CONVEXITY AND CONTINUITY PROPERTIES OF PREFERENCE FUNCTIONS

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ABSTRACT

This paper studies the convexity properties of the preferences of a perfectly competitive economy. These results are used to establish a relation between two alternative types of continuity of preferences used by R. Aumann and K. Vind. This discussion is related to the problem of establishing conditions for the equivalence of two kinds of blocking used in the definition of the core of an economy.

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1. Introduction

An important tool, in the study of economies with large numbers of traders, is the preference function. This function assigns, to each coalition E of traders, a preference relation $\geqq_E$ on the set of all allocations of commodity bundles[1] among the coalitions. One can derive, from this preference function, for each coalition E and for each allocation α , a set $P_\alpha(E)$ of commodity bundles which can be divided up among the subcoalitions of E in such a way as to make everyone better off than he would be receiving the bundle assigned to him by α . Thus, if E is an individual consumer, then $P_\alpha(E)$ is the set above E's indifference curve through the bundle $\alpha(E)$.

This paper demonstrates that, under "perfect competition" and under certain regularity conditions on the preference function, the set $P_\alpha(E)$ and certain other related subsets of the commodity space S are convex for every E and α. This result is well known,[2] but this paper treats the problem somewhat more thoroughly than earlier papers. Further, these results are used to find conditions under which the sets $P_\alpha(E)$ are topologically open. Interest in this problem originated in the observation that, although Vind[15] assumed $P_\alpha(E)$ was open, this assumption could not be easily related to the corresponding assumptions

[1] A "commodity bundle" is a vector in a finite dimensional (real) vector space, S . The concept of a preference function was developed by Debreu in [7].

[2] For example, see the Appendix of Vind's article [15].

made in other models of economies with large numbers of traders; most notably, the model of Aumann [2].[3] The openness of $P_\alpha(E)$ is a very useful tool in the study of general equilibrium economic models.[4]

The final result of this paper concerns the connection between the relations $>_E$ and $\gg_E$. $\beta \gg_E \alpha$ means that every (nonzero) subcoalition of E prefers β to α; $\beta >_E \alpha$ means that some (nonzero) subcoalition F prefers β to α and $E \setminus F$ is indifferent between β and α. When $\beta >_E \alpha$ holds, we shall say β is <u>semibetter</u> for E than α and, when $\beta \gg_E \alpha$ holds, we shall say β is <u>strictly better</u> for E than α.

2. Specification of the Model

The set of coalitions, $\mathcal{A}$, is a σ-algebra with operations join $\cup$, meet $\cap$ and complementation $'$. $\mathcal{A}$ has a unit element A, which represents the set of all individual economic agents, and a zero element $0 = A' \neq A$, which represents the empty set of agents. It is also assumed that there exists a probability measure μ[5] on $\mathcal{A}$ such that $\mu(E) = 0$ implies $E = 0$. (The letters E, F, G, H will always denote coalitions in $\mathcal{A}$.) This structure of a σ-algebra $\mathcal{A}$ containing a unit A and supporting a strictly positive measure μ is called a <u>σ-algebra of coalitions.</u>

For any two coalitions E and F we define

$$E \setminus F = E \cap (F') .$$

E and F are disjoint if $E \cap F = 0$. A partial order $\subset$ is defined on $\mathcal{A}$ by

[3] The author wishes to acknowledge the influence of Prof. Debreu's insistence on the importance of reconciling these assumptions.

[4] For example, see [4] and [5].

[5] The measure μ need not be given a probabilistic interpretation, however. The economic meaning of this measure is developed in [5] and [11].

$$E \subset F \quad \text{if} \quad E \cap F = E$$

and E is then a subcoalition of F. If F is a nonzero coalition, then $\mathcal{A}_F$ denotes the sub-σ-algebra of coalitions consisting of the subcoalitions of F. $\mathcal{A}$ is nonatomic if every nonzero coalition contains a nonzero subcoalition which is distinct from the given coalition. This is equivalent to requiring that the measure μ specified above be nonatomic. The relation of this condition to the idea of perfect competition has been given in [4].

We assume there is a consumption correspondence X on $\mathcal{A}$ which assigns to each coalition E a closed, convex subset $X(E)$ of S. $X(E)$ is the collection of commodity bundles which E is able to consume. It is assumed that $X(0) = \{0\}$ and $0 \in X(E)$ for every E. Further, X is countably additive; that is, if $\{E_n\}$ is a sequence of disjoint coalitions and $E = \cup E_n$, then

$$X(E) = \Sigma X(E_n)$$

where
$$\Sigma X(E_n) = \{z \in S: \text{ for each } n = 1,2,\ldots \text{ there exists } z_n \in X(E_n) \text{ such that } \sum_{n=1}^{k} z_n \text{ converges absolutely in } k \text{ to } z\}.$$

Finally, we assume that X is bounded below; that is, there exists in S a cone with vertex 0, P, which is closed, convex and pointed, i.e. P contains no straight lines (one-dimensional affine manifolds) and there exists an S-valued measure β on $\mathcal{A}$ such that for every E, $X(E) - \beta(E) \subset P$.

The economic meaning of these assumptions has been given in [4]. If γ is in $\mathcal{M}_X$, which means that γ is an S-valued measure on $\mathcal{A}$ satisfying

$$\gamma(E) \in X(E), \qquad E \text{ in } \mathcal{A},$$

then γ is called an allocation. The above assumptions on X imply that to every point z in $X(E)$, there corresponds an allocation γ with $\gamma(E) = z$. (see Theorem 1 in [3]).

We assume that there is an allocation ω , called the resources allocation, which specifies each coalition's holdings of commodities before trade or production occur. There is another correspondence Y from $\mathcal{Q}$ to S called the production correspondence. For each E , $Y(E)$ is the set of vectors y such that if E possessed the inputs required by y , that is, the negative coordinates of y , then E has the technology to obtain the outputs of y , which are the positive coordinates of y . We assume Y satisfies the same assumptions as X . $\mathcal{M}_Y$ denotes the set of measures υ from $\mathcal{Q}$ to S satisfying $\upsilon(E) \in Y(E)$ for every coalition E .

The preferences of the consumers of the economy are specified by an $\mathcal{M}_X$-preference function on $\mathcal{Q}$. This function assigns to each coalition E a binary relation on $\mathcal{M}_X$. The relation $\beta \geqq_E \alpha$ means "E prefers β at least as much as α." The following conventions are used:

$\alpha \leqq_E \beta$ is equivalent to $\beta \geqq_E \alpha$,

$\alpha =_E \beta$ means $\alpha \geqq_E \beta$ and $\beta \geqq_E \alpha$,

$\alpha >_E \beta$ means $\alpha \geqq_E \beta$ is true but $\beta \geqq_E \alpha$ is false,

$\alpha >>_E \beta$ means $\alpha >_F \beta$ holds for every nonzero subcoalition F of E and $E \neq 0$.

