s-1} \frac{N_i \sum_{j=i+1}^{s} N_j}{N-N_i} p_{i(i+1,\ldots,s)} + \sum_{i=2}^{s} \frac{N_i \sum_{j=1}^{i-1} N_j}{N-N_i} p_{i(1,\ldots,i-1)} \right] . \tag{7.22}$$

In a similar way we obtain for (7.17)

$${}_sP_{(k)} = \frac{4}{s(s-1)} \sum_{i=1}^{s-1} (s-i) \, p_{i(i+1,\ldots,s)} \tag{7.23}$$

7.3. Intensity of Transvariation.

7.3.1. We denote by $m_{r,ij}$ and $m_{r,ij,M}$ the r^{th} moment of transvariation and its maximum between the reduced random variables (7.6) and (7.7) and the parameters (7.8) and (7.9). Then we have, in the realization of the combined r.e. (7.4), for the moment of transvariation

(7.24) $$ {}_s m_r(k) = \sum_{i \neq j} \frac{N_i N_j m_{r,ij}}{N(N-N_i)} $$

its maximum

(7.25) $$ {}_s m_r(k)_M = \sum_{i \neq j} \frac{N_i N_j m_{r,ij,M}}{N(N-N_i)} $$

and for the intensity of transvariation

(7.26) $$ {}_s I_r(k) = \left(\frac{{}_s m_r(k)}{{}_s m_r(k)_M}\right)^{\frac{1}{r}} \quad ; \qquad r=1,2,\ldots \quad . $$

If $N_1 = N_2 \ldots = N_s$ we have

(7.27) $$ {}_s m_r(k) = \frac{2}{s(s-1)} \sum_{i<j} m_{r,ij} $$

(7.28) $$ {}_s m_r(k)_M = \frac{2}{s(s-1)} \sum_{i<j} m_{r,ij,M} $$

and

(7.29) $$ {}_s I_r(k) = \left(\frac{\sum_{i<j} m_{r,ij}}{\sum_{i<j} m_{r,ij,M}} \right)^{\frac{1}{r}} \quad . $$

If the parameter (7.8) and (7.9) are the means of their corresponding random variables we have, for $m_{1,ij,M}$

(7.30) $$ m_{i,ij,M} = \frac{1}{2} \beta_{1,ij} = \frac{1}{2} \Delta_{ij} $$

where Δ_{ij} is now Gini's mean difference between the reduced random variables (7.6) and (7.7). Therefore, we have for the first order intensity of transvariation in (7.26)

(7.31) $$ {}_sI_{1(k)} = \frac{\sum_{i \neq j} \frac{N_i N_j \Delta_{ij} I_{1,ij}}{N(N-N_i)}}{\sum_{i \neq j} \frac{N_i N_j \Delta_{ij}}{N(N-N_i)}} $$

and for (7.29)

(7.32) $$ {}_sI_{1(k)} = \frac{\sum_{i<j} m_{1,ij}}{\sum_{i<j} m_{1,ij,M}} = \frac{\sum_{i<j} \Delta_{ij} I_{1,ij}}{\sum_{i<j} \Delta_{ij}} \ . $$

7.3.2. Gini's aggregative method. If we denote by T_{ij} the sum of the absolute differences of all the transvariations between the random variables (7.6) and (7.7) and their corresponding means, we have

(7.33) $$ I_{1,ij} = \frac{m_{1,ij}}{m_{1,ij,M}} = \frac{2\,T_{ij}}{N_i N_j \Delta_{ij}} \ . $$

Ordering the s reduced random variables $\eta_{(k)i}$ $(i=1,\ldots,s)$ according to the increasing or decreasing order of their corresponding means, and taking account of the relation (7.33), we obtain

