# INTEGRATING MEASUREABLE

# AND

# CONTINUOUS CORRESPONDENCES

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## ABSTRACT

On a measure space $(A, \mathcal{A}, \mu)$, a correspondence $\varphi$ on $A$ is a function which assigns to each $a$ in $A$ a nonempty subset $\varphi(a)$ of $R^n$. Aumann has defined an integral of correspondences and has shown that if $\varphi$ has certain properties then $\Phi(E) = \int_E \varphi \, d\mu$, $E \in \mathcal{A}$ defines a countably additive correspondence on $\mathcal{A}$. This paper offers a proof of the converse result; namely, if a correspondence $\Phi$ on $\mathcal{A}$ satisfies certain properties, then a correspondence $\varphi$ on $A$ exists such that $\int_E \varphi \, d\mu = \Phi(E)$, $E \in \mathcal{A}$. This paper also provides conditions on $\varphi$ such that every point in the set $\int_E \varphi \, d\mu$ is in fact the integral of a continuous function $f$ such that $f(a) \in \varphi(a)$ a.e.

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Suppose $\mathcal{A}$ is a $\sigma$-algebra with unit A and supporting a probability measure $\mu$. Let S be a finite dimensional, real vector space. If Z is a correspondence from $\mathcal{A}$ to S, then Z(E) is a subset of S for each set E.[1] Let $\mathcal{M}_Z$ denote the set of Z-valued measures on $\mathcal{A}$; that is, $\zeta \in \mathcal{M}_Z$ implies $\zeta(E) \in Z(E)$ for every E in $\mathcal{A}$. This paper established conditions under which

$$Z(E) = \mathrm{eval}_E(\mathcal{M}_Z)^2, \quad E \in \mathcal{A}.^3$$

If $\mathcal{A}$ is a $\sigma$-field of subsets of a set A, then the preceding result is used to provide conditions under which there exists a correspondence $\varphi: A \Rightarrow S$ such that

$$Z(E) = \int_E \varphi d\mu, \quad E \in \mathcal{A}$$

where

$$\int_E \varphi d\mu = \{z \in S: z = \int_E f d\mu, f \in \mathcal{L}_\varphi\}^4$$

where $\mathcal{L}_\varphi$ is the set of integrable functions $f: A \to S$ with $f(a) \in \varphi(a)$ a.e. on A.

This Radon-Nikodym type of result can be strengthened if A is a topological space and if $\mathcal{A}$ is the $\sigma$-field of Borel subsets of A. Letting

$\mathcal{L}^c_\varphi = \{f \in \mathcal{L}_\varphi: f \text{ is continuous}\}$, conditions are stated for

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[1] The letters E,F,G,H will always refer to elements of $\mathcal{A}$. These elements will be called "sets". The operations in $\mathcal{A}$ are "join", $\cup$, "meet", $\cap$, and "complementation", $'$. We define $E \setminus F = E \cap F'$ and $E \subset F$ if $E \cap F = E$. E is then called a "subset" of F. The zero element of $\mathcal{A}$ is denoted $\emptyset$. E is null if $\mu(E) = 0$.

[2] $\mathrm{eval}_E(\mathcal{M}_Z) = \{z \in S: z = \zeta(E), \zeta \in \mathcal{M}_Z\}$.

[3] This result, which is contained in Theorem 1 below, was first discovered by G. Debreu and was described to this author in a version akin to Theorem 2.

[4] This definition of the integral of a correspondence is due to Aumann [1]. It has been related to another definition by Debreu [7].

$$\int_E \varphi \, d\mu = \int_E^c \varphi \, d\mu$$

where $\int_E^c \varphi \, d\mu = \{z \in S: z = \int_E f \, d\mu, \ f \in \mathcal{L}_\varphi^c\}$. These results are useful in the analysis of economies with large numbers of traders.[5]

A correspondence $Z$ on $\mathcal{Q}$ is <u>$\mu$-continuous</u> if $\mu(E) = 0$ implies $Z(E) = \{0\}$. $Z$ is <u>countably additive</u> if, for any disjoint sequence $\{E_n\}$ of sets,

$$Z(\cup E_n) = \Sigma Z(E_n)$$

where

$$\Sigma Z(E_n) = \{z = \lim_k \sum_{n=1}^{k} z_n, z_n \in Z(E_n),$$

where $\sum_{n=1}^{k} z_n$ converges absolutely to $z\}$. $Z$ is <u>nonempty</u> if $Z(E)$ is nonempty for every $E$. $Z$ is <u>closed - and convex - valued</u> if $Z(E)$ is closed and convex for every $E$. Because $\emptyset$ is disjoint from itself, the assumptions that $Z(\emptyset)$ is convex and contains $0$ and that $Z$ is (countably) additive imply that $Z(\emptyset)$ is a convex cone.

The correspondence $Z$ is <u>bounded below</u> if there exists a continuous, antisymmetric vector order $\geqq$ on $S$ and there exists an $S$-valued measure $\beta$ such that for every $E$,

$$z \in Z(E) \text{ implies } z \geqq \beta(E).$$

This condition is equivalent to requiring that there be a closed, convex cone P with a vertex $0$ and containing no lines (one-dimensional linear manifolds) such that for every $E$, $Z(E) - \beta(E) \subset P$. $Z$ is <u>contained in P</u> if $Z(E) \subset P$ for every $E$. The condition that $Z$ be bounded below defines a collection of correspondences whose images may be unbounded in the usual sense, but for which

[5] For example, see [4], [5], [6], [8], [13].

many of the same results hold as for correspondences whose images are bounded. In particular, the following result holds:[6]

THEOREM 1: Let $Z$ be a nonempty, closed- and convex-valued, countably additive, bounded below, $\mu$-continuous correspondence from $\mathcal{A}$ to $S$. Then, for every set $E$,

$$Z(E) = \text{eval}_E(\mathcal{M}_Z) .$$

PROOF: The proof uses an induction argument on the dimension of $S$. It is necessary to prove two preparatory lemmas:

LEMMA 1: Let $Z$ be a nonempty, countably additive correspondence and let $p \in S$. Define a function $\sigma_p(\cdot)$ from $\mathcal{A}$ to $R \cup \{+\infty\}$ by $\sigma_p(E) = \sup p \cdot Z(E)$,[7] $E \in \mathcal{A}$.
Then $\sigma_p(\cdot)$ is countably additive.

