

THE USE OF PRICES TO CHARACTERIZE

THE CORE OF AN ECONOMY

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ABSTRACT

The core of an economy is the set of outcomes of trading and production which have the property that no coalition of traders and producers can do better for its members. When the traders and producers are "perfectly competitive," any such outcome can be achieved by using an appropriate price system. This paper establishes this well-known result under very general conditions by building on a model due to Vind. Particular attention is paid to the problem of representing "perfect competition" and of describing production in a nonmarket economy. All assumptions in this model are formulated so as to be easily compared with the assumptions made by Aumann in his model of a perfectly competitive economy.

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1. Introduction

The core of an economy is the collection of allocations of commodities among the consumers of the economy which can be attained by trading among the consumers and producers and which no coalition of consumers and producers can improve on. This gives a concept of economic efficiency which is independent of any particular economic institutions. Further, under certain conditions the core of a finite economy is nonempty [31]. However, the trading process which would be necessary to find an allocation in the core would in general be very complex. It has long been thought¹ that, if there were a large number of traders, a price system could be introduced which would simplify this trading process. This possibility of using prices to find allocations in the core will be explored in this paper.

It has become common, when analyzing the core of an economy, to suppose that the possible coalition are elements of a σ -field of subsets of the set of all economic agents. This paper develops a different, but logically equivalent, approach which permits a simpler explanation of why the measure-theoretic analogue of "perfect competition" is a "nonatomic" economy. This approach derives from work of Vind [34] and is presented here as a complement to the work of Aumann [1] and [2], Hildenbrand [21], [22], [23] and others. By making use of

¹See [17]. This work was "discovered", after being neglected for some time, by Shubik [32].

a recent result by Debreu [9] on preference functions, it is possible to compare the assumptions made here with the assumptions made by Aumann or Hildenbrand. In particular, we remark that no monotonicity of preferences is assumed.

The next Section describes a representation of the coalition structure of an economy. In Section 3 the consumption possibilities and the preferences of the coalitions are described. Section 4 discusses the problem of representing production possibilities in a nonmarket economy. Alternative concepts of economic efficiency are defined in Section 5. This Section also describes the representation of the assumption of perfect competition and discusses the significance which the results of this paper have for more general types of economies. Finally, in Section 6 the main results are proven.

I am indebted to Karl Vind for discussions on the interpretation of his model [34] of the core of an economy. I am also indebted to Werner Hildenbrand for conversations on the concept of blocking in an economy with production. Finally, I am grateful to Gerard Debreu for his insistence on the need to make the assumptions of the model presented here comparable with earlier work such as that by Aumann [1].

2. The Coalition Structure

The basic structure in any trading process is the collection of coalitions which are able or are allowed to trade. This set is often assumed² to be a particular collection of subsets of the set of all "individual" economic agents in the economy. Thus the concept of a coalition is derived from that of an "individual" economic agent. In this paper we shall adopt the reverse procedure

²See [1], [2], [9], [21].

of taking the collection of objects called coalitions as given and shall derive the concept of an individual agent from the concept of a coalition. This procedure offers certain advantages which will be indicated later.

The primitive concept in this approach is a set \mathcal{A} of objects called coalitions. \mathcal{A} is a σ -algebra with unit element denoted A , with zero element denoted 0 and with operations denoted $U, \cap, ' .^3$ These operations have most of the analogous properties of set-theoretic union, intersection and complementation, respectively, and may be so interpreted in this paper.

It is also assumed that the algebraic structure of \mathcal{A} supports a strictly positive measure μ ; ⁴ that is, $\mu(B) \geq 0$ for every coalition B and if $\mu(B) = 0$ then $B = 0$. In the terminology of the next Section, this assumption is equivalent to the requirement that \mathcal{A} permit the specification of an allocation which assigns the 0 consumption plan only to coalition 0 . In summary, a σ -algebra of coalitions is a σ -algebra which has a unit and which supports at least one strictly positive measure.

The following additional operations can be defined in \mathcal{A} : A binary operation \setminus is defined on any two coalitions E and F by

$$E \setminus F = E \cap (F')$$

E and F will be called disjoint if $E \cap F = 0$.

An order \subset is defined on \mathcal{A} :

$$E \subset F \quad \text{if} \quad E \cap F = E$$

E is called a subcoalition of F if $E \subset F$.

³Halmos [20] gives a good introduction to σ -algebras and Sikorski [33] gives a more complete discussion of them.

⁴The measure μ is a function from \mathcal{A} to \mathbb{R} such that if $\{B_n\}$ is a sequence of disjoint coalitions, then $\mu(U B_n) = \sum \mu(B_n)$. The requirement that \mathcal{A} support a strictly positive measure ⁿ can be stated algebraically (see Kelly [27]). There are σ -algebras which do not satisfy this requirement. Consider, for example, the σ -field of Lebesgue subsets of $[0,1]$. See Theorem 2.4 in Horn and Tarski [24].

To complete the description of a coalition structure, define an atom of an algebra \mathcal{A} to be any nonzero coalition E whose only subcoalitions are 0 and E . The atoms of \mathcal{A} are those coalitions which cannot be divided. It is shown in Section 3 that atoms may be interpreted as "individual" economic agents. If \mathcal{A} is finite, then \mathcal{A} has atoms and, in fact, \mathcal{A} is isomorphic to the field of subsets of the set of atoms of \mathcal{A} .⁵ Thus if \mathcal{A} is nonatomic, i.e., if \mathcal{A} has no atoms, then \mathcal{A} is infinite. It will be shown at the end of Section 5 that the assumption that \mathcal{A} is nonatomic corresponds to the idea of "perfect competition".

The preceding description of the coalition structure can be connected with the usual treatment of coalitions as the set \mathcal{A}° of subsets of the set A° of individual traders. If A° is finite, so is \mathcal{A}° and, in fact, \mathcal{A}° is a σ -algebra of coalitions as defined above. The converse result has already been remarked on; namely, if \mathcal{A} is a finite σ -algebra of coalitions, then it may be identified with the collection of subsets of the set of atoms of \mathcal{A} . In this case the unit A (resp., the zero 0) of \mathcal{A} is identified with the set of all traders (resp. the empty set of traders).

If \mathcal{A}° , resp. \mathcal{A} , is infinite, this identification procedure is slightly more complex.⁶ If A° is a set, \mathcal{A}° a σ -field of subsets of A° , and μ° a nonnegative, nonzero measure on \mathcal{A}° , then we may define an equivalence relation \equiv_{μ° on \mathcal{A}° by

$$E^\circ \equiv_{\mu^\circ} F^\circ \quad \text{if} \quad \mu^\circ(E^\circ \Delta F^\circ) = 0$$

where

$$E^\circ \Delta F^\circ = (E^\circ \setminus F^\circ) \cup (F^\circ \setminus E^\circ).$$

⁵See Theorem 9.1 and Example D, page 28 in Sikorski [33].

⁶If \mathcal{A} is nonatomic and supports a strictly positive measure, then \mathcal{A} cannot be a σ -field. See Sikorski [33], page 105.