(7.34) $$ {}_sI_{1(k)} = \frac{n_1 + n_2}{d_1 + d_2} $$

where

$$ n_1 = \sum_{i=1}^{s-1} \frac{N_i \sum_{j=i+1}^{s} N_j}{N-N_i} \Delta_{i(i+1,\ldots,s)} I_{1,i(i+1,\ldots,s)} $$

$$ n_2 = \sum_{i=2}^{s} \frac{N_i \sum_{j=1}^{i-1} N_j}{N-Ni} \Delta_{i(1,\ldots,i-1)} I_{1,i(1,\ldots,i-1)} $$

$$ d_1 = \sum_{i=1}^{s-1} \frac{N_i \sum_{j=i+1}^{s} N_j}{N-N_i} \Delta_{i(i+1,\ldots,s)} $$

$$ d_2 = \sum_{i=2}^{s} \frac{N_i \sum_{j=1}^{i-1} N_j}{N-N_i} \Delta_{i(1,\ldots,i-1)} $$

and for (7.32)

(7.35) $$ {}_s I_{1(k)} = \frac{\sum_{i=1}^{s-1} (s-i)\, \Delta_{i(i+1,\ldots,s)} I_{1,i(i+1,\ldots,s)}}{\sum_{i=1}^{s-1} (s-i)\, \Delta_{i(i+1,\ldots,s)}} $$

where the notations $\Delta_{i(i+1,\ldots,s)}$, $\Delta_{i(1,\ldots,i-1)}$, $I_{1,i(i+1,\ldots,s)}$ and $I_{1,i(1,\ldots,i-1)}$ have an interpretation similar to the transvariabilities $p_{i(i+1,\ldots s)}$ and $p_{i(1,\ldots,i-1)}$ in section 7.2.2.

7.4. Hypothesis of normal distributions. If the random variables in (7.1), $i=1,\ldots,s$, have a k-variate normal distribution, then, the reduced random variables (7.6) and (7.7), by application of Lemmas 2 and 3, have univariate normal distributions with means, respectively

(7.36) $$ \mu_{(k)i} = E(\eta_{(k)i}) = \sum_{h=1}^{k} E(\eta_{ih}) = \sum_{h=1}^{k} \frac{\lambda_{ih}}{\Delta_{hh}} $$

(7.37 $$ \mu_{(k)j} = E(\eta_{(k)j}) = \sum_{h=1}^{k} E(\eta_{jh}) = \sum_{h=1}^{k} \frac{\lambda_{jh}}{\Delta_{hh}} $$

and variances

(7.38) $$ \mathrm{var}(\eta_{(k)i}) = \sum_{h=1}^{k} \frac{\mathrm{var}(\xi_{ih})}{\Delta^2_{hh}} + 2 \sum_{h<\ell} \frac{\mathrm{cov}(\xi_{ih},\xi_{i\ell})}{\Delta_{hh}\,\Delta_{\ell\ell}} $$

(7.39) $$ \mathrm{var}(\eta_{(k)j}) = \sum_{h=1}^{k} \frac{\mathrm{var}(\xi_{jh})}{\Delta^2_{hh}} + 2 \sum_{h<\ell} \frac{\mathrm{cov}(\xi_{jh},\xi_{j\ell})}{\Delta_{hh}\,\Delta_{\ell\ell}} . $$

Therefore, the linear transformation

(7.40) $$ \eta_{(k)ij} = \eta_{(k)j} - \eta_{(k)i} $$

by Lemma 4, has a normal distribution with mean

(7.41) $$\mu_{(k)ij} = E(\eta_{(k)ij}) = \mu_{(k)j} - \mu_{(k)i}$$

and variance

(7.42) $$\sigma_{(k)ij} = \text{var}(\eta_{(k)ij}) = \text{var}(\eta_{(k)i}) + \text{var}(\eta_{(k)j}) - 2\,\text{cov}(\eta_{(k)i}, \eta_{(k)j}).$$

We then substitute in (7.16) and (7.17) the transvariability $p_{ij}(i,j=1,\ldots,s; i\neq j)$ for the result obtained in the bivariate normal distribution [Dagum 1960a and 1968] and already applied in section 5.3.2. We deal similarly with the moment of transvariation and its maximum in (7.24), (7.25), (7.27) and (7.28).