Possible assumptions on this preference function are:

(P.1) Unanimity. For every $(E,F) \in \mathcal{Q} \times \mathcal{Q}$ and $(\alpha,\beta) \in \mathcal{M}_X \times \mathcal{M}_X$, if $E \subset F$ and $\alpha \leqq_F \beta$ then $\alpha \leqq_E \beta$.

(P.2) Additivity. For every sequence $\{E_n\}$ of coalitions and for every $(\alpha,\beta) \in \mathcal{M}_X \times \mathcal{M}_X$, if $\alpha \geqq_{E_n} \beta$ for all n then $\alpha \geqq_{\cup E_n} \beta$.

(P.3) Completeness. For every $(\alpha,\beta) \in \mathcal{M}_X \times \mathcal{M}_X$ there exist two coalitions A_1 and A_2 such that $A_1 \cup A_2 = A$, $\alpha \geqq_{A_1} \beta$ and $\beta \geqq_{A_2} \alpha$.

It should be noted that (P.1) - (P.3) imply that $\geqq_E$ is a reflexive relation for every E. (P.1) and (P.3) imply that $\beta \geqq_0 \alpha$ and $\alpha \geqq_0 \beta$ both hold for any pair of allocations (α,β). Thus $\beta >_0 \alpha$ and $\beta >>_0 \alpha$ never hold.

(P.4) No externalities of consumption. For every coalition F and $(\alpha,\beta,\gamma) \in \mathcal{M}_X \times \mathcal{M}_X \times \mathcal{M}_X$, if $\alpha \leqq_F \beta$ and if $\gamma|_F = \beta|_F$, then $\alpha \leqq_F \gamma$, where $\beta|_F$ is the allocation defined by $\beta|_F(H) = \beta(F \cap H)$, $H \in \mathcal{A}$.

(P.5) Continuity. For every coalition E, the set $\{(\alpha,\beta) \in \mathcal{M}_X \times \mathcal{M}_X : \beta \leqq_E \alpha\}$ is a closed subset of $\mathcal{M}_X \times \mathcal{M}_X$ with respect to the product topology on $\mathcal{M}_X \times \mathcal{M}_X$, where $\mathcal{M}_X$ has the topology corresponding to the norm $\| \;\|$ defined on $\mathcal{M}_X$ by $\|\alpha\| = \sup\{|\alpha(E)| : E \in \mathcal{A}\}$.[6]

(P.6) Transitivity. For every coalition E, $\geqq_E$ is transitive.

(P.7) Weak Convexity. For any coalition E and any two allocations α and β if $\beta \geqq_E \alpha$ and if γ is a convex combination of α and β then $\gamma \geqq_E \alpha$.

It is easily seen that (P.7), together with (P.2), (P.3) and (P.6) imply that for any coalition E and any allocation α the sets $\{\beta \in \mathcal{M}_X : \beta \geqq_E \alpha\}$ and $\{\beta \in \mathcal{M}_X : \beta >>_E \alpha\}$ are convex. To conclude that the set $\{\beta \in \mathcal{M}_X : \beta >_E \alpha\}$ is convex, it appears to be necessary to make a slightly stronger convexity assumption; namely, if E is a coalition and α and β are allocations satisfying $\beta >>_E \alpha$, then any convex combination γ of α and β satisfies $\gamma >>_E \alpha$. Additional relations among these alternative convexity assumptions are established in [9].

[6] $|\cdot|$ is the Euclidean norm on S.

For the next assumption, we choose some coordinate system for S. Let N be the dimension of S and let the i^{th} coordinate of a vector z be denoted z^i. Each coordinate represents quantitites of a "commodity" (see [9], Chapter 2).

(P.8) Monotonicity There exists a nonempty subset $\mathcal{D}$ of $\{1,\ldots,N\}$ such that if E is a nonzero coalition and if α and β are two allocations such that for every nonzero $F \subset E$ there exists $i \in \mathcal{D}$ such that

$$\beta^i(F) > \alpha^i(F)$$

$$\beta^j(F) = \alpha^j(F) \qquad \text{for } j \neq i,$$

then

$$\beta >>_E \alpha .$$

$\mathcal{D}$ is the collection of commodities which are always desirable to all consumers. For a similar assumption, see [8], page 271.

A final possible characteristic of preferences is contained in the following definition. An allocation α is nonsatiating if there exists another allocation β satisfying $\beta >>_A \alpha$. We say that α is locally nonsatiating if for every $\delta > 0$ and for every nonzero coalition E there exists an allocation β such that $\beta >>_E \alpha$ and $|\beta(E) - \alpha(E)| < \delta$.

We shall now define several correspondences which arise in the study of the core of an economy and whose convexity properties we shall study in the next section. For any coalition E and allocation α, define:

$$P_\alpha(E) = \{z \in S: z = \beta(E) \text{ for some allocation } \beta \text{ satisfying } \beta >>_E \alpha\}$$

$$Q_\alpha(E) = \{z \in S: z = \beta(E) \text{ for some allocation } \beta \text{ satisfying } \beta >_E \alpha\}$$

$$S_\alpha(E) = P_\alpha(E) - \omega(E) - Y(E)$$

$$T_\alpha(E) = Q_\alpha(E) - \omega(E) - Y(E) .$$

$P_\alpha(E)$ is the set of commodity bundles in $X(E)$ which can be allocated among the subcoalitions of E in such a way that each subcoalition is better off than it would be if α were chosen. $S_\alpha(E)$ is the set of net trades which E could make and end up after production with a commodity bundle in $P_\alpha(E)$.

These concepts are related to the core allocations by the fact that if α is an allocation which is feasible for the whole economy (that is, $\alpha(A) \in \omega(A) + Y(A)$), then α is in the core if and only if

$$0 \notin S_\alpha(\mathcal{A}) \equiv \cup \{S_\alpha(E): E \in \mathcal{A}\} .$$

This fact leads eaily to a price-characterization of the core allocations if the set $S_\alpha(\mathcal{A})$ is convex.[7]

3. Convexity

THEOREM 1: If preferences satisfy (P.1) - (P.4) and if $\mathcal{A}$ is nonatomic then for every allocation α, $S_\alpha(\mathcal{A})$ is convex.

PROOF: Suppose z_1 and z_2 are two points in $S_\alpha(\mathcal{A})$. Then for $i=1,2$ there exists a coalition E_i, there exists an allocation β_i satisfying $\beta_i >>_{E_i} \alpha$ and, by Theorem 1 of [3], there exists υ_i in $\mathcal{M}_Y$ such that

$$z_i = \beta_i(E_i) - \omega(E_i) - \upsilon_i(E_i) .$$

If t is any scalar in $(0,1)$, we want to show that

$$tz_1 + (1-t)z_2 \in S_\alpha(\mathcal{A}) .$$

For $i=1,2$ define a measure $\zeta_i = \beta_i - \omega - \upsilon_i$. If $E_1 \setminus E_2 \neq 0$, set $\zeta_1' = \zeta_1 |_{E_1 \setminus E_2}$. $\mathcal{A}$ is a nonatomic algebra supporting strictly positive

[7]This characterization of core allocations is the subject of [4].

measures so by Lyapunov's Theorem [12] the range of ζ_1' is convex. Thus there exists a nonzero subcoalition F_1 of $E_1 \setminus E_2$ satisfying

$$\zeta_1(F_1) = \zeta_1'(F_1) = t\,\zeta_1'(E_1 \setminus E_2) = t\,\zeta_1(E_1 \setminus E_2) .$$

If $E_1 \setminus E_2 = 0$, let $F_1 = 0$.