Let \mathcal{A} be the collection of equivalence classes so defined on A° , let A be the class of all elements of \mathcal{A}° which are equivalent to A° and let μ be defined on \mathcal{A} by:

$$\mu(E) = \mu^\circ(E^\circ), \quad \text{any } E^\circ \text{ in the equivalence class } E.$$

Then \mathcal{A} is a σ -algebra with unit A and supporting the strictly positive measure μ . We see that the unit A (resp., the zero 0) represents those subsets of individual economic agents which are μ° -equivalent to A° (resp., to the empty subset of A°). The assumption that μ° -equivalent subsets of A° are, in fact, economically equivalent has been made explicit in earlier measure theoretic work and has recently been justified.⁷

Thus every σ -field of subsets of a set supporting a nonzero, nonnegative measure can be identified with a σ -algebra of coalitions. The reverse identification can always be made also (see Sikorski [33], page 117).

3. Consumption Possibilities and Preferences

A finite dimensional, Euclidean space, S , will be the commodity space of the economy.⁸ In order to specify the consumption and production possibilities that are available to the coalitions, the idea of a correspondence Z from \mathcal{A} to S is introduced; that is, for each coalition E , $Z(E)$ is a (nonempty) subset of S . For any sequence $\{E_n\}$ of coalitions, define

$$\Sigma Z(E_n) = \{z \in S: \text{for each } n=1,2,\dots \text{ there exists } z_n \in Z(E_n) \\ \text{such that } \sum_{n=1}^k z_n \text{ converges absolutely in } k \text{ to } z\}.$$

⁷The economic equivalence of any two coalitions which are μ° -equivalent usually derives from the requirement that economic relations only be valid almost everywhere. This is demonstrated in [9], page 112. A justification for this procedure can be found in the recent work of Hildenbrand [23].

⁸The idea of a commodity space is developed in [12], Chapter 2.

Z is called finitely (countably) subadditive if for every finite (countable) set $\{E_n\}$ of disjoint coalitions, $\sum Z(E_n) \subset Z(\cup E_n)$. Z is called finitely (countably) additive if for every finite (countable) set $\{E_n\}$ of disjoint coalitions, $\sum Z(E_n) = Z(\cup E_n)$.

A concept closely related to that of a correspondence Z is the collection \mathcal{M}_Z of S -valued measures ζ on \mathcal{A} such that for every E , $\zeta(E) \in Z(E)$. \mathcal{M}_Z will be called the collection of measures induced by Z . If \mathcal{M}_Z is considered as a set of mappings from \mathcal{A} to S , then it is natural to define

$$\text{eval}_E(\mathcal{M}_Z) = \{z \in S : z = \zeta(E) \text{ for some } \zeta \in \mathcal{M}_Z\}.$$

In [6] a basic theorem due to Debreu is given which presents conditions on Z such that, if \mathcal{M}_Z is the set of measures induced by Z , then $Z(E) = \text{eval}_E(\mathcal{M}_Z)$ for each coalition E . In particular, conditions will be stated under which \mathcal{M}_Z is nonempty.

The importance in economics of the concept of an additive correspondence comes from its usefulness in representing the consumption and production possibilities of the various coalitions. The rest of this section explains this representation of the consumption possibilities of an economy.

For each coalition E there is a subset $X(E)$ of S which is interpreted as the set of points in S which coalition E is able to consume. Any point in $X(E)$ chosen by E is called a consumption plan for E . The correspondence X on \mathcal{A} which maps E into $X(E)$ is a consumption correspondence if it satisfies:

(X.1) X is countably subadditive

(X.2) For every E , $0 \in X(E)$,

(X.3) $X(0) = \{0\}$.

The assumption that X is subadditive corresponds to the assumption that coalition E can choose a point in $X(E)$ independently of what consumption plans are chosen by coalitions disjoint from E . Thus it is assumed that there are no interactions between the choices of one coalition and the set of choices available to another, disjoint coalition. A study of a finite economy where such interactions are permitted is made in [13].

The set of measures induced by X is denoted \mathcal{M}_X and is called the set of allocations. An allocation γ specifies, for each E , a consumption plan $\gamma(E) \in X(E)$. The requirement that $\gamma \in \mathcal{M}_X$ means it is possible for each coalition to carry out its plan; that is, for each E the inputs (positive coordinates) of the vector $\gamma(E)$ are such that E can survive long enough to consume them and to give up the outputs (negative components) of the vector $\gamma(E)$.⁹ One particular element of \mathcal{M}_X will be denoted by ω and will be interpreted as the initial resources allocation. For each E , $\omega(E)$ is interpreted as the bundle of commodities with which E is endowed before trading or production occur.

The allocations represent the alternative ways in which, for any E , the points of $X(E)$ can be divided among the subcoalitions of E . In fact, if X satisfies conditions (X.4) - (X.7) given in the following paragraph as well as (X.1) - (X.3), then by Theorem 1 of [6], $X(E) = \text{eval}_E(\mathcal{M}_X)$ and hence $X(E)$ can be interpreted as the set of points z which can be divided among the subcoalitions of E in such a way that each subcoalition is able to carry out its share of the plan z .

⁹The usual convention is adopted here of considering a consumer's outputs to be the negative coordinates of his consumption plan and a consumer's inputs to be the positive coordinates. An output may be the performance of a given type of labor service at a particular date and location and an input may be the amount of a nonlabor commodity consumed.

This paper will make the following assumptions on the consumption correspondence:

- (X.4) X is countably additive
- (X.5) For every E , $X(E)$ is a closed subset of S .
- (X.6) X is bounded below; that is, there exists in S a closed, convex,¹⁰ pointed cone P with vertex 0 and there exists an S -valued measure β on \mathcal{Q} such that, for every E , $X(E) - \beta(E) \subset P$.
- (X.7) For every E , $X(E)$ is convex.

Assumption (X.4) implies (X.1) and thus has the same economic implications as (X.1). However, (X.4) also implies that all consumption is carried out privately; that is, no coalition can consume more than the "sum" of what its subcoalitions can consume. There are no "public goods"¹¹ for they can only be consumed jointly by the subcoalitions of some coalition. An effort to extend value theory to include public goods has recently been made by Foley [19].

Assumption (X.6) means that there exists a continuous, antisymmetric vector order \geq _≡¹² on S and there exists an S -valued measure β on \mathcal{Q} such that for every E

$$z \geq_{\equiv} \beta(E) \quad \text{for every } z \in X(E) .$$

¹⁰A cone P is "pointed" if it contains no straight lines (one dimensional linear manifolds).

¹¹"Public goods" are defined, for example in [19].