We can easily develop the corresponding result in (7.22), (7.23), (7.34) and (7.35) under the assumption of normal distributions. For those cases we work with the following normal random variables

(7.43) $$\eta_{(k)i(i+1,\ldots,s)} = \eta_{(k)(i+1,\ldots,s)} - \eta_{(k)i} \; ; \; i=1,\ldots,s-1.$$

and

(7.44) $$\eta_{(k)i(1,\ldots,i-1)} = \eta_{(k)(1,\ldots,i-1)} - \eta_{(k)i} \; ; \; i=2,\ldots,s \; .$$

8. Sample Estimators and Their Variances

8.1. The unbiased sample estimators for the parameters of transvariation in the bivariate case and their corresponding variances were obtained elsewhere [Dagum 1960a and 1968]. Because of the linear transformation introduced in section 6, we may transform the multivariate distributions into bivariate cases. Then we can apply to these transformations the sample estimators and the corresponding variances obtained for the bivariate distributions.

8.2. Case of several distributions. The transvariability, the moment of transvariation and their corresponding maximums are linear combinations of their bivariate estimators.

Assuming independence among the parameters of transvariation corresponding to each combined r.e.

$$\Lambda_{ij} = (\Lambda_i, \Lambda_j) \; ; \quad i,j=1,\ldots,s; \quad i \neq j \; ,$$

we obtain the following variances of their estimators:

$$(8.1) \qquad \mathrm{var}({}_s p_{(k)\Lambda}) = \sum_{i \neq j} A_{ij}^2 \; \mathrm{var}\,(p_{ij,\Lambda})$$

$$(8.2) \qquad \mathrm{var}({}_s p_{M(k)\Lambda}) = \sum_{i \neq j} A_{ij}^2 \; \mathrm{var}\,(p_{ij,M,\Lambda})$$

$$(8.3) \qquad \mathrm{cov}({}_s p_{(k)\Lambda}, \; {}_s p_{M(k)\Lambda}) = \sum_{i \neq j} A_{ij}^2 \; \mathrm{cov}(p_{ij,\Lambda}, \; p_{ij,M,\Lambda})$$

$$(8.4) \qquad \mathrm{var}({}_s m_{r(k)\Lambda}) = \sum_{i \neq j} A_{ij}^2 \; \mathrm{var}(m_{r,ij,\Lambda})$$

$$(8.5) \qquad \mathrm{var}({}_s m_{r(k)M,\Lambda}) = \sum_{i \neq j} A_{ij}^2 \; \mathrm{var}(m_{r,ij,M,\Lambda})$$

$$(8.6) \qquad \mathrm{cov}({}_s m_{r(k)\Lambda}, \; {}_s m_{r(k)M,\Lambda}) = \sum_{i \neq j} A_{ij}^2 \; \mathrm{cov}(m_{r,ij,\Lambda}, \; m_{r,ij,M,\Lambda})$$

where A_{ij} is given in (7.5).

The variance estimator of the sample probability of transvariation is

$$(8.7) \qquad \mathrm{var}({}_s P_{(k)\Lambda}) = {}_s P_{(k)}^2 \Big[\frac{\mathrm{var}({}_s p_{(k)\Lambda})}{{}_s p_{(k)}^2} + \frac{\mathrm{var}({}_s p_{M(k)\Lambda})}{{}_s p_{M(k)}^2} - \frac{2 \cos({}_s p_{(k)\Lambda}, {}_s p_{M(k)\Lambda})}{{}_s p_{(k)} \; {}_s p_{M(k)}} \Big]$$

where the variances and covariance of the right hand member are given by (8.1) to (8.3).

The variance estimator of the sample intensity of transvariation is

$$(8.8)\quad var({}_sI_{r(k)\Lambda}) = \frac{{}_sI^2_{r(k)}}{r^2}\left[\frac{var({}_sm_{r(k)\Lambda})}{{}_sm^2_{r(k)}} + \frac{var({}_sm_{r(k)M,\Lambda})}{{}_sm^2_{r(k)M}} - \frac{2cov({}_sm_{r(k)\Lambda}, {}_sm_{r(k)M,\Lambda})}{{}_sm_{r(k)}\ {}_sm_{r(k)M}}\right]$$

where the variances and the covariance of the right hand member are given by (8.4) to (8.6).

The variances of the sample estimators, when $N_1 = N_2 = \ldots = N_s$, can be easily obtained.