PROOF: Let $\{E_n\}$ be a sequence of disjoint sets and let $E = \cup E_n$. If $z \in Z(E)$ then there exist $z_n \in Z(E_n)$ such that $z = \Sigma z_n$ (where $\sum_{n=1}^{k} z_n$ converges absolutely to $z$ as $k \to \infty$). Hence

$$p \cdot z = \lim_k \sum_{n=1}^{k} p \cdot z_n \leqq \sum_{n=1}^{\infty} \sigma_p(E_n) ,$$

so $\sigma_p(E) \leqq \Sigma\ \sigma_p(E_n)$. If $\sigma_p(E) = +\infty$, we are finished. Suppose $\sigma_p(E) < \infty$ and suppose $\sigma_p(E_n) = +\infty$ for some $n$. Then there exists a sequence $\{x_k\} \subset Z(E_n)$ with $p \cdot x_k \underset{k}{\to} +\infty$. Since $Z$ is nonempty, there exists $y \in Z(E \setminus E_n)$ so $z_k = x_k + y \in Z(E)$ and $p \cdot z_k \to +\infty$. This contradicts the assumption that $\sigma_p(E) < +\infty$ so $\sigma_p(E_n) < +\infty$ for all $n$.

[6]This result can also be obtained from recent work of Rieffel, [10] and [11], if $Z$ is compact-valued.

[7]If $p \in S$ and $K \subset S$ then

$$\sup p \cdot K = \sup \{p \cdot x : x \in K\} .$$

For any $\epsilon > 0$, choose $z_n$ in $Z(E_n)$ satisfying

$$\sigma_p(E_n) - p \cdot z_n < \frac{\epsilon}{2^n} .$$

If $\sum_{n=1}^{\infty} \sigma_p(E_n) = +\infty$ then $\sum_{n=1}^{\infty} p \cdot z_n = +\infty$.

Let $x$ be any point in $Z(E) = \Sigma Z(E_n)$. Then $x$ is the absolute sum of $\{x_n\}$ where $x_n \in Z(E_n)$.

Define $y_k = \sum_{n=1}^{k} z_n + \sum_{n=k+1}^{\infty} x_n$. Then $y_k \in Z(E)$ for every $k$. However,

$$p \cdot y_k = \sum_{n=1}^{k} p \cdot z_n + \sum_{n=k+1}^{\infty} p \cdot x_n \underset{k}{\rightarrow} +\infty$$

because

$$\left| \sum_{n=k+1}^{\infty} p \cdot x_n \right| \leqq \sum_{n=k+1}^{\infty} |p \cdot x_n| \underset{k}{\rightarrow} 0$$

since

$$\sum_{n=k+1}^{\infty} |x_n| \underset{k}{\rightarrow} 0 .$$

This contradicts the assumption that $\sigma_p(E) < +\infty$, so $\sum_{n=1}^{\infty} \sigma_p(E_n) < +\infty$.

Choose $k_0$ large enough so that

$$\left| p \cdot y_{k_0} - \sum_{n=1}^{\infty} p \cdot z_n \right| < \epsilon ,$$

which is possible since $\sum_{n=1}^{\infty} p \cdot z_n$ is finite. But then

$$\left| p \cdot y_{k_0} - \sum_{n=1}^{\infty} \sigma_p(E_n) \right| < 2 \epsilon .$$

Further,

$$p \cdot y_{k_0} \leqq \sigma_p(E) \leqq \sum_{n=1}^{\infty} \sigma_p(E_n)$$

so

$$\left| \sigma_p(E) - \sum_{n=1}^{\infty} \sigma_p(E_n) \right| < 2 \epsilon .$$

Letting $\epsilon \to 0$ , we have

$$\sigma_p(E) = \sum_{n=1}^{\infty} \sigma_p(E_n) .$$

This completes the proof of LEMMA 1. A correspondence $S_p$ can be defined for each $p$ in $S$ by:

$$\begin{aligned} S_p(E) &= \{z \epsilon Z(E): \ p\cdot z = \sigma_p(E)\}, \quad E \epsilon \mathcal{A} \\ &= p^{-1}(\sigma_p(E)) \cap Z(E).^{8} \end{aligned}$$

<u>LEMMA 2:</u> Let $Z$ be a nonempty, countably additive correspondence and let $p \epsilon S$ .

(i) If $S_p(E)$ is not empty for some set $E$ , then $S_p(F)$ is not empty for any subset $F$ of $E$ .

(ii) If $S_p$ is nonempty, then it is countably additive.

<u>PROOF:</u> If $\{E_n\}$ is a disjoint sequence of sets and if $z \epsilon S_p(\cup E_n)$, then there exist $z_n$ in $Z(E_n)$ such that $z = \Sigma z_n$ . Suppose that for some $n_0$ , $z_{n_0} \notin S_p(E_{n_0})$ . Then there exists $y \epsilon Z(E_{n_0})$ with $p\cdot y > p\cdot z_{n_0}$ . But if $z' = y + \sum_{n \neq n_0} z_n$ , then $z' \epsilon Z(\cup E_n)$ and $p\cdot z' > p\cdot z$ . This contradicts the fact that $p\cdot z = \sigma_p(\cup E_n)$ . Thus $z_n \epsilon S_p(E_n)$ for every $n$ . This proves (i) and half of (ii): $S_p(\cup E_n) \subset \Sigma S_p(E_n)$ .