¹²There is a one-to-one relation between vector orders \geq on S and convex cones P with vertex zero in S given by the relation $x \geq 0$ if and only if $x \in P$. A continuous order \geq_{\equiv} satisfies the condition: If $\{x_n\}$ and $\{y_n\}$ are two sequences of vectors satisfying

$$\begin{aligned} y_n &\geq_{\equiv} x_n \\ y_n &\rightarrow \text{some } y \\ x_n &\rightarrow \text{some } x , \end{aligned}$$

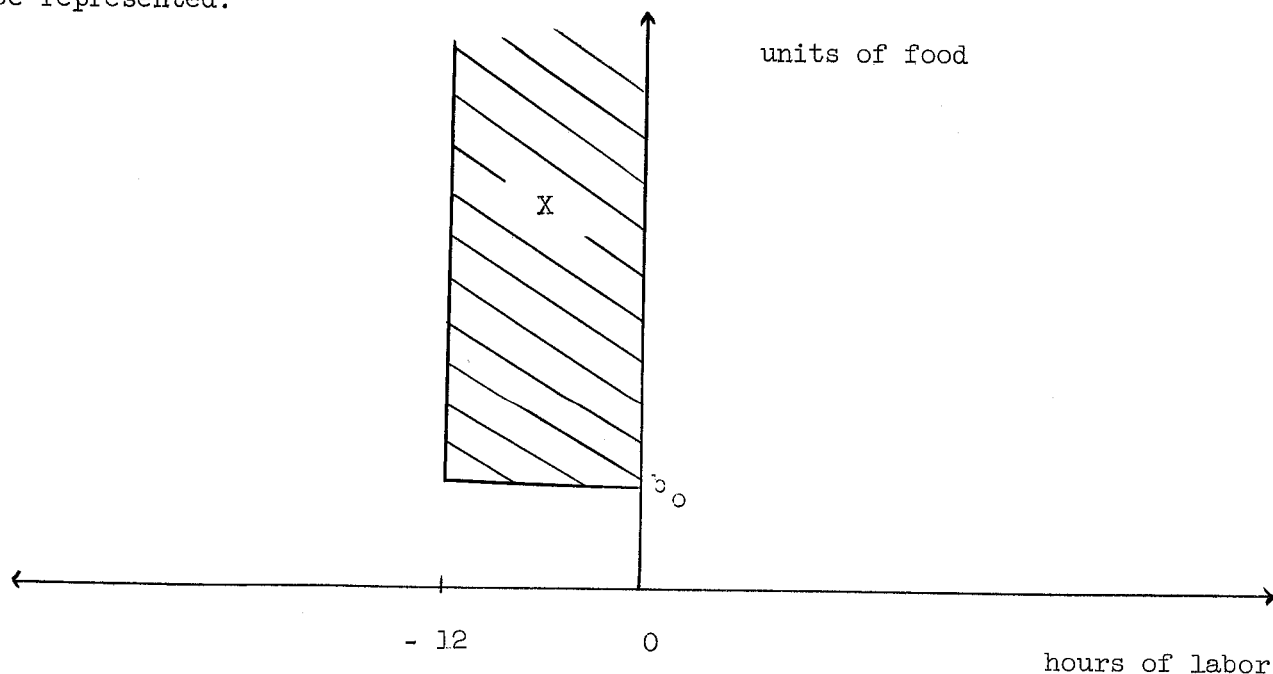
then $y \geq_{\equiv} x$. An order \geq_{\equiv} is continuous if and only if its corresponding convex cone is closed. An antisymmetric order \geq_{\equiv} satisfies the condition;

$$x \geq_{\equiv} y \quad \text{and} \quad y \geq_{\equiv} x \quad \text{imply} \quad x = y .$$

\geq_{\equiv} is antisymmetric if and only if $P \cap (-P) = \{0\}$. If P is a closed convex cone

This condition is fulfilled if $X(E)$ is a subset of the usual positive orthant of S , but assumption (X.6) also allows for the consumption of negative amounts of certain commodities, like labor services, as long as there is a lower bound to these amounts. This is a reasonable assumption (see [12] page 53).

Assumption (X.7) requires that no consumable commodity be available only in indivisible units. If z is a feasible consumption plan for E , then (X.2) and (X.7) imply that λz is also a feasible consumption plan for every $\lambda \in [0,1]$. (X.7) also requires that the technology of consumption show no increasing returns to scale. The significance of this assumption is illustrated by supposing there are two commodities, labor and food. A consumption possibilities set X might be represented:



This diagram is based on the assumption that this consumer can perform this type of labor for no more than 12 hours no matter how much food he has. Further, there is a subsistence level b_0 of food consumption such that the consumer will

with vertex zero, then the condition $P \cap (-P) = \{0\}$ is equivalent to the condition that P contain no one-dimensional linear manifolds.

not work if he receives less than b_0 units of food. The line segment $[(0,0), (0,b_0)]$ is in the consumption possibilities set so the set is nonconvex. This example illustrates that (X.7) has substantial economic significance. A similar example is given in [12] page 52.

The preferences of the consumers of the economy are specified by a \mathcal{M}_X -preference function on \mathcal{A} .¹³ This function assigns to each coalition E a binary relation \succeq_E on \mathcal{M}_X . The relation $\beta \succeq_E \alpha$ is interpreted as "E prefers β at least as much as α ". Thus it is supposed that for every coalition E there is a decision process which yields preferences among some, though in general not among all, pairs of allocations. The nature of this decision procedure is clarified by the following assumptions which are made on preferences. In these assumptions the following conventions are used:

$\alpha \simeq_E \beta$ means $\alpha \succeq_E \beta$ and $\beta \succeq_E \alpha$,

$\alpha \preceq_E \beta$ is equivalent to $\beta \succeq_E \alpha$,

$\alpha \succ_E \beta$ means $\alpha \succeq_E \beta$ is true but $\beta \succeq_E \alpha$ is not true,

$\alpha \succ \succ_E \beta$ means $\alpha \succ_F \beta$ holds for every nonzero subcoalition F of E and $E \neq \emptyset$.

The assumptions are:

(P.1) For every $(E,F) \in \mathcal{A} \times \mathcal{A}$ and $(\alpha,\beta) \in \mathcal{M}_X \times \mathcal{M}_X$, if $E \subset F$ and $\alpha \preceq_F \beta$ then $\alpha \preceq_E \beta$.

This is a type of unanimity rule for preferences of coalitions and reflects the assumption that the formation of coalitions is voluntary. Under assumption (P.1), the relation $\alpha \succ \succ_E \beta$ is equivalent to the statement: $\alpha \succ_E \beta$ and if F is a subcoalition of E and $\beta \succeq_F \alpha$ then $F = \emptyset$.

¹³This concept and its relation to the more usual idea of the preferences of individual economic agents were developed by G. Debreu [9].

(P.2) For every sequence $\{E_n\}$ of coalitions and for every $(\alpha, \beta) \in \mathcal{M}_X \times \mathcal{M}_X$ if $\alpha \geq_{E_n} \beta$ for all n then $\alpha \geq \bigcup E_n \beta$.

(P.3) For every $(\alpha, \beta) \in \mathcal{M}_X \times \mathcal{M}_X$ there exists $(A_1, A_2) \in \mathcal{A} \times \mathcal{A}$ such that $A_1 \cup A_2 = A$, $\alpha \geq_{A_1} \beta$ and $\beta \geq_{A_2} \alpha$.