Similarly, if $E_2 \setminus E_1 \neq 0$, there exists a nonzero subcoalition F_2 of $E_2 \setminus E_1$ such that

$$\zeta_2(F_2) = (1-t)\,\zeta_2(E_2 \setminus E_1) .$$

If $E_2 \setminus E_1 = 0$, let $F_2 = 0$.

Consider a measure (ζ_1, ζ_2) on $\mathcal{A}$ with values in $S \times S$ defined by

$$(\zeta_1,\zeta_2)(H) = (\zeta_1(H), \zeta_2(H)), \qquad H \in \mathcal{A} .$$

The procedure used previously yields $F_3 \subset E_1 \cap E_2$ such that

$$\zeta_1(F_3) = t\,\zeta_1(E_1 \cap E_2)$$

$$\zeta_2(F_3) = t\,\zeta_2(E_1 \cap E_2) .$$

Further, if $E_1 \cap E_2 \neq 0$ then F_3 can be chosen to satisfy $F_3 \neq 0$ and $F_4 = (E_1 \cap E_2) \setminus F_3 \neq 0$. Clearly,

$$\zeta_2(F_4) = (1-t)\,\zeta_2(E_1 \cap E_2) .$$

Define two nonzero coalitions $G_1 = F_1 \cup F_3$ and $G_2 = F_2 \cup F_4$ and define a measure:

$$\zeta = \zeta_1\big|_{G_1} + \zeta_2\big|_{G_1'} .$$

Then $\zeta(G_1 \cup G_2) = \zeta_1(G_1) + \zeta_2(G_2)$

$$= t z_1 + (1-t)\, z_2 .$$

But

$$\zeta = \zeta_1|_{G_1} + \zeta_2|_{G_1'}$$

$$= \beta_1|_{G_1} + \beta_2|_{G_1'} - \omega - (\upsilon_1|_{G_1} + \upsilon_2|_{G_1'}) .$$

$\upsilon_1|_{G_1} + \upsilon_2|_{G_1'} \in \mathcal{M}_Y$ by the additivity of Y and $\beta_1|_{G_1} + \beta_2|_{G_1'} \in \mathcal{M}_X$ by the additivity of X. Further, $\beta_i >>_{G_i} \alpha$ because $0 \neq G_i \subset E_i$. By (P.1) - (P.4)

$$\beta_1|_{G_1} + \beta_2|_{G_2} >>_{G_1 \cup G_2} \alpha .$$

Thus

$$t\, z_1 + (1-t)\, z_2 = \zeta(G_1 \cup G_2) \in S_\alpha(\mathcal{A}) .$$

<u>COROLLARY:</u> If preferences satisfy (P.1) - (P.4) and if $\mathcal{A}$ is nonatomic, then, for any allocation α and any coalition E, $P_\alpha(\mathcal{A})$ and $P_\alpha(E)$ are convex.

<u>PROOF:</u> Theorem 1 implies that $P_\alpha(\mathcal{A})$ is convex if we apply it to an economy where, for each coalition F, $\omega(F) = 0$ and $Y(F) = \{0\}$.

$P_\alpha(E)$ is trivially convex if it is empty. Otherwise, there exists an allocation β satisfying $\beta >>_E \alpha$. Define a new correspondence:

$$R_\alpha(F) = P_\alpha(F) - \beta(F), \quad F \in \mathcal{A}_E .$$

Then $R_\alpha(G) \subset R_\alpha(F)$ when $G \subset F$. This implies

$$R_\alpha(E) = R_\alpha(\mathcal{A}_E) .$$

But letting $\omega = \beta$ and $Y \equiv \{0\}$ in Theorem 1 we conclude that $R_\alpha(\mathcal{A}_E)$ is convex. Hence so is $P_\alpha(E) = R_\alpha(E) + \beta(E)$.

<u>THEOREM 2:</u> If preferences satisfy (P.1) - (P.4) and if $\mathcal{A}$ is nonatomic, then for any allocation α and any coalition E, $T_\alpha(\mathcal{A})$, $Q_\alpha(\mathcal{A})$ and $Q_\alpha(E)$ are convex.

<u>PROOF:</u> Two points of $T_\alpha(\mathcal{A})$ are specified by

$$z_i = \beta_i(E_i) - \omega(E_i) - \upsilon_i(E_i), \qquad i=1,2,$$

where $E_i \in \mathcal{A}$, $\upsilon_i \in \mathcal{M}_Y$, $\beta_i \in \mathcal{M}_X$ and $\beta_i >_{E_i} \alpha$. Let

$$\mathcal{H}_i = \{H \in \mathcal{A}: \beta_i \leqq_H \alpha\}, \qquad i=1,2 .$$

By (P.2), $\mathcal{H}_i$ is closed under countable unions and hence there exists an element H_i of $\mathcal{H}_i$ such that

$$\mu(H_i) = \sup\{\mu(H): H \in \mathcal{H}_i\} .$$

Define $F_i = E_i \cap H_i$ and $G_i = E_i \setminus H_i$, $i=1,2$. Then $\beta_i =_{F_i} \alpha$ and $\beta_i >>_{G_i} \alpha$. Finally, define

$$w_1 = \beta_1(F_1) - \omega(F_1) - \upsilon_1(F_1)$$

$$w_2 = \beta_1(G_1) - \omega(G_1) - \upsilon_1(G_1)$$

$$w_3 = \beta_2(F_2) - \omega(F_2) - \upsilon_2(F_2)$$

$$w_4 = \beta_2(G_2) - \omega(G_2) - \upsilon_2(G_2) .$$

For any t in $(0,1)$,

$$t z_1 + (1-t)z_2 = t w_1 + (1-t)w_3 + t w_2 + (1-t) w_4 .$$

By Theorem 1 there exists $G \subset G_1 \cup G_2$, $\gamma'' \in \mathcal{M}_X$ and $\upsilon'' \in \mathcal{M}_Y$ with $\gamma'' >>_G \alpha$ and

$$t w_2 + (1-t) w_4 = \gamma''(G) - \omega(G) - \upsilon''(G) .$$

By a similar procedure, there exists $F \subset F_1 \cup F_2$, $\gamma' \in \mathcal{M}_X$ and $\upsilon' \in \mathcal{M}_Y$ with $\gamma' =_F \alpha$ and

$$t\, w_1 + (1-t)\, w_3 = \gamma'(F) - \omega(F) - \upsilon'(F) .$$

If $E = F \cup G$, $\gamma = \gamma'\big|_F + \gamma''\big|_{F'}$ and $\upsilon = \upsilon\big|_F + \upsilon\big|_{F'}$ then $\gamma >_E \alpha$ and

$$t\, z_1 + (1-t)\, z_2 = \gamma(E) - \omega(E) - \upsilon(E) \in T_\alpha(\mathcal{A}) .$$

Thus $T_\alpha(\mathcal{A})$ is convex. The convexity of $Q_\alpha(\mathcal{A})$ and $Q_\alpha(E)$ is established by the method employed in the proof of the Corollary to Theorem 1 .