The proof that $\Sigma S_p(E_n) \subset S_p(\cup E_n)$ is made by supposing $\{z_n\}$ is a sequence of vectors satisfying $z_n \epsilon S_p(E_n)$ , $n=1,2,\ldots$ and $\sum_{n=1}^{k} z_n$ converges absolutely to some $z$ . Then $z \epsilon Z(\cup E_n)$ and

$$p\cdot z = \sum_{n=1}^{\infty} p\cdot z_n = \sum_{n=1}^{\infty} \sigma_p(E_n) = \sigma_p(\cup E_n)$$

by LEMMA 1.

[8] $p^{-1}$ denotes the function which is inverse to the function $p(\cdot): \ x \mapsto p\cdot x$ .

To prove Theorem 1, it suffices to consider only the case where $Z$ is contained in a closed convex cone with vertex zero and containing no lines, because if $Z$ is bounded below, then there exists a measure $\beta$ such that $Z-\beta$ is contained in such a cone. But if

$$(Z-\beta)(E) = \text{eval}_E(\mathcal{M}_Z-\beta), \quad \text{then} \quad Z(E) = \text{eval}_E \mathcal{M}_Z .$$

The relation $\text{eval}_E \mathcal{M}_Z \subset Z(E)$ is obvious. To prove the opposite inclusion, assume that the dimension $N$ of the vector space $S$ is at least equal to 2. Assume also that <u>if</u> $Z'$ is a nonempty, closed, convex, countably additive correspondence from $\mathcal{Q}$ to $R^{N-1}$ satisfying $Z'(0) = \{0\}$ and, for every $E$, $Z'(E)$ is contained in a fixed closed convex cone $P'$ with vertex $0$ containing no lines <u>then</u> $Z'(E) = \text{eval}_E (\mathcal{M}_{Z'})$. This is the induction hypothesis.

Because $Z(A)$ is a closed convex set containing no lines, any point in $Z(A)$ is a (finite) convex combination of points in ext $Z(A)$[9], where ext $Z(A)$ is the union of the set of extreme points of $Z(A)$ with the extreme rays of $Z(A)$.[10] Further, $\mathcal{M}_Z$ is convex. Thus it suffices to show that if $z \in \text{ext } Z(A)$, then there exists $\zeta \in \mathcal{M}_Z$ with $\zeta(A) = z$.

If $z \in \text{ext } Z(A)$, then $z \in Z(A) \cap H$, where $H$ is some supporting hyperplane to $Z(A)$.[11] Hence there exists a nonzero vector $p$ in $R^N$ such that $H = p^{-1}(\sigma_p(A))$ and $z \in S_p(A)$. By LEMMA 2, $S_p$ is a nonempty, countably additive correspondence. Further, for each $E$, $S_p(E)$ is a closed convex subset

[9] See, for example, THEOREM 6.13, page 54 in [12].

[10] A point $x$ in a set $K$ is an <u>extreme point</u> of $K$ if there is no line segment contained in $K$ and containing $x$ in its interior. A ray $\{z = x_0 + tx_1, t \geqq 0\}$ is an <u>extreme ray</u> if every line segment, which is contained in $K$ and which intersects the ray at a point in the interior of the segment, is entirely contained in the ray.

[11] See [12], THEOREM 7.11, page 66.

of $P$. Let $H_0$ be the $N-1$ dimensional subspace parallel to $H$. The remainder of the proof of the induction step consists of showing that, for each $E$, $S_p(E)$ can be projected into $H_0$ so as to yield a correspondence satisfying the conditions of the induction hypothesis. This requires the following result.

<u>LEMMA 3:</u> $S \setminus (H_0 \cup P \cup (-P))$ is not empty.

<u>PROOF:</u> If this were not true, then $H_0 \cup P \cup (-P) = S$ and hence

$$P \cup (-P) \supset (H_0 \cup P \cup (-P)) \setminus H_0 = S \setminus H_0 .$$

Let $x \in H_0^+$, the open half space above $H_0$. Then $x \in P$ or $x \in -P$. Suppose $x \in P$ and let $y$ be any other element of $H_0^+$. Suppose $y \in -P$. Now the line segment $[x,y]$ connecting $x$ and $y$ is in $H_0^+$ and $0 \notin [x,y]$. But then

$$[x,y] = \{[x,y] \cap P\} \cup \{[x,y] \cap (-P)\}$$

is a union of two nonempty, disjoint closed sets which contradicts the connectedness of $[x,y]$. Thus $y \in P$ so $H_0^+ \subset P$ and $H_0^- \subset -P$. This contradicts the assumption that $P$ contains no lines. Similarly, if $x \in -P$, then $H_0^+ \subset -P$ and $H_0^- \subset P$ which is also impossible. Thus the LEMMA is correct.

Let $L$ be the line through $0$ generated by any point in $S \setminus (H_0 \cup P \cup (-P))$. Then $L \cap (H_0 \cup P \cup (-P)) = \{0\}$. Let $P'$ be the projection of $P$ into $H_0$ along $L$. $P'$ is a convex cone with vertex $0$ because projection is a linear mapping. Further,

<u>LEMMA 4:</u> $P'$ is closed.

PROOF: Let $y$ be any point in $H_0$ which is not in $P'$. This means that $y + L$ is disjoint from $P$. We want to find a neighborhood $V$ in $H_0$ of $y$ which is disjoint from $P'$. Let

$$W_\epsilon = \{w \in S : |w-(y+z)| < \epsilon \quad \text{for some} \quad z \in L\}$$

be an $\epsilon$-neighborhood of $y + L$. Then it suffices to show that for some $\epsilon > 0$, $W_\epsilon \cap P = \emptyset$.