A similar procedure is applied in order to obtain the variance of the sample estimators corresponding to sections 7.2.2 and 7.3.2 where we deduced a simplified result for the parameters of transvariation by an aggregative method.

9. Applications

9.1. The problem. The applications herein suggested are illustrative of the possibilities that the transvariation theory offers to quantitative analysis in economics.

These applications deal with comparative cost-of-living indexes, by family size and income before taxes, covering twenty cities of the United States[4], in 1963. These indexes are meant to apply to families residing in the suburbs of the 20 metropolitan statistical areas for which consumer price indexes have been computed. These families own their own homes; they range in size from two to six persons and, in income, from $6,000 to $24,000 before taxes. The indexes of the various areas are expressed as a percentage of the indexes of Washington, D.C., which is taken to be 100.

[4]SOURCE: United States, Bureau of the Census, Statistical Abstract of the United States, Department of Commerce, 1967.

For our purposes, a more meaningful statistical approach would be to express the indexes of each in terms of a percentage of the indexes for that city in some base year; e.g. 1959. Then, the result would be a quantitative statement regarding the relative changes of intersize (by family size), and|or interincome (by level of income).

9.2. The random variables: We have three random variables, whose domains are the three dimensional spaces Ω_i' (i=1,2,3) that are the nonnegative orthants of Euclidean space R_3. We denote their corresponding statistical observations by

(9.1) $$x_i = (x_{i1}, x_{i2}, x_{i3}) \; ; \quad i=1,2,3$$

where, the first subscript i stands for the size of the families, in our case: two, four and six, respectively. The second subscript stands for the level of income; in our case: six, twelve and eighteen thousand dollars per annum respectively. Therefore, x_{23} e.g., stands for a vector of twenty observations of cost-of-living indexes corresponding to families of four, with an income of $18,000 per annum.

Given the criterion of the Bureau of Census used in the formulation of these indexes, one may expect a high degree of transvariation. Much more interesting would be the case of indexes arrived at according to the criterion previously suggested.

9.3. The cases estimated: The parameters of transvariation were estimated for nine cases out of a larger number of possible combinations. These nine cases were worked out and their corresponding sample means are given in Table 9.1.

In section 9.4 we performed an application of transvariation between several distributions by considering the three univariate marginal distributions of the r.v. vector $x_1 = (x_{11}, x_{12}, x_{13})$.

The nine estimates were in the following correspondance: a) three to the binary combainations of the trivariate vector $x_1 = (x_{11}, x_{12}, x_{13})$; b) three to the combinations (x_{1j}, x_{3j}), j=1,2,3, between the trivariate vectors x_1 and x_3 ; and c) three to the combinations $(x_{1i} + x_{1j}, x_{3i} + x_{3j})$; i,j=1,2,3; $i < j$. Case a) can be regarded as transvariations among the components of a multivariate random variable; case b) represents marginal transvariations of the first order between x_1 and x_3; and case c) marginal transvariations of the second order

Table 9.1

Variables	λ
$x_{11} - x_{12}$	-0.160
$x_{11} - x_{13}$	-0.055
$x_{13} - x_{12}$	-0.105
$x_{31} - x_{11}$	-0.025
$x_{12} - x_{32}$	-0.270
$x_{13} - x_{33}$	-0.210
$(x_{11}+x_{12}) - (x_{31}+x_{32})$	-0.245
$(x_{11}+x_{13}) - (x_{31}+x_{33})$	-0.185
$(x_{12}+x_{13}) - (x_{32}+x_{33})$	-0.480

between x_1 and x_3 . All the estimates are nonparametrics. For the parametric cases we will follow the method set out in [Dagum 1960.a and 1968]. In this particular case it was not necessary, for the estimates of the marginal transvariation of the second order, to introduce Gini's mean difference (as was pointed out in (6.2) and (6.3)) because the components of the vectors are already additive. In fact, they are index numbers and therefore dimensionless.

The following nonparametric sample estimators were applied to the nine cases listed in Table 9.1: marginal transvariability, its maximums; the r^{th} marginal moment of transvariation for r=1 and 2, their maximums; and the moment $m_{\overline{1+1},M}$. The estimates of these are given in Table 9.2.