4. Continuity properties of preference functions.

The basic result on the continuity of preferences is contained in Lemma 1 below. In order to state this result, it is necessary to define a $R \cup \{\infty\}$-valued function on the set $\mathcal{M}$ of S-valued measures on $\mathcal{A}$. For any α in $\mathcal{M}$,

$$\|\alpha\|_\mu = \sup\left\{ \frac{|\alpha|(E)}{\mu(E)} : 0 \neq E \in \mathcal{A} \right\}$$

where $|\alpha|(E)$ is the total variation of α on E . $\|\ \|_\mu$ is a norm on the subset of $\mathcal{M}$ consisting of those measures α for which $\|\alpha\|_\mu < \infty$. The relation of $\|\ \|_\mu$ to $\|\ \|$ is discussed in the appendix.

<u>LEMMA 1:</u> If preferences satisfy (P.1) - (P.5) then for every $\eta > 0$ and for every (α, β, E) in $\mathcal{M}_X \times \mathcal{M}_X \times \mathcal{A}$ such that $\beta >>_E \alpha$, there exists a subcoalition F of E with $\mu(E \setminus F) < \eta$ and there exists ϵ in $(0,1)$ such that if (γ, δ) is any pair of allocations satisfying $\|\alpha - \gamma\|_\mu \vee \|\beta - \delta\|_\mu < \epsilon$ then $\delta >>_F \gamma$.

PROOF: For ϵ in $(0,1)$ define:

$$\mathcal{G}(\epsilon) = \{G \in \mathcal{Q} : G \subset E \text{ and there exists } (\gamma,\delta) \in \mathcal{M}_X \times \mathcal{M}_X \text{ such that } \|\alpha-\gamma\|_\mu \vee \|\beta-\delta\|_\mu \leqq \epsilon \text{ and } \delta \leqq_G \gamma\} .$$

It will now be shown that $\mathcal{G}(\epsilon)$ is closed under countable unions. Suppose $\{G_n, n=1,2,\ldots\}$ is a countable subset of $\mathcal{G}(\epsilon)$. Then set $H_n = G_n \setminus \bigcup_{k=1}^{n-1} G_k$ for $n=1,2,\ldots$ and $G = \bigcup H_n = \bigcup G_n$. To each G_n there corresponds a pair (γ_n,δ_n) in $\mathcal{M}_X \times \mathcal{M}_X$ such that $\|\alpha-\gamma_n\|_\mu \vee \|\beta-\delta_n\|_\mu \leqq \epsilon$ and, by (P.1) $\delta_n \leqq_{H_n} \gamma_n$. For any H in $\mathcal{Q}$ and for any n,

$$\sum_{k=1}^{n} |(\gamma_k-\alpha)(H_k \cap H)| \leqq \sum_{k=1}^{n} |\gamma_k-\alpha|(H_k \cap H) \leqq \sum_{k=1}^{\infty} \epsilon \cdot \mu(H_k \cap H) \leqq \epsilon \cdot \mu(H) .$$

Thus $\alpha(H) + \sum_{k=1}^{n} (\gamma_k-\alpha)(H_k \cap H)$ converges absolutely to some limit, denoted $\gamma(H)$. Then

$$|\alpha-\gamma|(H) \leqq \sum_{n=1}^{\infty} |(\gamma_n-\alpha)|(H_n \cap H)$$
$$\leqq \epsilon \cdot \mu(H)$$

so $\|\alpha-\gamma\|_\mu \leqq \epsilon$. In particular, if $\{B_n\}$ is a sequence of elements of $\mathcal{Q}$ satisfying $B_{n+1} \subset B_n$ and $\cap B_n = 0$ then

$$|\gamma(B_n)| \leqq |\gamma|(B_n) \leqq |\alpha|(B_n) + |\alpha-\gamma|(B_n)$$
$$\leqq |\alpha|(B_n) + \epsilon \cdot \mu(B_n) \rightarrow 0 \text{ and } n \rightarrow \infty .$$

Thus, since γ is clearly finitely additive, it is easily seen that γ is countably additive. Finally, for any coalition H,

$$\gamma(H) = \alpha(H) + \Sigma (\gamma_n(H_n \cap H) - \alpha(H_n \cap H))$$
$$= \alpha(H \setminus G) + \Sigma \gamma_n(H_n \cap H) \in X(H)$$

by the countable additivity of X. Thus $\gamma \in \mathcal{M}_X$.

Similarly, define $\delta = \beta + \Sigma (\delta_n - \beta)|_{H_n}$. Then $\|\beta - \delta\|_\mu \leqq \epsilon$ and $\delta \in \mathcal{M}_X$. Finally, by (P.4) $\delta \leqq_{H_n} \gamma$ for every n and so by (P.2) $\delta \leqq_G \gamma$. Thus $G \in \mathcal{G}(\epsilon)$ so $\mathcal{G}(\epsilon)$ is closed under countable unions.

Let $s_\epsilon = \sup\{\mu(G): G \in \mathcal{G}(\epsilon)\}$. Since $\mathcal{G}(\epsilon)$ is closed under countable unions, and since $\mathcal{G}(\epsilon)$ is not empty $(0 \in \mathcal{G}(\epsilon))$, there exists $G(\epsilon) \in \mathcal{G}(\epsilon)$ such that $\mu(G(\epsilon)) = s_\epsilon$. Let $F(\epsilon) = E \setminus G(\epsilon)$.

It will now be shown that given any $\eta > 0$, ϵ can be chosen so that $\mu(E \setminus F(\epsilon)) = \mu(G(\epsilon)) < \eta$. It suffices to show that $\inf\{\mu(G(\epsilon)): \epsilon > 0\} = 0$. Let $0 < \epsilon' < \epsilon$ be arbitrary scalars. Then $G(\epsilon') \in \mathcal{G}(\epsilon)$. Because $\mathcal{G}(\epsilon)$ is closed under countable unions, $G(\epsilon') \cup G(\epsilon) \in \mathcal{G}(\epsilon)$ so by the μ-maximality of $G(\epsilon)$, $G(\epsilon') \setminus G(\epsilon) = 0$. Let $G(0) = \bigcap_{n=1}^{\infty} G(1/n)$. Then by the preceding remarks and since μ is finite,

$$\mu(G(0)) = \lim_{n \to \infty} \mu(G(1/n)) = \inf\{\mu(G(\epsilon)): \epsilon > 0\}.$$

If (α_n, β_n) is the element of $\mathcal{M}_X \times \mathcal{M}_X$ corresponding to $G(1/n)$ and satisfying $\beta_n \leqq_{G(1/n)} \alpha_n$ and $\|\alpha - \alpha_n\|_\mu \vee \|\beta - \beta_n\|_\mu \leqq 1/n$, then by (P.1) the relation $\beta_n \leqq_{G(0)} \alpha_n$ holds for each n. Further,

$$\|\alpha - \alpha_n\| \vee \|\beta - \beta_n\| \leqq \mu(A) \cdot \|\alpha - \alpha_n\|_\mu \vee \|\beta - \beta_n\|_\mu \leqq \frac{\mu(A)}{n}$$

so $(\alpha_n, \beta_n) \to (\alpha, \beta)$ in the product topology on $\mathcal{M}_X \times \mathcal{M}_X$. Then by (P.5) $\beta \leqq_{G(0)} \alpha$. But then $G(0) = 0$ since $G(0) \subset E$ and $\beta \gg_E \alpha$.