Suppose this is not true, so for $n=1,2,\ldots$ there exist $x_n$ in $P$ and $z_n$ in $L$ such that

$$|x_n - y - z_n| < 1/n .$$

If $\{z_n\}$ has a convergent subsequence $\{z'_n\}$, then its limit $z$ is also in $L$. Further, for any $n$ and $m$,

$$|x'_n - x'_m| \leq |x'_n - y - z'_n| + |z'_n - z'_m| + |x'_m - y - z'_m|$$

$$\leq 1/n + 1/m + |z'_n - z'_m|$$

so $\{x'_n\}$ is Cauchy. If $x$ is the limit of $\{x'_n\}$, then $x$ is in $P$. Further

$$|x-y-z| \leqq |x-x'_n| + |x'_n - y - z'_n| + |z'_n - z| \underset{n}{\rightarrow} 0 .$$

Thus $y + z = x \in P$. But then $y \in P'$ which contradicts our hypothesis.

Thus $\{z_n\}$ has no convergent subsequence, and hence $\{z_n\}$ is unbounded. Choose a subsequence $\{z'_n\}$ for which $|z'_n| \to +\infty$. This subsequence contains a subsequence $\{z''_n\}$ such that $\frac{z''_n}{|z''_n|}$ converges to some $z$ in $L$. But

$$\left|z - \frac{x''_n}{|z''_n|}\right| \leqq \left|z - \frac{z''_n}{|z''_n|}\right| + \left|\frac{z''_n + y}{|z''_n|} - \frac{x''_n}{|z''_n|}\right| + \frac{|y|}{|z''_n|}$$

so $\frac{x''_n}{|z''_n|} \to z$ also. Hence $z \in P \cap L$ which is impossible since $z \neq 0$. Thus there exists $\epsilon > 0$ such that $W_\epsilon$ is disjoint from $P$.

LEMMA 5: $P'$ contains no lines.

PROOF: Suppose $P'$ contains a line, which will be represented:

$$L_1 = \{z = x + \lambda y, \quad \lambda \epsilon R\}$$

where $x$ is some vector in $P'$ and $y$ is some nonzero vector in $H_0$. It will be shown that the line

$$L_2 = \{z = \lambda y, \quad \lambda \epsilon R\}$$

is also in $P'$: Let $\lambda y$ be a point in $L_2$. Then set

$$y_t = t(x + \lambda y) + (1-t)\lambda y, \quad t \in (0,1).$$

Then

$$\frac{1}{t} y_t = x + \lambda y + \frac{1-t}{t} \lambda y = x + \frac{\lambda}{t} y \in L_1 \subset P'.$$

Thus $y_t$ is a convex combination of the vectors $0$ and $\frac{1}{t} y_t$, both of which are in $P'$, so $y_t \in P'$ for $t \in (0,1)$. But $y_t \to \lambda y$ as $t \to 0$ and $P'$ is closed so $\lambda y \in P'$. Since $\lambda$ was arbitrary, $L_2 \subset P'$.

This shows, in particular, that $y \in P'$ and $-y \in P'$. Hence there exist vectors $z$ and $w$ in $L$ such that $y + z \in P$ and $-y + w \in P$. If $z = -w$, then $0 \neq y + z \in P$ and $-(y + z) = -y + w \in P$. This is impossible since $P \cap (-P) = \{0\}$; that is, $P$ contains no lines. Thus

$$0 \neq \frac{1}{2}(z + w) = \frac{1}{2}(y + z) + \frac{1}{2}(-y + w) \in P.$$

But $\frac{1}{2}(z + w) \in L$ also, which contradicts the disjointness of $P \setminus \{0\}$ and $L \setminus \{0\}$. Thus $P'$ contains no lines.

In summary $P'$, which is the projection of $P$ into $H_0$, is a closed convex cone with vertex $0$ containing no lines. For each $E$, let $S_p'(E)$

be the projection of $S_p(E)$ into $H_0$ along $L$ and let $\delta(E)$ be the projection of $S_p(E)$ into $L$ along $H_0$. Because $S_p(E)$ is contained in a translate of $H_0$, $\delta(E)$ is a singleton. It is convenient to treat $\delta$ as a function, rather than a correspondence, on $\mathcal{A}$.

Let $\mathrm{proj}_L(\cdot)$ be the continuous linear function which maps each $x$ in $S$ into its projection into $L$ along $H_0$. To show that $\delta$ is a countably additive function, choose any $x \in S_p(\cup E_n)$ where $\{E_n\}$ is some disjoint sequence of sets. Because $S_p$ is countably additive, there exist $x_n$ in $S_p(E_n)$ such that the sum $\Sigma x_n$ converges absolutely to $x$. Then

$$\begin{aligned}\sum_{n=1}^{k} \delta(E_n) &= \sum_{n=1}^{k} \mathrm{proj}_L(x_n) \\ &= \mathrm{proj}_L\left(\sum_{n=1}^{k} x_n\right) \\ &\underset{k}{\rightarrow} \mathrm{proj}_L(x) = \delta(\cup E_n) \quad .\end{aligned}$$

In particular, the sum $\sum_{n=1}^{\infty} \delta(E_n)$ is independent of the order of summation.

For each $E$, $S_p'(E)$ is a translate of $S_p(E)$ which is nonempty, closed and convex. Hence $S_p'$ is nonempty, closed, convex and $P'$-valued. Further, $S_p' = S_p - \delta$ so $S_p'$ is countably additive. Thus $S_p'$ satisfies the conditions of the induction hypothesis. If $z$ is any point in $S_p(A) = Z(A) \cap H$, then $z = y + \delta(A)$ where $y \in S_p'(A)$. By the induction hypothesis, there exists a measure $\zeta'$ on $\mathcal{A}$ such that $\zeta'(E) \in S_p'(E)$ for all $E$ and $\zeta'(A) = y$. Then $\zeta' + \delta \in \mathcal{M}_Z$ and $(\zeta' + \delta)(A) = z$.