This assumption serves the same role for the preferences of coalitions as does the usual completeness assumption on the preferences of individual economic agents (see [9]). In general \geq_E is not complete; that is, there may exist allocations α and β for which neither $\alpha \geq_E \beta$ nor $\beta \geq_E \alpha$ is true. However, assumptions (P.1) and (P.3) assert that there exist subcoalitions $E_1 = E \cap A_1$ and $E_2 = E \cap A_2$ satisfying $E_1 \cup E_2 = E$, $\alpha \geq_{E_1} \beta$ and $\beta \geq_{E_2} \alpha$. In particular, for any pair (α, β) of allocations, the relations $\alpha \geq_0 \beta$ and $\beta \geq_0 \alpha$ both hold and the relations $\beta >_0 \alpha$ and $\beta \gg_0 \alpha$ do not hold. Further, (P.1), (P.2) and (P.3) imply that \geq_E is reflexive for every coalition E .

(P.4) For every coalition F and $(\alpha, \beta, \gamma) \in \mathcal{M}_X \times \mathcal{M}_X \times \mathcal{M}_X$ if $\alpha \leq_F \beta$ and if $\beta|_F = \gamma|_F$ then $\alpha \leq_F \gamma$, where for any coalition H we define $(\beta|_F)(H) = \beta(H \cap F)$.

This is a strong economic assumption because it rules out "externalities of consumption". It is assumed that the consumption of a coalition F disjoint from a coalition E does not affect the desirability to E of what E consumes.¹⁴

(P.5) For every coalition E , the set $\{(\alpha, \beta) \in \mathcal{M}_X \times \mathcal{M}_X : \beta \leq_E \alpha\}$ is a closed subset of $\mathcal{M}_X \times \mathcal{M}_X$ with respect to the product topology

¹⁴For a related discussion, see [3], Chapter II, Section 3.

on $\mathcal{M}_X \times \mathcal{M}_X$ where \mathcal{M}_X has the topology corresponding to the norm $\| \cdot \|$ defined on \mathcal{M}_X by $\| \alpha \| = \sup\{|\alpha(E)| : E \in \mathcal{A}\}$. $\| \cdot \|$ is the Euclidean norm on S .

This is a "continuity" assumption on the preference relations of each of the coalitions analogous to the more usual assumption that, for each consumer, the set of pairs of commodity bundles x and y such that the consumer prefers x at least as much as y is closed in the Euclidean topology on $S \times S$ (see [9] and [12]). This analogy is made more explicit in [7].

(P.6) For every coalition E , \succcurlyeq_E is transitive.

An allocation is satiating for a coalition E if it is a maximal element of \mathcal{M}_X with respect to \succcurlyeq_E ; α is nonsatiating for E if there exists an allocation β such that $\beta \succcurlyeq_E \alpha$; α is locally nonsatiating for E if, for every $\epsilon > 0$, there exists an allocation β satisfying

$$\beta \succcurlyeq_E \alpha \quad \text{and} \quad |\beta(E) - \alpha(E)| < \epsilon ;$$

α is (locally) nonsatiating if it is (locally) nonsatiating for all nonzero coalitions.

It will be shown how the concepts introduced above can be applied to the description of a finite economy. Suppose there are n consumers represented by

$$A = \{1, \dots, n\} .$$

For consumer i there is a consumption possibilities set X_i which is a subset of S containing the point 0 . Consumer i also has a preference relation \preceq_i on X_i . The structures X_i and \preceq_i are the same as those defined in Debreu [12], Chapter 4.

Let \mathcal{A} be the set of all subsets of A . The operations in \mathcal{A} are the usual set operations of union, intersection and complementation. Define a consumption correspondence by:

$$X(E) = \begin{cases} \{0\} & E = \emptyset \\ \Sigma \{X_i : i \in E\} & E \neq \emptyset \end{cases}$$

\mathcal{M}_X is induced by X and consists of all finitely additive functions γ from \mathcal{A} to S such that, for each E , $\gamma(E) \in X(E)$. Finally, a preference function $\{\geq_E, E \in \mathcal{A}\}$ is defined by letting, for each $\{i\} \in \mathcal{A}$,

$$\geq_{\{i\}} = \{(\alpha, \beta) \in \mathcal{M}_X \times \mathcal{M}_X; \alpha(\{i\}) \succeq_i \beta(\{i\})\}$$

or, equivalently, $\alpha \geq_{\{i\}} \beta$ holds if and only if $\alpha(\{i\}) \succeq_i \beta(\{i\})$ holds. For each coalition E , define

$$\geq_E = \bigcap \{ \geq_{\{i\}} : i \in E \}$$

or, equivalently, $\alpha \geq_E \beta$ holds if and only if $\alpha(\{i\}) \succeq_i \beta(\{i\})$ holds for all i in E . It is readily seen that these preferences satisfy (P.1), (P.2) and (P.4) for any binary relations \succeq_i . Further, if \succeq_i is complete for each i , then for any pair of allocations α and β let

$$A_1 = \{i \in A : \alpha(\{i\}) \succeq_i \beta(\{i\})\} \quad \text{and}$$

$$A_2 = A \setminus A_1.$$

Then $\alpha \geq_{A_1} \beta$ and $\beta \geq_{A_2} \alpha$. Finally, if for each i \succeq_i is continuous, that is, if \succeq_i is a closed subset of $X_i \times X_i$, then \geq_E is a closed subset of $\mathcal{M}_X \times \mathcal{M}_X$ for every E in \mathcal{A} ; that is, (P.5) is valid.

A partial converse to this example is provided by considering an arbitrary structure $(\mathcal{A}, X, \{\geq_E, E \in \mathcal{A}\})$ where \mathcal{A} , a σ -algebra of coalitions,

contains an atom E , where X is a consumption correspondence and where $\{ \succeq_E, E \in \mathcal{A} \}$ is an \mathcal{M}_X -preference function on \mathcal{A} . It will be shown that E can be interpreted as an individual consumer of the type described in Chapter 4 of Debreu [12]. $X(E)$ is the set of feasible consumption plans for E . Each allocation γ specifies a point in $X(E)$. Conversely, to each point x in $X(E)$ there corresponds an allocation γ_x defined by

$$\gamma_x(F) = \begin{cases} 0 & \text{if } F \cap E = 0 \\ x & \text{if } F \cap E \neq 0 \end{cases} .$$

A preference ordering $\underline{\preceq}$ may be specified on $X(E)$ by defining $x \underline{\preceq} y$ for x and y in $X(E)$ if and only if $\gamma_x \succeq_E \gamma_y$. If preferences satisfy (P.1) and (P.3) then $\underline{\preceq}$ is complete and if preferences satisfy (P.5) then

$$\{(x,y) \in X(E) \times X(E) : x \underline{\preceq} y\}$$

is a closed subset of $X(E) \times X(E)$. This discussion demonstrates that the interpretation of an atom of \mathcal{A} as an individual economic agent is reasonable.

4. Production Possibilities

The production possibilities of an economy can be conveniently represented by a production correspondence Y which is a correspondence from \mathcal{A} to S satisfying:

(Y.1) Y is countably subadditive.

(Y.2) For every E , $0 \in Y(E)$,

(Y.3) $Y(0) = \{0\}$.