This argument demonstrates, in particular, that ϵ can be chosen so that $F(\epsilon) \neq 0$. Let ϵ be such a scalar and let (γ, δ) be any element of $\mathcal{M}_X \times \mathcal{M}_X$ satisfying $\|\alpha - \gamma\|_\mu \vee \|\beta - \delta\|_\mu \leqq \epsilon$. Let A_1 and A_2 be the two coalitions specified in (P.3) such that $\gamma \leqq_{A_1} \delta$ and $\gamma \geqq_{A_2} \delta$. Set

$F_i = F(\epsilon) \cap A_i$. Then, by (P.1), $\gamma \geqq_{F_2} \delta$ so by the μ-maximality of $G(\epsilon)$, $F_2 = 0$. Thus $F(\epsilon) = F_1$ so $\delta \geqq_{F(\epsilon)} \gamma$. In fact, since $F(\epsilon) \neq 0$ and by the μ-maximality of $G(\epsilon)$, $\delta >>_{F(\epsilon)} \gamma$.

This concludes the proof of the Lemma. It might be useful to point out that the reason the norm $\| \ \|_\mu$ is used in the Lemma rather than the norm $\| \ \|$ is that if α_n and β_n are two sequences of S-valued measures on $\mathcal{a}$ with $\|\alpha_n - \beta_n\|_\mu \leqq \epsilon$ for all n and if there is a countable measurable partition $\{A_n\}$ of A such that the sums $\Sigma\, \alpha_n|_{A_n}$ and $\Sigma\, \beta_n|_{A_n}$ define measures α and β respectivly, then α and β also satisfy $\|\alpha - \beta\|_\mu \leqq \epsilon$. This would not be true for the norm $\| \ \|$.

Lemma 1 demonstrates that, if β is strictly better for E than α , then there is a subcoalition F arbitrarily "close" to E such that if δ is an allocation "close enough" to β , then δ is strictly better for F than α . In general, δ need not be strictly better for E than α . Simple examples can be constructed of allocations α , β , δ_n where $\beta >>_E \alpha$ and $\|\delta_n - \beta\|_\mu \underset{n}{\rightarrow} 0$ but for each n there exists a nonzero coalition $G_n \subset E$ such that $\delta \leqq_{G_n} \alpha$. A stronger result in the case where A is a topological space and where E is a compact subset of A has been established by Hildenbrand (Lemma 1, page 42 in [11]).

In order to apply Lemma 1 to study the openness of $P_\alpha(E)$, we need a technical result stating when there exist allocations which are close to each other in the sense of $\| \ \|_\mu$ and which assume certain values. For any subset K of the vector space S , let L(K) be the smallest affine manifold containing K . In the applications in this paper, K will always contain 0 and so L(K) will be a vector space. We shall let r i (K) denote the interior of K with respect to the relative topology on L(K).

LEMMA 2: For any allocation β, coalition E and y in $r\,i(X(E))$, there exists $\eta > 0$ and a subcoalition F_o of E such that $\mu(E\setminus F_o) < \eta$ and if $\epsilon > 0$ and if $F \subset F_o$ and $\mu(E\setminus F) < \eta$ then there exists $t \in (0,1)$ and there exists $\zeta_t \in \mathcal{M}_X$ satisfying

(i) $\|\zeta_t - \beta\|_\mu \leqq \epsilon$

(ii) $\zeta_t(F) = y_t - \beta(E\setminus F)$

where $y_t = (1-t)\beta(E) + ty$.

PROOF: Because $y \in r\,i\,X(E)$, there exist q points $x_1,\ldots,x_q$ of $X(E)$ such that $x_1 = \beta(E)$ and such that the convex hull $\langle x_1,\ldots,x_q\rangle$ of $\{x_1,\ldots,x_q\}$ is a neighborhood of y in $L(X(E))$. Let $\beta_1 = \beta$. By Theorem 1 in [3] there exist β_i in $\mathcal{M}_X$ such that $\beta_i(E) = x_i$, $i=2,\ldots,q$. For each coalition G, define a subset $K(G)$ of S:

$$K(G) = \langle \beta_1(G),\ldots,\beta_q(G)\rangle .$$

We will show first that if F is "close enough" to E, then $y-\beta(E\setminus F) \in K(F)$. But

$$y-\beta(E \setminus F) \in L(X(E)) + L(X(E \setminus F)) = L(X(E))$$

and

$$K(F) \subset L(X(F)) \subset L(X(E)),$$

so to show that $y-\beta(E \setminus F) \in K(F)$, it suffices to show that for every z in $L(X(E)) \setminus K(F)$ we have

$$|y-\beta(E \setminus F)-z| > 0 .$$

Since y is in the interior of $K(E)$ with respect to $L(X(E))$, there exists $d > 0$ such that:

(1) $$|y-z| > d \quad \text{for} \quad z \in L(X(E)) \setminus K(E) .$$

Choose $\eta > 0$ so that if F is a subcoalition of E satisfying $\mu(E \setminus F) < \eta$, then

(2) $$|\beta_i(E \setminus F)| < d/3 , \quad i=1,\dots,q .$$

Let F be any such subcoalition of E. If $z \in L(X(E)) \setminus K(E)$, then

(3) $$\begin{aligned} |y-\beta(E \setminus F)-z| &\geqq |y-z| - |\beta(E \setminus F)| \\ &> 2\,d/3 \end{aligned}$$

by (1) and (2). If $z \in [L(X(E)) \setminus K(F)] \cap K(E)$ then there exists $w \in L(X(E)) \setminus K(E)$ with

(4) $$|z-w| \leqq d/3 .$$

But then

(5) $$|y-\beta(E \setminus F) -z| \geqq |y-\beta(E \setminus F) -w| - |w-z| > d/3$$

by (3) and (4). Thus if $z \in L(X(E)) \setminus K(F)$ then by (3) and (5):

$$|y-\beta(E \setminus F) -z| > d/3.$$

Thus it has been shown that if $F \subset E$ and $\mu(E \setminus F) < \eta$, then $y-\beta(E \setminus F) \in K(F)$.