This completes the proof of the fact that if the induction hypothesis holds then $Z(A) = \mathrm{eval}_A(\mathcal{M}_Z)$. If $E$ is a proper subset of $A$, then the preceding remarks show that for each $z \in Z(E)$ there exists a measure $\zeta'$ on $\mathcal{A}_E$, the $\sigma$-algebra of subsets of $E$, such that $\zeta'(F) \in Z(F)$ for every $F \subset E$ and

$\zeta'(E) = z$. Similarly, there exists a measure $\zeta''$ on $\mathcal{A}_{A\setminus E}$ satisfying $\zeta''(F) \in Z(F)$ for every $F \subset A \setminus E$. If $\zeta'$ and $\zeta''$ are extended to all of $\mathcal{A}$ in an obvious way, then $\zeta' + \zeta'' \in \mathcal{M}_Z$ and $(\zeta' + \zeta'')(E) = z$. Thus, under the induction hypothesis, $Z(E) = \text{eval}_E(\mathcal{M}_Z)$.

To complete the proof of THEOREM 1, it is only necessary to consider the case where $N = 1$; that is, where $Z(E)$ is a subset of the real line. $Z(E)$ is nonempty, closed, convex and contains no lines so it is either a singleton, an interval or a half line. Define

$$\mathcal{H}_1 = \{H \in \mathcal{A}: \; Z(H) \text{ is a singleton}\}.$$

Then $\mathcal{H}_1$ is closed under countable unions and hence there exists a set $H_1 \in \mathcal{H}_1$ such that

$$\mu(H_1) = \sup\{\mu(H): \; H \in \mathcal{H}_1\}.$$

If $E$ is any subset of $H_1$, then $Z(E)$ is a singleton. If $E$ is any nonnull set disjoint from $H_1$, then $Z(E)$ contains more than one point. It is clear that $Z(E) = \text{eval}_E(\mathcal{M}_{Z|H_1})$ holds for any $E \subset H_1$, where $\mathcal{M}_{Z|H_1}$ is the set of $S$-valued measures $\zeta$ on $\mathcal{A}_{H_1}$ such that $\zeta(E) \in Z(E)$ for $E \in \mathcal{A}_{H_1}$.

Define

$$\mathcal{H}_2 = \{H \in \mathcal{A}: \; Z(H) \text{ is a bounded, nondegenerate interval}\},$$

and let $s = \sup\{\mu(H): H \in \mathcal{H}_2\}$. For every $n=1,2,\ldots$ there exists $G_n \in \mathcal{H}_2$ satisfying $s - \mu(G_n) < 1/n$. Let

$$H_2 = \bigcup G_n \quad \text{and} \quad F_n = G_n \setminus \left(\bigcup_{k=1}^{n-1} G_k\right).$$

It suffices to show for each $n$ that if $E \subset F_n$ then $Z(E) = \text{eval}_E(\mathcal{M}_{Z|F_n})$. But since $Z(F_n)$ is a bounded interval, there exist $p$ and $p'$ in $R$ such that $S_p(F_n)$ and $S_{p'}(F_n)$ are singletons and $Z(F_n) = \langle S_p(F_n), S_{p'}(F_n)\rangle$, the convex hull of $\{S_p(F_n), S_{p'}(F_n)\}$. By LEMMA 2, $S_p$ and $S_{p'}$ are countably additive on $\mathcal{A}_{F_n}$ and so can be considered elements of $\mathcal{M}_{Z|F_n}$. Further, $\mathcal{M}_{Z|F_n}$ is convex. It is then clear that $Z(E) = \text{eval}_E(\mathcal{M}_{Z|F_n})$ if $E \in \mathcal{A}_{F_n}$, $n = 1,2,\ldots$ .

Let $H_3 = A \setminus (H_1 \cup H_2)$[12]. If $H_3$ is null, the proof is finished. Otherwise, for every nonnull subset $F$ of $H_3$, $Z(F)$ is a half-line. If $p \in -P$ and if $z$ is any point in $Z(F)$, then $z \geqq S_p(F)$ (where $\geqq$ is the order induced on $R$ by $P$ ). Define

$$\zeta = S_p\big|_{H_3} + \frac{(z - S_p(F))}{\mu(F)} \mu\big|_{H_3} .$$

Then LEMMA 2 implies that $\zeta \in \mathcal{M}_{Z|H_3}$ and $\zeta(F) = z$ . Thus $Z(F) = \text{eval}_F(\mathcal{M}_{Z|H_3})$. This completes the proof of THEOREM 1.

It will now be shown that THEOREM 1 yields a Radon-Nikodym theorem for countably additive correspondences. The following LEMMA is needed:

<u>LEMMA 6:</u> If $\{p_n\}$ is any countable dense subset of $S$ and if $K$ is a nonempty closed convex set containing no lines, then $K = \bigcap_n H_n^-$ where $H_n^- = \{z \in S: p_n \cdot z \leq \sup p_n \cdot K\}$ .

<u>PROOF:</u> It is clear that $K$ is contained in the intersection specified above. Conversely suppose there exists $x \notin K$ . Then there exists a vector $p$ such that $\sup p \cdot K < p \cdot x$ . If $y$ is any element of $K$ , then there exists $z \in [x,y]$ such that $\sup p \cdot K < p \cdot z < p \cdot x$ . We can suppose without loss of generality that $z = 0$ by translating by $-z$ .

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[12] If $\mathcal{H}_2$ is empty, let $H_2$ be the empty subset of $A$ . The $\mu$-continuity of $Z$ implies that $\mathcal{H}_1$ is not empty.

Let $P$ be the smallest closed convex cone with vertex $0$ containing $K$; that is, $P$ is the projecting cone of $K$ with respect to $0$. If $A(K)$ is the asymptotic cone of $K$, then it can be shown (see [12] THEOREM 5.12) that

(1) $$P = \bigcup_{\lambda > 0} \lambda K \cup A(K) .$$

It can also be shown (see [12] THEOREM 5.7) that if $u$ is any vector in $K$, then

(2) $$A(K) = \{z \epsilon S: u + \lambda z \in K, \text{ for every } \lambda \geqq 0 \} .$$

This implies that $A(K) + u \subset K$.