$Y(E)$ is interpreted as the technology available to E independently of the rest of the economy. This means that if y is a point in $Y(E)$ and if E

has available the inputs required by y , which, by convention, are the negative coordinates of y , then E has the capability to obtain the outputs of y , which are the positive coordinates of y . Thus the set $Y(E) + \omega(E)$ is the set of commodity bundles which E can obtain from its own resources $\omega(E)$ independently of the other coalitions. The set $X(E) \cap (Y(E) + \omega(E))$ is the set of commodity bundles which E can produce and consume from its own initial resources $\omega(E)$.

The practice of representing the technological possibilities available to a producer by means of a subset of the commodity space is quite familiar (see [12]), but the idea of a coalition of consumers having its own technology is not. Production is assumed to be the responsibility of consumers rather than of producers because it is desirable to study the operation of an economy with no prices. In this setting, a producer's aims cannot be defined, although consumers would still want production to occur. The problem of defining an equilibrium in such an economy can be simplified, therefore, by assuming consumers are also producers.

Assumption (Y.1) that Y is subadditive reflects the assumption that if coalitions E and F are disjoint then the technologies $Y(E)$ and $Y(F)$ are independent; that is $Y(E)$ is interpreted to be the set of production plans which E can carry out if E has the requisite inputs regardless of which production plans are chosen by coalitions disjoint from E . Thus it can be supposed that two disjoint coalitions E and F jointly can do at least as much as they could do separately; that is, $Y(E \cup F) \supseteq Y(E) + Y(F)$. Assumption (Y.2) that 0 is in the technology set of every coalition E means that it is possible for E to carry out trades among its subcoalitions.

It may be desirable to allow for the possibility of decreasing returns to scale (see [4] p. 267). In this case there may be a coalition E for which

$Y(E)$ is not a subadditive set; that is, the relation $Y(E) + Y(E) \subset Y(E)$ may not hold. It is then necessary to "divide" E 's technology among its subcoalitions. This division is given by Y and will be accepted as a basic characteristic of an economy.

These ideas are illustrated by two examples. First, suppose there exists a technology $T \subset S$ which is available to all coalitions, which contains the point 0 and which has nondecreasing returns to scale; i.e. $T + T \subset T$. A production correspondence Y can be defined by

$$Y(E) = \begin{cases} \{0\} & E = 0 \\ T & E \neq 0 \end{cases} .$$

This example corresponds to the productive economy studied by Debreu and Scarf [14].

A slightly more general example can be obtained by supposing there are n firms, denoted by the index $j=1, \dots, n$. Firm j has a technology T_j which is a subadditive set containing 0 . Suppose that firm j is owned by coalition A_j . These coalitions need not be disjoint and it does not matter what share of firm j is owned by any particular subcoalition of A_j . A production correspondence may be defined by assuming $\bigcup A_j = A$ and defining:

and

$$J(E) = \{j=1, \dots, n: E \cap A_j \neq 0\}, \quad E \in \mathcal{A}$$

$$Y(E) = \sum_{j \in J(E)} Y_j, \quad E \in \mathcal{A} .$$

This production correspondence reflects the assumptions that each coalition has access to all the technologies of firms in which one of its subcoalitions has an ownership share and that the technologies of the various firms are independent. This type of production correspondence (though not the type of motivation presented here) is considered in [21], page 10.

In this paper the following additional assumptions are made on the production correspondence Y :

(Y.4) Y is countably additive.

(Y.5) For every E , $Y(E)$ is a closed subset of S .

(Y.6) Y is bounded below.

(Y.7) For every E , $Y(E)$ is convex.

Assumption (Y.4) implies (Y.1) and has the additional implication that the formation of coalitions does not, in itself, bring increasing returns to scale. Assumption (Y.5) requires that if $\{y_n\}$ is a sequence of production plans, which can be carried out if the requisite inputs are available, and if $|y_n - y| \rightarrow 0$, where $y \in S$ and where $|\cdot|$ is the Euclidean norm on S , then y is also a technologically feasible production plan.

Assumption (Y.6) is analogous to (X.6), but the ordering with respect to which Y is bounded below may differ from the ordering with respect to which X is bounded below. If, for some E , $Y(E)$ is a cone with vertex zero (that is, $y \in Y(E)$ implies $t y \in Y(E)$ for every nonnegative scalar t) then (Y.6) implies $Y(E) \cap -Y(E) = \{0\}$ and hence production is irreversible (see Debreu [12] page 40). Assumption (X.7) is a restrictive assumption because it implies that there are no increasing returns to scale and it implies that all inputs and outputs are divisible.

The following definitions can now be made. An economy is a structure $(\mathcal{A}, X, \{\succeq_E, E \in \mathcal{A}\}, \omega, Y)$ where \mathcal{A} is a σ -algebra of coalitions, X is a consumption correspondence, $\{\succeq_E, E \in \mathcal{A}\}$ is an \mathcal{M}_X -preference function on \mathcal{A} satisfying (P.1) - (P.5), ω is an allocation (by (X.2) there

exists at least one allocation) and Y is a production correspondence. A regular economy is an economy in which X satisfies (X.4) - (X.6) and Y satisfies (Y.4) - (Y.6). A convex economy is an economy in which X satisfies (X.7) and Y satisfies (Y.7). Finally, a nonatomic economy is an economy in which

\mathcal{A} is nonatomic; that is, to every nonzero coalition E there corresponds a nonzero subcoalition F such that $E \setminus F \neq \emptyset$. The implications of these assumptions are studied in the following Sections.

5. Optimality

It is now possible to describe several types of optimal or efficient allocations. For every coalition E let

$$\mathcal{F}(E) = \{ \alpha \in \mathcal{M}_X : \alpha(E) \in Y(E) + \omega(E) \} .$$

$\mathcal{F}(E)$ is the set of allocations attainable by E independently of $A \setminus E$. E is said to block an allocation α if there exists β in $\mathcal{F}(E)$ satisfying $\beta \succ_E \alpha$. Thus a coalition can block an allocation α if it can find another allocation which it prefers to α and which it can produce for itself. This definition of blocking requires that all subcoalitions of E strictly prefer β to α . A weaker definition of blocking has been used by some:¹⁵ E weakly blocks an allocation α if there exists β in $\mathcal{F}(E)$ satisfying $\beta \succ_E \alpha$. It is easier for a coalition to weakly block an allocation than to block the same allocation. Conditions for the equivalence of these two types of blocking have been studied in [7].

A Pareto allocation is any element of $\mathcal{F}(A)$ which cannot be weakly blocked by A .¹⁶ The core, or set of Edgeworth allocations \mathcal{C} , is the set

¹⁵For example, see [11].

¹⁶The definition of a Pareto allocation depends only on the value of ω at A and not on its values on subcoalitions. Thus the concept of Pareto optimality can be used even if the total resources of an economy are given without specifying the distribution of their ownership. In footnote 28 below, a reason is given for using weak blocking rather than blocking in the definition of the Pareto allocations.

of elements of $\mathcal{F}(A)$ which no coalition can block. Thus for α to be in \mathcal{E} , it is necessary that α be feasible for the economy as a whole, that is, $\alpha \in \mathcal{F}(A)$, and that no coalition be able to do better on its own. Finally, the strong core, \mathcal{E}^S is the set of elements of $\mathcal{F}(A)$ which no coalition can weakly block. It is clear that $\mathcal{E}^S \subset \mathcal{E}$.