If $t \in [0,1]$, then since $K(F)$ is convex and since $\beta(F) \in K(F)$ and $y-\beta(E \setminus F) \in K(F)$,

$$\begin{aligned} y_t-\beta(E \setminus F) &= (1-t)\,\beta(E) + ty -\beta(E \setminus F) \\ &= (1-t)\,\beta(F) + t[y-\beta(E \setminus F)] \in K(F) . \end{aligned}$$

Let $\mathcal{A}^{\circ}$ be a σ-field of subsets of a set A° and let μ° be a nonnegative measure on $\mathcal{A}^{\circ}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{A}^{\circ}$ modulo its μ°-null sets.[8] Each measure β_i on $\mathcal{A}$ induces a unique measure on $\mathcal{A}^{\circ}$ which

[8] The existence of such a σ-field $\mathcal{A}^{\circ}$ and measure μ° is assured by Loomis' Theorem. See [14], page 117, for example.

will also be denoted β_i. Let b_i be a Radon-Nikodym derivative of β_i with respect to μ^o, $i=1,\ldots,q$. There exists $M > 0$ and a subset F_0^o of E^o such that $\mu^o(E^o \setminus F_0^o) < \eta$ and for almost every $a \in F_0^o$ (F_0^o is an element of $\mathcal{A}^o$), $\max_{1 \leqq i \leqq q} |b_i(a)| \leqq M$. Choose any $\epsilon > 0$ and any subcoalition F of F_0 satisfying $\mu(E \setminus F) < \eta$ (F_0 is the coalition in $\mathcal{A}$ corresponding to F_0^o in $\mathcal{A}^o$).

For any $\epsilon' > 0$, the set

$$\{w \in K(F):\ w = \sum_{i=1}^{q} t_i \beta_i(F) \quad \text{for} \quad 0 \leqq 1-t_1 < \epsilon', \quad 0 \leqq t_i < \epsilon', \quad i=2,\ldots,q$$

$$\text{and} \quad \sum_{i=1}^{q} t_i = 1\}$$

is a neighborhood of $\beta(F)$ in $K(F)$. It has been shown that for all $t \in [0,1]$,

$$y_t - \beta(E \setminus F) \in K(F).$$

Further, $\lim_{t \to 0} y_t - \beta(E \setminus F) = \beta(F)$. Hence there exists $t_0 > 0$ such that $t \in (0,t_0)$ implies that it is possible to choose $s_{i,t}$ in $[0,1]$, $i=1,\ldots,q$, so that

a) $0 \leqq 1-s_{1,t} < \frac{\epsilon}{qM}$ and $0 \leqq s_{i,t} < \frac{\epsilon}{qM}$, $\quad i=2,\ldots,q$

b) $y_t - \beta(E \setminus F) = \sum_{i=1}^{q} s_{i,t}\, \beta_i(F)$.

But then, a.e. on $F^o \subset F_0^o$, $|b_1(a) - \sum_{i=1}^{q} s_{i,t}\, b_i(a)| < \epsilon$.

If $\zeta_t = \sum_{i=1}^{q} s_{i,t}\, \beta_i\big|_F + \beta_1\big|_{A \setminus F}$, then by the convexity and additivity of X, $\zeta_t \in \mathcal{M}_X$. Further, $\zeta_t(F) = y_t - \beta(E \setminus F)$ and

$$\|\zeta_t - \beta\|_\mu = \sup \left\{ \frac{\int_{H^o} |\sum_{i=1}^{q} s_{i,t}\, b_i - b_1|\, d\mu^o}{\mu^o(H^o)} : H^o \subset F^o,\ H^o \in \mathcal{A}^o,\ \mu(H^o) > 0 \right\}$$

$$\leqq \epsilon .$$

It is now possible to establish our main results on the openness of $P_\alpha(E)$:

THEOREM 3: If preferences satisfy (P.1) - (P.5), then for any coalition E and allocation α, if $x \in P_\alpha(E)$ and $y \in ri(X(E))$, then there exists t in $(0,1)$ such that

$$y_t = (1-t)x + ty \in P_\alpha(E) .$$

PROOF: There exists β in $\mathcal{M}_X$ such that $\beta(E) = x$ and $\beta >>_E \alpha$. Choose $\eta > 0$ and $F_0 \subset E$ by Lemma 2. Set $\eta_1 = \eta - \mu(E \setminus F_0)$. By Lemma 1 choose $\epsilon > 0$ and a subcoalition F_1 of E with $\mu(E \setminus F_1) < \eta_1$ and such that if $(\gamma,\delta) \in \mathcal{M}_X \times \mathcal{M}_X$ and $\|\gamma-\alpha\|_\mu \vee \|\delta-\beta\|_\mu \leqq \epsilon$ then $\delta >>_{F_1} \gamma$. Set $F = F_0 \cap F_1$. Then $\mu(E\setminus F) < \eta$ so by Lemma 2 there exists $t \in (0,1)$ such that there exists δ_t in $\mathcal{M}_X$ with $\delta_t(F) = y_t - \beta(E \setminus F)$ and $\|\delta_t - \beta\|_\mu \leqq \epsilon$. By redefining δ_t on $A \setminus F$, if necessary, it can be supposed that $\delta_t\big|_{A \setminus F} = \beta\big|_{A \setminus F}$. Thus $\delta_t(E) = y_t$. Further, by Lemma 1, $\delta_t >>_F \alpha$ and so by (P.1), (P.2) and (P.4), $\delta_t >>_E \alpha$. Thus $y_t \in P_\alpha(E)$.

THEOREM 4: If preferences satisfy (P.1) - (P.5) and if either $\mathcal{a}$ is nonatomic or preferences satisfy (P.6) and (P.7), then, for any coalition E and any allocation α, $P_\alpha(E) \cap ri(X(E))$ is open in $L(X(E))$.

PROOF: Suppose $x \in K \equiv P_\alpha(E) \cap ri(X(E))$. If x were in the boundary of K, there would be a hyperplane through x supporting K, since K is convex by the Corollary to Theorem 1 or by (P.6) and (P.7). Thus, to show $x \in ri(K)$ it suffices to show that $p \cdot x < \sup p \cdot K$[9] for every nonzero vector p in $L((X(E))$.

[9] $\sup p \cdot K = \sup\{p \cdot z : z \in K\}$.

Because $x \in ri(X(E))$, there exists a scalar $s > 0$ such that $y = x + s p \in ri(X(E))$. Theorem 3 implies that for some $t \in (0,1)$, $x + t s p = (1-t) x + t (x + sp) \in P_{\alpha}(E)$. But $p \cdot (x + t s p) = p \cdot x + tsp \cdot p > p \cdot x$ since $p \neq 0$. This completes the proof of Theorem 4.

This Theorem demonstrates that under the specified conditions, $P_{\alpha}(E) \cap ri(X(E))$ is open in $X(E)$. In fact, $P_{\alpha}(E)$ may be an open subset of $X(E)$. This is known to be true if the consumption correspondence is Ω-valued; that is, if $X(E)$ is equal to the closed positive orthant of S for every coalition E except 0, at which it assumes the value $\{0\}$.[10] It is not known how general this result is. It is also not known whether a converse result is true; namely, if preferences satisfy (P.1)-(P.4) and if $P_{\alpha}(E)$ is an open subset of $X(E)$ for every E and α, does (P.5) then hold? If $\mathcal{A}$ is finite, it is at least possible to conclude that the set

$$\{\beta \in \mathcal{M}_X : \beta \geqq_E \alpha\}$$

is closed in $\mathcal{M}_X$ for every E and α.