We shall show that $P$ contains no lines by assuming that $P$ contains a line $L$ and by then finding a contradiction. Because $P$ is a closed convex cone with vertex $0$, it suffices to consider the case where $0 \in L$.[13] By (1), we first consider the case $L \subset A(K)$. Then by (2), $L + u \subset A(K) + u \subset K$ which contradicts the assumption that $K$ contains no lines. Thus there exists $y_0 \in L$ satisfying $y_0 \notin A(K)$. This means that

$$L^+ = \{y \epsilon S: y = \lambda y_0, \quad \text{some} \quad \lambda > 0 \}$$

is disjoint from $A(K)$. Define

$$L^- = \{y \epsilon S: \quad y = \lambda y_0, \quad \text{some} \quad \lambda \leqq 0\} .$$

Suppose $L^- \subset A(K)$. Now $L^+ \subset \bigcup_{\lambda > 0} \lambda K$ so there exists $v \epsilon L^+ \cap K$.

Then $-v \epsilon L^- \subset A(K)$ so there exist $\lambda_n > 0$ and $x_n \in K$ such that $\lambda_n \to 0$ and $\lambda_n x_n \to -v$. For any $\epsilon > 0$, choose $M$ so $|\lambda_M x_M - (-v)| < \epsilon$. Define

[13] See the first part of the proof of LEMMA 5.

$$z = \frac{v}{1+\lambda_M} + \frac{\lambda_M}{1+\lambda_M} x_M$$

which is an element of K since K is convex. But

$$|z| \leqq |v + \lambda_M x_M| < \epsilon .$$

Since $\epsilon$ was arbitrary and K is closed, $0 \in K$. This is false and so we conclude that $L^- \cap A(K) = \{0\}$.

We assumed that $L^- \subset P$ so $L^- \subset \bigcup_{\lambda > 0} \lambda K$. In particular, $-v = \lambda_0 v_0$ for some $\lambda_0 > 0$ and $v_0 \in K$ where $v$ is the element of $L^+ \cap K$ chosen in the preceding paragraph. But then

$$\begin{aligned} 0 &= v-v \\ &= v + \lambda_0 v_0 \\ &= \frac{1}{1+\lambda_0} v + \frac{\lambda_0}{1+\lambda_0} v_0 \in K . \end{aligned}$$

But $0 \notin K$ so $L^- \not\subset P$. Thus P contains no lines.

Let $P^0 = \{z \in S: z \cdot y \leqq 0 \text{ for every } y \in P\}$ be the polar of P, let $S' = P^0 - P^0$ be the subspace generated by $P^0$ and let $(S')^\perp$ be the orthogonal complement of S'. If $x \in (S')^\perp$ then $x \in P^{00} = P$ so $(S')^\perp \subset P$. Because P contains no lines, $(S')^\perp$ has dimension 0 and $S' = S$. Thus $P^0$ has a non-empty interior.

Because $\sup p \cdot K < 0$, $K \subset \{z \in S: p \cdot z \leqq 0\}$ which is a closed convex cone with vertex 0. Hence $P \subset \{z \in S: p \cdot z \leqq 0\}$ so $p \in P^0$, where p was chosen in the first paragraph of this proof. Because $P^0$ is convex and has a nonempty interior, any neighborhood of p contains a set which is a subset of $P^0$ and which is an open set in S. But then this neighborhood contains an element of the sequence $\{p_n'\}$ which is dense in S. Thus there exists a subsequence $\{p'_n\}$ of $\{p_n\}$ such that $p'_n \in P^o$ and $p'_n \to p$. Because $p'_n \in P^o$, $\sup p'_n \cdot K \leqq 0$ for every $n$. Because $p'_n \to p$, there exists $n_0$ such that $p'_{n_0} \cdot x > 0$. Thus $x \notin \bigcap_{n=1}^{\infty} H_n^-$. This ends the proof of LEMMA 6.

Let $\mathcal{A}$ be a $\sigma$-field of subsets of a set $A$ and let $\mu$ be a probability measure on $\mathcal{A}$. $\mathcal{A}_\mu$ will denote the $\mu$-completion of $\mathcal{A}$; that is, $E \in \mathcal{A}_\mu$ if and only if there exist $F$ and $G$ in $\mathcal{A}$ such that $\mu(G) = 0$ and $E = F \cup H$ for some subset $H$ of $G$. A correspondence $\varphi$ from $A$ to $S$ is <u>measureable</u> if the graph of $\varphi$,

$$G_\varphi = \{(a,x) \in A \times S : x \in \varphi(a)\},$$

is an element of $\mathcal{A}_\mu \otimes \mathcal{B}$, the product $\sigma$-field on $A \times S$ generated by $\mathcal{A}_\mu$ and $\mathcal{B}$, the Borel subsets of $S$. $\mathcal{L}_\varphi$ is defined to be the collection of $\mu$-integrable functions $f$ from $A$ to $S$ such that for almost every $a \in A$, $f(a) \in \varphi(a)$. We then define the integral of a measureable correspondence $\varphi$ on $A$ by

$$\int_E \varphi \, d\mu = \{\int_E f \, d\mu : \text{some } f \text{ in } \mathcal{L}_\varphi\}.$$

If $Z$ is $\mu$-continuous, define $\mathcal{L}_Z$ to be the collection of $\mu$-integrable functions $f$ from $A$ to $S$ such that there exists $\zeta \in \mathcal{M}_Z$ satisfying $f(a) = \frac{d\zeta}{d\mu}(a)$ for almost every $a \in A$.