A characterization of \mathcal{E} and \mathcal{E}^S which is analytically convenient can be obtained by defining for each coalition E and for each allocation α :

$$P_\alpha(E) = \{x \in S : x = \beta(E) \text{ for some allocation } \beta \text{ satisfying } \beta \succ_E \alpha\}$$

$$Q_\alpha(E) = \{x \in S : x = \beta(E) \text{ for some allocation } \beta \text{ satisfying } \beta \succ_E \alpha\}$$

$$S_\alpha(E) = P_\alpha(E) - \omega(E) - Y(E)$$

$$T_\alpha(E) = Q_\alpha(E) - \omega(E) - Y(E)$$

$P_\alpha(E)$ is the set of commodity bundles in $X(E)$ which can be allocated among the subcoalitions of E in such a way that each subcoalition is better off than it would be if α were chosen. $S_\alpha(E)$ is the set of net trades which E could make and end up after production with a commodity bundle in $P_\alpha(E)$. The usefulness of these definitions stems from the fact that if $\alpha \in \mathcal{F}(A)$, then:

$$(i) \alpha \in \mathcal{E} \text{ if and only if } 0 \notin S_\alpha(A) \equiv \bigcup \{S_\alpha(E) : E \in \mathcal{A}\}$$

$$(ii) \alpha \in \mathcal{E}^S \text{ if and only if } 0 \notin T_\alpha(A) \equiv \bigcup \{T_\alpha(E) : E \in \mathcal{A}\} .$$

In a regular, convex, nonatomic economy, the sets $S_\alpha(A)$ and $T_\alpha(A)$ are convex (see Theorems 1 and 2 in [7]).

Correspondences P_α , Q_α , S_α and T_α can be defined on \mathcal{A} as the mappings which take E into, respectively, $P_\alpha(E)$, $Q_\alpha(E)$, $S_\alpha(E)$ and $T_\alpha(E)$.

In a regular convex economy, these correspondences are finitely additive on the set of nonzero coalitions. Each of the correspondences maps the zero coalition into the empty set.

Another type of optimality may be defined in an economy where there are prices. A price vector p is an element of S . For each coalition E and each price vector p it is possible to define a point in \bar{R} ¹⁷ by

$$\pi(p,E) = \sup p \cdot Y(E) = \sup\{p \cdot y : y \in Y(E)\} .$$

π is the profit function. For each p and E , it gives the supremum of the profits attainable by E . Similarly, a wealth function may be defined by

$$w(p,E) = p \cdot \omega(E) + \pi(p,E) .$$

The set \mathcal{Q} of quasi-competitive allocations is the set of α in $\mathcal{F}(A)$ to each of which there corresponds a nonzero price vector p such that for every coalition E :

- (i) $p \cdot \alpha(E) \leq w(p,E)$,
- (ii) Either x in $P_\alpha(E)$ implies $p \cdot x > w(p,E)$ or $w(p,E) = \inf p \cdot X(E)$,
- (iii) $p \cdot [\alpha(A) - \omega(A)] = \pi(p,A)$.

Condition (i) requires that for every coalition E , its consumption plan $\alpha(E)$ satisfy its budget constraint.¹⁸ Condition (ii) requires that, unless $w(p,E) = \inf p \cdot X(E)$, E can find no other bundle in its budget set which is

¹⁷ \bar{R} is the extended real line: $R \cup \{+\infty\} \cup \{-\infty\}$. This use of the profit function π was suggested by W. Hildenbrand.

¹⁸It should be recalled that since $\alpha(E)$ includes as negative components the labor services rendered by E , then $p \cdot \alpha(E)$ is the cost of E 's expenditures net of labor income.

"preferable". That is, α is a "maximal" element¹⁹ for E in the set of allocations γ satisfying $p \cdot \gamma(E) \leq w(p, E)$. Condition (iii) requires that "profits" $p \cdot [\alpha(A) - \omega(A)]$ be the maximum attainable by the economy as a whole. Such an allocation α is "competitive" in the sense that coalitions take p as given in maximizing their satisfactions and that profits are maximized with respect to given p .

If $\alpha \in \mathcal{Q}$ and if, for every coalition E , $x \in P_\alpha(E)$ implies $p \cdot x > w(p, E)$, then α is called a Walras allocation. The set of such allocations will be denoted \mathcal{W} . Of course a quasi-competitive allocation is a Walras allocation if ω and p are such that for all nonzero E ,

$$w(p, E) > \inf p \cdot X(E) .$$

In this paper it is assumed that voluntary bargaining will prevent any allocation from being chosen which is not in the core. The influence which any particular coalition E with preferences \succeq_E has on the choice of the final allocation depends on the set $\mathcal{F}(E)$. Every nonzero coalition E will in general have some blocking capability and thus will have some influence in the choice of the final allocation. In particular, atoms will be able to affect the choice of an allocation. However, in a perfectly competitive economy no individual economic agent, that is, no atom, has blocking power. Thus an economy is perfectly competitive only if it is nonatomic.

¹⁹Just as there are two types of blocking, based on $>$ and $>>$, so there are two possible types of maximality. Let a price p and a wealth function w be given. α is weakly maximal with respect to p and w if γ in \mathcal{M}_X , E in \mathcal{A} and $\gamma >_E \alpha$ imply $p \cdot \gamma(E) > w(p, E)$. α is strongly maximal with respect to p and w if γ in \mathcal{M}_X , E in \mathcal{A} and $\gamma >_E \alpha$ imply $p \cdot \gamma(E) > w(p, E)$. If α is strongly maximal then it is weakly maximal. Conversely, if preferences satisfy (P.1) - (P.3) and are transitive (P.6), and if α is locally nonsatiating and weakly maximal, then α is strongly maximal. To see this, suppose that there exists an allocation γ satisfying $\gamma >_E \alpha$. Then there exists a μ -maximal sub-coalition E_1 of E such that $\gamma \leq_{E_1} \alpha$. Let $E_2 = E \setminus E_1$. Then $\gamma >>_{E_2} \alpha$ and $\gamma =_{E_1} \alpha$. Because α is weakly maximal, $p \cdot \gamma(E_2) > w(p, E_2)$. Because α is locally nonsatiating and preferences are transitive, $p \cdot \gamma(E_1) \geq w(p, E_1)$. Finally, because w is finitely additive (see Lemma 1 in [6]), $p \cdot \gamma(E) > w(p, E)$. Thus α is strongly maximal.

This definition of perfect competition refers only to the coalition structure of the traders in the economy and not to a definite market mechanism. The surprising result is that this apparently weak characterization of perfect competition guarantees that in a regular convex economy any locally nonsatiating allocation α which can be arrived at through voluntary bargaining (that is, $\alpha \in \mathcal{E}$) can also be obtained by specifying a price system and by restricting all coalitions to trade at those prices (that is, $\alpha \in \mathcal{Q}$). Thus price systems arise very naturally even when no initial specification of a market is made. This result is contained in Theorem 2 in the next Section and has been established in several different forms by others ([1], [21] and [34]).