The variances and covariances corresponding to the first four columns estimates of Table 9.2 are given in Table 9.3.

Table 9.3

Variables	p	p_M	m_1	m_{1M}	m_2	m_{2M}	$m_{\overline{1+1},M}$
$x_{11} - x_{12}$	0.480	0.493	2.793	2.871	24.288	25.194	24.735
$x_{11} - x_{13}$	0.485	0.490	2.767	2.794	23.244	23.549	23.396
$x_{12} - x_{13}$	0.495	0.518	2.301	2.354	17.203	17.692	17.445
$x_{31} - x_{11}$	0.488	0.493	2.889	2.902	27.569	27.714	27.497
$x_{12} - x_{32}$	0.489	0.505	2.199	2.333	15.321	16.544	14.727
$x_{13} - x_{33}$	0.498	0.513	2.062	2.168	13.676	14.564	13.243
$(x_{11}+x_{12}) - (x_{31}+x_{32})$	0.496	0.518	5.060	5.185	78.636	81.146	77.397
$(x_{11}+x_{13}) - (x_{31}+x_{33})$	0.505	0.512	4.889	4.984	73.188	75.014	72.283
$(x_{12}+x_{13}) - (x_{32}+x_{33})$	0.494	0.508	4.246	4.486	57.534	61.725	55.496

Table 9.3

Variables	$var(p_\Lambda)$	$var(p_{M,\Lambda})$	cov $(p_\Lambda, p_{M,\Lambda})$	$var(m_{1,\Lambda})$	$var(m_{1M,\Lambda})$	cov $(m_{1\Lambda}, m_{1M,\Lambda})$
$x_{11} - x_{12}$	0.000624	0.000626	0.000611	0.0413	0.0425	0.0419
$x_{11} - x_{13}$	0.000624	0.000626	0.000620	0.0391	0.0395	0.0393
$x_{12} - x_{13}$	0.000625	0.000626	0.000599	0.0298	0.0305	0.0301
$x_{31} - x_{11}$	0.000626	0.000626	0.000620	0.0482	0.0485	0.0479
$x_{12} - x_{32}$	0.000626	0.000627	0.000606	0.0263	0.0278	0.0241
$x_{13} - x_{33}$	0.000627	0.000626	0.000608	0.0236	0.0247	0.0220
$(x_{11}+x_{12}) - (x_{31}+x_{32})$	0.000627	0.000626	0.000600	0.1329	0.1360	0.1282
$(x_{11}+x_{13}) - (x_{31}+x_{33})$	0.000627	0.000626	0.000617	0.1235	0.1258	0.1201
$(x_{12}+x_{13}) - (x_{32}+x_{33})$	0.000626	0.000626	0.000609	0.0990	0.1043	0.0914

In Table 9.4 we have the estimates of the marginal probability of transvariation, the first order marginal intensity of transvariation and their corresponding mean square errors.

Table 9.4

Variables	P	I_1	σ_P	σ_{I_1}
$x_{11} - x_{12}$	0.975	0.973	0.011	0.001
$x_{11} - x_{13}$	0.990	0.990	0.007	0.0004
$x_{12} - x_{13}$	0.956	0.978	0.014	0.001
$x_{31} - x_{11}$	0.990	0.996	0.007	0.010
$x_{12} - x_{32}$	0.968	0.938	0.012	0.032
$x_{13} - x_{33}$	0.971	0.951	0.012	0.030
$(x_{11}+x_{12}) - (x_{31}+x_{32})$	0.959	0.976	0.015	0.021
$(x_{11}+x_{13}) - (x_{31}+x_{33})$	0.985	0.981	0.008	0.019
$(x_{12}+x_{13}) - (x_{32}+x_{33})$	0.973	0.946	0.011	0.031

According to the statistical criterion introduced in [Dagum 1968], the differences between the distributions considered above are not statistically significant.