<u>THEOREM 2.</u> If $Z$ is a nonempty, closed, convex, bounded below, countably additive, $\mu$-continuous correspondence on $\mathcal{A}$, then there exists a measureable, closed-and convex-valued correspondence $\varphi$ on $A$ such that for every coalition $E$,

$$Z(E) = \int_E \varphi d\mu .$$

PROOF: Let $\{p_n\}$ be a countable dense subset of $S$. For each $n$, define

$$\sigma_n(E) = \sup p_n \cdot Z(E).$$

$\sigma_n$ is $R \cup \{+\infty\}$-valued, countably additive (by LEMMA 1) and $\mu$-continuous. Thus there exists a measureable function $s_n$ from $A$ to $R \cup \{+\infty\}$ such that for every $E$, $\sigma_n(E) = \int_E s_n \, d\mu$.[14] $\varphi_n$ is defined by:

$$\varphi_n(a) = \{x \in S: p_n \cdot x \leqq s_n(a)\}, \quad a \in A,$$

and $\varphi$ is defined by:

$$\varphi(a) = \bigcap_{n=1}^{\infty} \varphi_n(a), \quad a \in A.$$

Since $G_\varphi = \bigcap_n G_{\varphi_n}$, it suffices to show $G_{\varphi_n} \in \mathcal{A}_\mu \otimes \mathcal{B}$ for each $n$. To simplify notation, the subscript $n$ will be dropped from $s_n$. $\varphi_n$ will be replaced by $\psi$, so that

$$\psi(a) = \{x \in S: p \cdot x \leqq s(a)\} \quad a \in A.$$

For any two elements $c$ and $d$ of $R \cup \{+\infty\}$, let

$$[s \leqq c] = \{a \in A: s(a) \leqq c\}$$
$$[s \geqq c] = \{a \in A: s(a) \geqq c\}$$
$$[s = c] = [s \leqq c] \cap [s \geqq c]$$
$$[s < c] = [s \leqq c] \setminus [s \geqq c]$$
$$[c < s \leqq d] = [s \leqq d] \setminus [s \leqq c]$$
$$[c < s < d] = [s < d] \setminus [s \leqq c].$$

[14] This because of the usual Radon-Nikodym Theorem. See Proposition IV. 1.4, page 111 in Neveu [9].

For each $m = 1,2,\ldots$ define a simple measureable function from $A$ to $R \cup \{+\infty\}$ by

$$f_m = -m\,\chi_{[s \leqq -m]} + \sum_{k=-m2^m+1}^{m2^m} \frac{k}{2^m}\,\chi_{\left[\frac{k-1}{2^m} < s \leqq \frac{k}{2^m}\right]} + m\,\chi_{[m < s < +\infty]} + \infty\,\chi_{[s=\infty]}\,,$$

where $\chi_B$ is the characteristic function of the set $B$.
Then $f_m(a) \to s(a)$ for every $a \in A$ and for every $a \in A$ there exists $k$ such that $m \geqq k$ implies $f_m(a) \geqq s(a)$.

Define

$$G_m = \{(a,x) \in AxS:\ p\cdot x \leqq f_m(a)\}$$
$$= \bigcup_{i=1}^{k_m} A_i x p^{-1}((-\infty,\ c_i])$$

where the $c_i$ are scalars in $R \cup \{+\infty\}$ and the $A_i$ are disjoint measureable sets such that $f_m = \sum_{i=1}^{k_m} c_i\,\chi_{A_i}$. It is clear that $G_m \in \mathcal{A}_\mu \otimes \mathcal{B}$ since it is a union of measureable rectangles. Further,

$$G_\psi = \bigcup_{k=1}^{\infty} \bigcap_{m \geqq k} G_m$$

so $G_\psi \in \mathcal{A}_\mu \otimes \mathcal{B}$, as was to be shown.

To complete the proof of the THEOREM, it will be shown that $\mathcal{L}_\varphi = \mathcal{L}_Z$ and hence $Z(E) = \mathrm{eval}_E(\mathcal{M}_Z) = \int_E \varphi\, d\mu$ for every $E$ in $\mathcal{A}$, by THEOREM 1. The relation $\mathcal{L}_Z \subset \mathcal{L}_\varphi$ is clear. Conversely, suppose $f \notin \mathcal{L}_Z$. Then for some nonnull $E$ in $\mathcal{A}$, $\int_E f\, d\mu \notin Z(E)$. Because $Z(E)$ is a closed convex nonempty set containing no lines, LEMMA 6 implies there exists $n$ such that $p_n \cdot \int_E f\, d\mu > \sigma_n(E)$. But then $f(a) \notin \varphi_n(a) \supset \varphi(a)$ for every $a$ in some nonnull subset of $E$. Thus $f \notin \mathcal{L}_Z$ implies $f \notin \mathcal{L}_\varphi$.

We now assume that A is a compact topological space and that $\mathcal{a}$ is the σ-field of Borel subsets of A . A correspondence $\varphi : A \Longrightarrow S$ is upper-semi-continuous (usc) if, for every open subset G of S , the set

$$\varphi^{+}(G) = \{a \in A: \varphi(a) \subset G\}$$

is open; φ is lower-semi-continuous (lsc) if G open in S implies

$$\varphi^{-}(g) = \{a \in A: \varphi(a) \cap G \neq \emptyset\}$$

is open. φ is continuous if it is usc and lsc.

THEOREM 3: Suppose that $\varphi : A \Longrightarrow S$ is a convex- and compact-valued, continuous, nonempty correspondence. Then for every Borel subset E of the compact set A:

$$\int_{E} \varphi \, d\mu = \int_{E}^{c} \varphi \, d\mu .$$

PROOF: The proof uses induction on the dimension of S .

Since φ is compact valued and is a continuous correspondence, it is a continuous function from A to $\mathcal{K}$ , the nonempty, compact subsets of S with the Hausdorff metric topology.[15] Hence the image of A under φ is compact in $\mathcal{K}$ and hence bounded in S . This implies that φ is integrably bounded which implies that $\int_{E} \varphi \, d\mu$ is compact in S for every Borel E ([1], THEOREM 4).

[15]This topology on $\mathcal{K}$ is defined in [3] pp. 132-133. The continuity of φ with respect to the Hausdorff metric on $\mathcal{K}$ is implied by THEOREM 1, page 133 in [3].