This may appear to be a poor model for an economy if it does not permit the existence of individual economic agents. However, it is one of the virtues of this analysis in that by establishing the precise implications of the idea of perfect competition, it is seen that, not only is it unnecessary to work with individual economic agents, but their individuality is lost. It is only in a nonatomic economy that Walrasian analysis of general equilibrium can be carried out. In order to relate the idea of perfect competition to finite economies composed of individual economic agents, it seems to be necessary to view a nonatomic economy as a limit of finite economies. Progress in this direction has been made by Kannai [25], Drèze, Gepts and Gabszewicz [15], Hildenbrand [23] and the present author [8]. This work is especially interesting, because it connects the general equilibrium analysis deriving from Walras with the game-theoretic analysis of von Neumann and Morgenstern.

6. Price Characterizations of Edgeworth and Pareto Allocations.

THEOREM 1: (i) $\mathcal{W} \subset \mathcal{E}$

(ii) Under (P.1) - (P.3) and (P.6), if α is a locally nonsatiating allocation in \mathcal{W} , then $\alpha \in \mathcal{E}^S$.

PROOF: (i) Let $\alpha \in \mathcal{W}$, let p be the corresponding price vector and let E be any coalition. If $x \in P_\alpha(E)$, then $p \cdot x > w(p, E)$. If $x \in Y(E) + \omega(E)$, then $p \cdot x \leq w(p, E)$. Thus $P_\alpha(E) \cap [Y(E) + \omega(E)]$ is empty for every E so $\alpha \in \mathcal{E}$.

(ii) We want to show that $P_\alpha(E) \cap [Y(E) + \omega(E)]$ is empty for every E . But the argument in footnote number 19 shows that under the given hypothesis, a procedure analogous to that used to demonstrate (i) can also be used here.

We note that the proof of (i) in Theorem 1 used only the definitions of \mathcal{W} and \mathcal{E} . No assumptions were made about the economy.

THEOREM 2: In a nonatomic, regular, convex economy, every locally nonsatiating allocation in \mathcal{E} is also in \mathcal{Q} .

PROOF: Let α be a locally nonsatiating allocation in \mathcal{E} . Then $0 \notin S_\alpha(\mathcal{A})$ which is nonempty because α is nonsatiating and which is convex by Theorem 1 in [7]. Hence there exists a nonzero price vector p in S such that if $x \in S_\alpha(\mathcal{A})$ then $p \cdot x \geq 0$. That is, for any given coalition $E \neq \emptyset$ and for every $(x, y) \in [P_\alpha(E) - \omega(E)] \times [Y(E)]$, $p \cdot y \leq p \cdot x$ so $\pi(p, E) = \sup p \cdot Y(E) \leq \inf p \cdot [P_\alpha(E) - \omega(E)]$.²⁰ But since α is locally nonsatiating, for every $n=1, 2, \dots$ there exists $\beta_n \in \mathcal{M}_X$ with $|\beta_n(E) - \alpha(E)| < 1/n$ and $\beta_n \succ_E \alpha$. This implies that for every $E \in \mathcal{A}$:

$$(a) \quad \pi(p, E) \leq p \cdot [\alpha(E) - \omega(E)].$$

²⁰We recall that $\inf p \cdot K = \inf \{p \cdot z : z \in K\}$ and $\sup p \cdot K = \sup \{p \cdot z : z \in K\}$.

Since $\alpha(A) - \omega(A) \in Y(A)$ then (a) implies that $p \cdot [\alpha(A) - \omega(A)] = \pi(p, A)$ so that the profit maximization condition (iii) for $\alpha \in \mathcal{Q}$ is verified.

To show that condition (i) is satisfied, suppose that for some coalition E , $p \cdot [\alpha(E) - \omega(E)] > \pi(p, E)$. Then by (a):

$$\begin{aligned} \pi(p, A \setminus E) &\leq p \cdot [\alpha(A \setminus E) - \omega(A \setminus E)] \\ &= p \cdot [\alpha(A) - \omega(A)] - p \cdot [\alpha(E) - \omega(E)] \\ &< \pi(p, A) - \pi(p, E) = \pi(p, A \setminus E) \end{aligned}$$

since $\pi(p, \cdot)$ is finitely additive by Lemma 1 in [6]. But this is a contradiction. Thus, for every coalition E , $p \cdot \alpha(E) \leq w(p, E)$. In fact, by (a), equality holds.

To show that condition (ii) holds, let $x \in P_\alpha(E)$ for some coalition E . By (a), $p \cdot x \geq w(p, E)$. Suppose $p \cdot x = w(p, E) > \inf p \cdot X(E)$. Then there exists $z \in X(E)$ such that $p \cdot z < w(p, E)$. In fact, since $p(\cdot)$ is continuous on $X(E)$, it can be assumed that z is in $\text{ri}(X(E))$.²¹ Let $z_t = (1-t)x + tz$. Then by Theorem 3 in [7] there exists $t_0 \in (0, 1)$ such that $z_{t_0} \in P_\alpha(E)$. But $p \cdot z_{t_0} < p \cdot x = w(p, E)$ which is impossible by (a). Thus either $p \cdot x > w(p, E)$ or $w(p, E) = \inf p \cdot X(E)$.

This completes the proof of Theorem 2. This result is not entirely satisfactory, because one really wants conditions under which $\mathcal{E} \subset \mathcal{W}$. Two approaches are possible: The first is to make assumptions which guarantee that even if $w(p, E) = \inf p \cdot X(E)$ for some E , then $x \in P_\alpha(E)$ still implies $p \cdot x > w(p, E)$. Aumann [1] adopts this approach by assuming:

²¹If K is a subset of S , then $\text{ri}(K)$, the relative interior of K , is the interior of K with respect to the smallest linear manifold containing K . This is studied by Eggleston [18], for example.

$X(E)$ equals the closed positive orthant for every nonzero E ,

$Y(E) = \{0\}$, $E \in \mathcal{A}$,

preferences are strictly monotonic,²²

$\omega(A) \gg 0$.

These assumptions ensure that $p \gg 0$.

The second approach which can be adopted is to make assumptions which guarantee that p can be chosen so that $w(p,E) > \inf p \cdot X(E)$ for each nonzero E . This approach has been developed by Debreu [10]. We shall present a variant of his approach which is adapted to an infinite economy and which avoids a free disposal assumption on the production possibilities in the economy. Two further assumptions must be stated for this approach. For the first assumption, we assume that the commodity space S is N -dimensional. For any scalar ρ and for any integer h satisfying $1 \leq h \leq N$, $h(\rho)$ denotes the vector in S all coordinates of which equal zero except for the h^{th} , which equals ρ .

(E.1) There exists a nonempty subset $\mathcal{D} \subset \{1, \dots, N\}$ such that for any nonzero E :

(i) for any $\rho > 0$ and $h \in \mathcal{D}$ and $\alpha \in \mathcal{M}_X$, $\alpha(E) + h(\rho) \in P_\alpha(E)$,

(ii) there exists $\rho > 0$ and $h \in \mathcal{D}$ such that $\omega(E) - h(\rho) \in X(E)$.