The preceding results are not valid if $P_{\alpha}(E)$ is replaced by $Q_{\alpha}(E)$. A counter example is easily constructed by letting Ω^2 be the closed, positive orthant of R^2 and letting X be Ω^2-valued. Define a preference function on $\mathcal{A}$ by specifying two nonzero elements A_1 and A_2 of $\mathcal{A}$ such that $A_1 \cap A_2 = 0$ and $A_1 \cup A_2 = A$. For any subcoalition E of A_1 and any pair α, β of allocations let $\beta \geqq_E \alpha$ hold if and only if $\beta^1(F) - \beta^2(F) \geqq \alpha^1(F) - \alpha^2(F)$ for every subcoalition F of E. More simply, every member of A_1 prefers β at least as much as α if the excess of commodity 1 over commodity 2 in β exceeds the corresponding excess for α. Every member of coalition A_2 has opposite tastes. It is readily verified that these preferences satisfy (P.1)-(P-7) but not (P.8).

[10] This is established in Theorem 6 in [6].

Let a and b be two strictly positive scalars and let α and β be defined by choosing a strictly positive measure $\lambda(A_1) = \lambda(A_2) = 1$ and by setting

$$\alpha(H) = (0,b)\cdot\lambda(A_2 \cap H), \quad H \in \mathcal{A}$$

$$\beta(H) = (a,0)\cdot\lambda(A_1 \cap H) + (0,b)\cdot\lambda(A_2 \cap H), \quad H \in \mathcal{A}.$$

Then $\beta >_A \alpha$ and $\beta(A) = (a,b)$ is in the interior of Ω^2. But for no $\epsilon > 0$ does there exist an allocation β_ϵ satisfying $\beta_\epsilon(A) = (a,b-\epsilon)$ and $\beta_\epsilon >_A \alpha$: Suppose $\beta_\epsilon >_A \alpha$ for some $\epsilon > 0$. Then $\beta_\epsilon \geqq_{A_2} \alpha$ and hence $\beta^2_\epsilon(A_2) - \beta^1_\epsilon(A_2) \geqq \alpha^2(A_2) - \alpha^1(A_2) = b$ so $\beta^2_\epsilon(A_2) \geqq b$. But

$$\beta^2_\epsilon(A_1) = \beta^2_\epsilon(A) - \beta^2_\epsilon(A_2)$$
$$\leqq b - \epsilon - b < 0$$

which contradicts the requirement that β_ϵ be Ω^2-valued.

This demonstrates that even if X is Ω-valued and even if preferences satisfy (P.1) - (P.7), then for arbitrary α in $\mathcal{M}_X$ and E in $\mathcal{A}$ the set $Q_\alpha(E) = \{z \in \Omega:$ there exists β in $\mathcal{M}_X$ with $\beta(E) = z$ and $\beta >_E \alpha\}$ need not be open in Ω.

This problem is closely related to the problem of finding conditions on preferences such that if there exist α and β in $\mathcal{M}_X$ and E in $\mathcal{A}$ satisfying $\beta >_E \alpha$, then there also exists δ in $\mathcal{M}_X$ satisfying $\delta(E) = \beta(E)$ and $\delta >>_E \alpha$. That is, when a particular commodity bundle can be allocated among the members of E in a way which is semibetter for E than α, is it also possible to allocate this bundle among the members of E in a way which is strictly better for E than α? The example of a preference function given in the preceding paragraph shows that this need not always be true.[11] However, this example depended on the fact that each coalition A_1 and A_2 received a bundle in the boundary of its set of feasible consumption vectors. The following lemma shows that this is the only way this could occur.

<u>LEMMA 3:</u> Suppose preferences satisfy (P.1) - (P.6), suppose that either $\mathcal{A}$ is nonatomic or preferences satisfy (P.7) and suppose there is a subspace S' of S such that $S' = L(X(E))$ for every nonzero E. If α and β are two allocations and if E is a coalition such that $\beta >_E \alpha$ and such that β is locally nonsatiating and satisfies

$$\beta(F) \in \text{ri}(X(F)) \qquad \text{for every } F \subset E ,$$

then there exists an allocation γ with $\gamma(E) = \beta(E)$ and $\gamma >>_E \alpha$.

[11] This example can be modified slightly to give an example where $>$ and $>>$ are not equivalent and where preferences are strictly convex (i.e., $\alpha \geqq_E \beta$, $\gamma \geqq_E \beta$ and $\alpha(F) \neq \gamma(F)$ for $0 \neq F \subset E$ imply $t\alpha + (1-t)\gamma >>_E \beta$ for any t in $(0,1)$). Treating A_1 as an individual economic agent (an atom of $\mathcal{A}$), let A_1 have the indifference curve through $(0,0)$ consisting of $\{(x,y) \in \Omega^2 : x = y + y^2\}$. Let his indifference curves "above" this one be horizontal translates of this curve and his indifference curves "below" this one be vertical translates of this curve. Similarly, let A_2 have the indifference curve through $(0,0)$ consisting of $(\{x,y): y = x + x^2\}$. Points "below" or to the right of this curve are worse for A_2 and points above are better. Thus A_1 likes commodity 1 and A_2 likes commodity 2 . These preferences are strictly convex and give rise to the same problems as the preferences defined in the text. These examples contradict a remark by Mishan [13], page 163. This discussion of the relation between $>_E$ and $>>_E$ has benefitted from conversations with Alan Kirman.

PROOF: Let E_1 be the μ-maximal subcoalition of E such that $\beta \leqq_{E_1} \alpha$ (see the beginning of the proof of Lemma 1, page 12). Then $\beta >_E \alpha$ implies $\beta =_{E_1} \alpha$ and $\beta >>_{E_2} \alpha$ where $E_2 = E \setminus E_1$. Because $\beta(E_2) \in ri(X(E_2))$, we know by Theorem 4 that there exists $\delta > 0$ such that if $z \in S'$ and $|z| < \delta$ then $\beta(E_2) - z \in P_\alpha(E_2)$. Because β is locally nonsatiating, there exists $z^* \in S'$ such that $|z^*| < \delta$ and $\beta(E_1) + z^* \in P_\beta(E_1) = P_\alpha(E_1)$. This means there exist two allocations γ_1 and γ_2, such that

$$\gamma_1(E_1) = \beta(E_1) + z^*, \quad \gamma_1 >>_{E_1} \alpha$$

$$\gamma_2(E_2) = \beta(E_2) - z^*, \quad \gamma_2 >>_{E_2} \alpha .$$

Define another allocation γ:

$$\gamma = \gamma_1\Big|_{E_1} + \gamma_2\Big|_{A \setminus E_1} .$$

Then $\gamma(E) = \beta(E)$ and $\gamma >>_E \alpha$.