If $z$ is in the compact, convex set $\int_E \varphi \, d\mu$ , then $z$ is a convex combination of a finite number of points $z_i$ in $\mathrm{Ext} \int_E \varphi \, d\mu$ , the set of points in $\int_E \varphi \, d\mu$ through which some supporting hyperplane passes. Suppose that for each $i$ there exists a continuous function $f_i$ in $\mathcal{L}_\varphi$ with $z_i = \int_E f_i \, d\mu$ . Since $z = \Sigma \lambda_i z_i$ , where $\Sigma \lambda_i = 1$, $\lambda_i > 0$ , then $f = \Sigma \lambda_i f_i \in \mathcal{L}_\varphi$ and $f$ is continuous. But then $z = \int_E f \, d\mu \in \int_E^c \varphi \, d\mu$ .

Thus it suffices to show that if $z \in \mathrm{Ext} \int_E \varphi \, d\mu$ , then $z \in \int_E^c \varphi \, d\mu$ . There exists a nonzero $p$ in $S$ such that $p.z \geqq p.y$, $y \in \int_E \varphi \, d\mu$ . Define

$$\begin{aligned} s(a) &= \sup p \cdot \varphi(a) \\ \psi(a) &= \{x \in \varphi(a): \; p \cdot x = s(a)\} \\ &= \varphi(a) \cap \Delta(a) \end{aligned}$$

where

$$\Delta(a) = \{y \in S: \; p \cdot y \geqq s(a)\} .$$

By a well-known result ([3], p. 122), $s(\cdot)$ is a continuous function and $\psi$ is usc . It is easy to show that $\psi$ is also lsc: Suppose $a_o \in \Delta^-(G) = \{a \in A: \; G \cap \Delta(a) \neq \emptyset\}$ for a given open subset $G$ of $S$ . Then there exists $y_o \in G$ with $p \cdot y_o \geqq s(a_o)$. Since $G$ is open and $p \neq 0$ , there exists $y_1 \in G$ with $p \cdot y_1 > s(a_o)$. Since $s(\cdot)$ is continuous, there is a neighborhood $U(a_o)$ such that $a \in U(a_o)$ implies $s(a) < p \cdot y_1$. Thus $y_1 \in \Delta(a) \cap G$ for $a$ in $U(a_o)$, so $U(a_o) \subset \Delta^-(G)$; that is, $\Delta^-(G)$ is open in $A$ . Thus $\Delta$ is lsc and hence so is $\psi = \varphi \cap \Delta$ ([3], THEOREM 3, p. 120).

Now $z \in [\int_E \varphi \, d\mu] \cap [p^{-1}(\sup p \cdot \int_E \varphi \, d\mu)]$. But it is easily seen that this intersection equals $\int_E \psi \, d\mu$ .[16] Thus it suffices to show

$$\int_E \psi \, d\mu = \int_E^c \psi \, d\mu .$$

[16] This demonstration requires the use of the Measureable Choice Theorem. See [2], page 4.

This relation will be shown to hold by the induction hypothesis.

Let $L$ be the line generated by a vector in the algebraic complement of $H = p^{-1}(0)$. Let $\text{proj}_H$ be the mapping projecting $S$ into $H$ along $L$ and let $\text{proj}_L$ be the mapping projecting $S$ into $L$ along $H$. Define

$$t(a) = \text{proj}_L \psi(a) \qquad a \in A,$$

$$\theta(a) = \text{proj}_H \psi(a) \qquad a \in A.$$

Since $\psi$ is a compact-, convex-valued continuous correspondence, so is $\theta$, by the continuity and linearity of $\text{proj}_H$. By the induction hypothesis,

$$\int_E \theta \, d\mu = \int_E^c \theta \, d\mu .$$

But

$$\int_E \psi \, d\mu = \int_E [\theta + t] \, d\mu$$

$$= \int_E \theta \, d\mu + \int_E t \, d\mu$$

$$= \int_E^c \theta \, d\mu + \int_E t \, d\mu$$

$$= \int_E^c \psi \, d\mu$$

because $t$ is a continuous function on $A$.

To complete the proof of the theorem, it is necessary to consider the case where $S = R$. Define two functions on $A$:

$$u(a) = \max \{x \in \varphi(a)\}$$

$$b(a) = \min \{x \in \varphi(a)\} .$$

Then $u(a) = \sup p \cdot \varphi(a)$ where $p = 1$ and $b(a) = -\sup p \cdot \varphi(a)$ where $p = -1$. Thus $u(\cdot)$ and $b(\cdot)$ are continuous. They are both integrable since A is compact. Further, if

$$z \in \int \varphi \, d\mu$$

then $z = \lambda \int b \, d\mu + (1-\lambda) \int u \, d\mu$ for some $\lambda$ in $[0,1]$. Then

$$z = \int [\lambda b + (1-\lambda) u] \, d\mu \in \int^{c} \varphi \, d\mu .$$

This completes the proof of THEOREM 3.

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On a measure space $(A, \mathcal{A}, \mu)$, a correspondence $\varphi$ on $A$ is a function which assigns to each $a$ in $A$ a nonempty subset $\varphi(a)$ of $R^n$. Aumann has defined an integral of correspondences and has shown that if $\varphi$ has certain properties then $\Phi(E) = \int_E \varphi \, d\mu$, $E \in \mathcal{A}$ defines a countably additive correspondence on $\mathcal{A}$. This paper offers a proof of the converse result; namely, if a correspondence $\Phi$ on $\mathcal{A}$ satisfies certain properties, then a correspondence $\varphi$ on $A$ exists such that $\int_E \varphi \, d\mu = \Phi(E)$, $E \in \mathcal{A}$. This paper also provides conditions on $\varphi$ such that every point in the set $\int_E \varphi \, d\mu$ is in fact the integral of a continuous function f such that $f(a) \in \varphi(a)$ a.e.

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