9.4. Transvariation among several distributions: As an illustrative example of the transvariation among several distributions we will deal with the three components of the r.v. vector $x_1 = (x_{11}, x_{12}, x_{13})$. Because $N_1 = N_2 = N_3 = 20$, we apply (7.15) to the probability of transvariation and (7.32) to the first order intensity of transvariation. Hence from Table 9.2 we have

$$_3P(1) = \frac{p_{12} + p_{13} + p_{23}}{p_{12,M} + p_{13,M} + p_{23,M}} = \frac{1.460}{1.501} = 0.97$$

and

$$_3I_1(1) = \frac{m_{1,12} + m_{1,13} + m_{1,23}}{m_{1,12,M} + m_{1,13,M} + m_{1,23,M}} = \frac{7.861}{8.019} = 0.98 \quad .$$

Their corresponding variances are, from Tables 9.2 and 9.3

$$\mathrm{var}(_3P_{(1),\Lambda}) = 0.000040$$

and

$$\mathrm{var}(_3I_{1(1)}) = 0.00000144$$

therefore

$$\sigma_{_3P(1)} = 0.0063$$

and

$$\sigma_{_3I_1(1)} = 0.0012 \quad .$$

ABSTRACT

We have developed the theory of transvariation for multivariate distribution functions. Further, we have defined and deduced the parameters of transvariation; first, without any assumption of parametric distribution functions (nonparametric case); and second, under the assumption of multivariate normal distributions. Our research continued with the study of the theory of transvariation among several (three or more) multivariate distribution functions.

For the purpose of simplifying computation, we have introduced a linear transformation that allows the application of bivariate transvariation theory to the transformed variables. The multivariate normal distribution is considered after the proof of two Lemmas regarding the distribution of a linear function of correlated normal random variables.

For the case of more than two multivariate distributions, Gini's aggregative method is applied to simplify further the computations in the applications.

The applications of this paper are in the field of comparative static economics. They affirm the fruitfulness of transvariation theory as a quantitative method in comparative statics (intertemporal and interspatial comparative analysis).

REFERENCES

1. Dagum, Camilo, (1960.a): Teoría de la transvariación, sus applicaciones a la economía. METRON, Istituto di Statistica, Università di Roma, vol. XX, no. 1-4.

2. ____________, (1960.b): Transvariación entre mas de dos distribuciones. Studi in Onore di Corrado Gini, vol. I, Università di Roma.

3. ____________, (1960.c): Transvariazione fra più di due distribuzioni. (Paper in Ref. 11).

4. ____________, (1961): Transvariación en la hipótesis de variables aleatorias normales multidimensionales. Bulletin of the International Statistical Institute, vol. XXXVIII, part. IV, Tokyo.

5. ____________, (1965): Probabilité et intensité de la transvariation dans l'space a N dimensions. Economie Appliquée, No. 4.

6. ____________, (1966): Wahrscheinlichkeit und ausmaβ der transvariation im n-dimensionalen raum. Statistische Hefte, 7 Jahrgang, Heft 1/2.

7. ____________, (1968): Nonparametric and Gaussian Bivariate Transvariation Theory. Its Application to Economics. Research Memorandum No. 99, Econometric Research Program, Princeton University.

8. Gini, Corrado, (1916): Il concetto di transvariazione e le sue prime applicazioni. Giornale degli Economisti e Rivista di Statistica. (Also in Ref. 11).

9. ____________, (1942): Per la determinazione delle probabilità di transvariozione tra più gruppi. Atti della V riunione della Società Italiana di Statistica. (Also in Ref. 11).

10. ____________, (1951): Della misura sintetica della transvariazione rispetto ad n caratteri (transvariazione n-dimensionale). Atti della XI riunione della Società Italiana di Statistica. (Also in Ref. 11).

11. ____________, (ed.) (1960): Transvariazione. Memorie di metodologia statistica, Vol. II, a cura di Giuseppe Ottaviani. Roma: Libreria Goliardica.

12. Gini, C. e G. Livada, (1943.a): Transvariazione a più dimensioni. Atti della VI riunione della Società Italiana di Statistica.

13. ________________, (1943.b): Nuovi comtributti alla teoria della transvariazione. Atti della VII riunione della Società Italiana di Statistica. (Also in Ref. 11).

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