The preceding Lemma demonstrates that the basic conditions necessary for $>_E$ and $>>_E$ to be equivalent are local nonsatiation and continuity of preferences, (P.5). The assumption that every nonzero coalition has a consumption set $X(E)$ which generates the same subspace S' of S is satisfied if loosely speaking, the same commodities can be feasibly consumed by all coalitions. This is a strong assumption, but the proof of the preceding Lemma makes it clear that it can be replaced by a much more acceptable assumption; namely, that for every coalition there is local nonsatiation in a "direction" which is "feasible" for all coalitions. More precisely, the condition is that there exist an unbounded ray R such that $R \subset L(X(E))$ for every nonzero E and such that for any allocation α, for any nonzero coalition E and $\delta > 0$ there exists y in R with $|y| < \delta$ and $\alpha(E) + y \in P_\alpha(E)$.

This condition of local nonsatiation just stated is similar to the monotonicity assumption (P.8) on preferences. It is probably evident that some type of monotonicity assumption makes $>_E$ and $>>_E$ equivalent. This is made precise in the next Lemma. In this Lemma, the notation $i(\epsilon)$ will denote the element of Ω which has all components equal to 0 except the i^{th} which will equal the positive scalar ϵ.

<u>LEMMA 4:</u> If X is Ω-valued, if preferences satisfy (P.1) - (P.6) and (P.8) and if (α, β, E) is an element of $\mathcal{M}_X \times \mathcal{M}_X \times \mathcal{A}$ such that $\beta >_E \alpha$ and such that for every nonzero subcoalition F of E there exists $i \in \mathcal{D}$ with $\beta^i(F) > 0$, then there exists an allocation γ with $\gamma >>_E \alpha$ and $\gamma(E) = \beta(E)$.

<u>PROOF:</u> Let E_1 be the μ-maximal subcoalition of E such that $\alpha \geqq_{E_1} \beta$. Then $\beta >_E \alpha$ implies $\beta =_{E_1} \alpha$ and $\beta >>_{E_2} \alpha$ where $E_2 = E \setminus E_1$. Because $\beta >_E \alpha$, $E_2 \neq 0$, so for some $i \in \mathcal{D}$, $\beta^i(E_2) > 0$. Because $P_\alpha(E_2)$ is an open subset of Ω, there exists $\epsilon > 0$ such that $\beta(E_2) - i(\epsilon) \in P_\alpha(E_2)$. Let γ_2 be the element of $\mathcal{M}_X$ such that $\gamma_2 >>_{E_2} \alpha$ and $\gamma_2(E_2) = \beta(E_2) - i(\epsilon)$. Define an element γ_1 of $\mathcal{M}_X$ by

$$\gamma_1(H) = \beta(H) + i(\epsilon) \cdot \frac{\mu(H \cap E_1)}{\mu(E_1)}, \quad H \in \mathcal{A}.$$

Then by the monotonicity assumption (P.8), $\gamma_1 >>_{E_1} \beta$ and by (P.6), $\gamma_1 >>_{E_1} \alpha$. By (P.1), (P.2) and (P.4), $\gamma = \gamma_1|_{E_1} + \gamma_2|_{A \setminus E_1} >>_E \alpha$. Finally, $\gamma(E) = \beta(E)$.

It is easy to establish that if the collection $\mathcal{D}$ specified in the statement of (P.8) is the set $\{1,\ldots,N\}$, if X is Ω-valued and if preferences satisfy (P.1) - (P.6) and (P.8), then $Q_\alpha(E) = P_\alpha(E)$ for every coalition E and allocation α. In this case, a definition of "blocking" which is based on $>_E$ is equivalent to a definition based on $>>_E$.[12] However, (P.8) is then a very restrictive monotonicity assumption. Weaker conditions for the equivalence of these two kinds of blocking are not known.

Appendix on normed spaces of measures.

In this paper, two norms were used on the set $\mathcal{M}$ of measures with domain $\mathcal{A}$ and range S. The first, denoted $\| \ \|$, was defined by

$$\|\alpha\| = \sup \{ |\alpha(E)| : E \in \mathcal{A} \} .$$

The second (perhaps infinite-valued) norm, $\| \ \|_\mu$, is based on a positive measure μ on $\mathcal{A}$:

$$\|\alpha\|_\mu = \sup \left\{ \frac{|\alpha|(E)}{\mu(E)} : E \in \mathcal{A} \text{ and } \mu(E) \neq 0 \right\}$$

where $\alpha \in \mathcal{M}$ and where $|\alpha|(E)$ is the total variation of α on E; that is,

$$|\alpha|(E) = \sup \sum_{i=1}^{n} |\alpha(E_i)|$$

where the sup is taken over all possible finite partitions $\{E_1, \ldots, E_n\}$ of E. These two norms will be compared by relating them to norms on the set of μ-integrable, S-valued functions on A. For this reason, it is convenient, in this appendix, to assume that $\mathcal{A}$ is a σ-field of subsets of the set A (see footnote 8).

[12] We say E blocks an allocation α if $0 \in S_\alpha(E)$ and E weakly blocks α if $0 \in T_\alpha(E)$. These two definitions of blocking are developed in [4].

The norm $\| \ \|$ generates the topology of uniform convergence on $\mathcal{A}$. It is not difficult to show that $\| \ \|$ is equivalent to the total variation norm $\| \ \|_t$ defined on $\mathcal{M}$ by

$$\|\alpha\|_t = |\alpha|(A) .$$

Suppose $\mathcal{M}_\mu$ is the collection of measures in $\mathcal{M}$ which are μ-continuous. Then every α in $\mathcal{M}_\mu$ has a Radon-Nikodym derivative f_α with respect to μ (see [10], page 181). It is a standard result that

$$\|\alpha\|_t = \int_A |f_\alpha| \, d\mu .$$

Thus the space $(\mathcal{M}_\mu , \| \ \|)$ is homeomorphic to $L^1(A, \mathcal{A}, S, \mu)$.

Let $\mathcal{M}_\mu^\infty$ designate those elements of $\mathcal{M}_\mu$ for which $\|\alpha\|_\mu < \infty$. Then it can be shown that $(\mathcal{M}_\mu^\infty , \| \ \|_\mu)$ is homeomorphic to $L^\infty(A, \mathcal{A}, S, \mu)$. Further, if λ and μ are two positive measures on $\mathcal{A}$, each of which is continuous with respect to the other, then the topologies generated on $\mathcal{M}$ by $\| \ \|_\lambda$ and $\| \ \|_\mu$ are equivalent if and only if $\max \{\|\frac{d\lambda}{d\mu}\|_\infty, \|\frac{d\mu}{d\lambda}\|_\infty\} < \infty$, where $\frac{d\lambda}{d\mu}$ is the Radon-Nikodym derivative of λ with respect to μ. Thus the topologies induced on $\mathcal{M}$ by different positive measures μ are not, in general, equivalent.

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13. ABSTRACT

This paper studies the convexity properties of the preferences of a perfectly competitive economy. These results are used to establish a relation between two alternative types of continuity of preferences used by R. Aumann and K. Vind. This discussion is related to the problem of establishing conditions for the equivalence of two kinds of blocking used in the definition of the core of an economy.

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