(E.2) $\text{ri}(X(A)) \cap \text{ri}[Y(A) + \omega(A)]$ is not empty.²¹

²²Preferences are strictly monotonic if for any nonzero E and any allocations α and β the condition $[\beta(F) \gg \alpha(F)]$ for every nonzero $F \subset E$ implies $\beta \gg_E \alpha$. The symbol \gg with no subscript refers to the usual vector ordering on S ; that is, $x \gg y$ if each coordinate of x is greater than the corresponding coordinate of y .

In assumption (E.1), \mathcal{D} is the collection of commodities which are always desired by everyone. We assume that each nonzero coalition could subtract a positive amount ρ from its initial holdings $\omega^h(E)$ of some commodity h in \mathcal{D} and still have a feasible consumption plan $\omega(E) - h(\rho)$.²³ (E.1) also implies that if x is any consumption plan in $X(E)$, $E \neq 0$ and if we increase the h^{th} coordinate of x by any amount and for any h in \mathcal{D} , then the resulting consumption plan is also in $X(E)$. This is equivalent to requiring that the cone C spanned by the unit coordinate vectors indexed by \mathcal{D} satisfies $X(E) + C \subset X(E)$ for every nonzero E . Assumption (E.2) is a slightly strengthened version of the condition that there exist allocations which are feasible for the economy as a whole.²⁴ In particular, for a pure trade economy where $Y(A) = \{0\}$, (E.2) implies that $\omega(A) \in \text{ri}(X(A))$. In the case where, for each nonzero E , $X(E)$ is the closed positive orthant of S , the analogous condition is that $\omega(A) \gg 0$. For a productive economy, the condition $\omega(A) \in \text{ri}(X(A))$ is sufficient to ensure that (E.2) is valid, but is not necessary.

THEOREM 3: In a nonatomic, regular, convex economy satisfying (E.1) and (E.2), every locally nonsatiating allocation α in \mathcal{E} is also in \mathcal{W} .

In particular, if (P.6) also holds, then $\alpha \in \mathcal{E}^S$.

PROOF: By Theorem 2, if α is a locally nonsatiating Edgeworth allocation, then it is also a quasi-competitive allocation. Let p be the price vector associated with α . We shall show first:

²³If there is a set \mathcal{Q} of "always productive" commodities (see [10], page 271), then part (ii) of (E.1) can be weakened to require that each nonzero coalition be able to subtract a positive amount from its initial holdings of some commodity in $\mathcal{D} \cup \mathcal{Q}$. However, the only known derivation of condition (b) in the proof of Theorem 3 below requires the assumption of free disposal ($z \leq 0 \Rightarrow z \in Y(A)$). This assumption guarantees that any price vector associated with an allocation which maximizes profit on $Y(A)$ is nonnegative.

²⁴The requirement that there exist allocations which are feasible for the whole economy is guaranteed by (Y.1) and the requirement that $\omega(A) \in X(A)$.

(b) If $w(p,E) = \inf p \cdot X(E)$ for some nonzero E , then $w(p,A) = \inf p \cdot X(A)$.²⁵

We assume $w(p,E) = \inf p \cdot X(E)$ for some nonzero E . Now suppose p^h , the h^{th} coordinate of p , is greater than 0 for every $h \in \mathcal{D}$. Then

$$\begin{aligned} w(p,E) &\geq p \cdot \omega(E) \\ &> p \cdot [\omega(E) - h(\rho)] \\ &\geq \inf p \cdot X(E) \end{aligned}$$

for some $\rho > 0$ and $h \in \mathcal{D}$ by (E.1). Thus $p^h \leq 0$ for some $h \in \mathcal{D}$. But then by (E.1) there exists $z = \alpha(A) + h(\rho) \in P_\alpha(A)$ satisfying

$$p \cdot z \leq p \cdot \alpha(A) \leq w(p,A).$$

But then $\alpha \in \mathcal{Q}$ implies $w(p,A) = \inf p \cdot X(A)$. This completes the proof of (b).

Let L be the smallest linear manifold containing the set $X(A) - \omega(A) - Y(A)$.²⁶ Because $0 \in Y(A)$ and $\omega(A) \in X(A)$ we have

$$0 \in X(A) - \omega(A) - Y(A).$$

But then L is a linear subspace of S . Further

$$\begin{aligned} 0 \in Y(A) &\text{ implies } X(A) - \omega(A) \subset L && \text{and} \\ 0 \in X(A) - \omega(A) &\text{ implies } Y(A) \subset L. \end{aligned}$$

Now the correspondence Z defined by $Z(E) = X(E) - \omega(E)$ is finitely additive and $Z(E)$ contains zero for every E . Hence $Z(E) \subset Z(A) \subset L$ for every E . Similarly, $Y(E) \subset Y(A) \subset L$ for every E . Thus, for every E ,

$$S_\alpha(E) \subset X(E) - \omega(E) - Y(E) \subset L.$$

²⁵This is the same as condition (e) on page 270 in [10].

²⁶The rest of the proof of Theorem 3 is a simple adaptation to our problem of the proof of the Proposition in [10].

Thus $S_\alpha(\mathcal{A})$ is a convex subset of L . But then in the proof of Theorem 2, we could choose p to be a nonzero vector in L .

Suppose $p \cdot \alpha(E) = w(p, E) = \inf p \cdot X(E)$ occurs for a nonzero E . Then by (b),

$$p \cdot \alpha(A) = w(p, A) = \inf p \cdot X(A).$$

But we also have

$$p \cdot \alpha(A) = \sup p \cdot (Y(A) + \omega(A)).$$

But then $X(A)$ lies "above" the hyperplane H with normal p through $\alpha(A)$ and $Y(A) + \omega(A)$ lies "below" H . Further, H cannot contain both $X(A)$ and $Y(A) + \omega(A)$, because H would then contain 0 and so be a proper linear subspace of L . But then $X(A) - \omega(A) - Y(A) \subset H$ which contradicts the choice of L . Thus either $X(A)$ or $Y(A) + \omega(A)$ contains points which are not in H . But then the relative interior of this set is disjoint from H and hence disjoint from the other set. This contradicts (E.2). Thus $w(p, E) = \inf p \cdot X(E)$ occurs for no nonzero E .

This completes the proof of Theorem 3. We have shown that under (E.1) and (E.2), the price vector p can be chosen in the proof of Theorem 2 so as to yield $\alpha \in \mathcal{W}$. A corollary of this proof is that $p^h > 0$ for every h in \mathcal{S} . Because $e^s \subset e$, the price characterizations given in Theorems 2 and 3 also apply to elements of e^s .

To derive a similar characterization of the Pareto allocations, it is convenient to let $e(\omega)$ (resp., $e^s(\omega)$) denote the core (resp., strong core) of the economy with an initial resources allocation ω . Similarly, let $\mathcal{Q}(\omega)$ (resp., $\mathcal{W}(\omega)$) denote the set of quasi-competitive allocations (resp., Walras allocations) of an economy with an initial resources allocation ω .

LEMMA: In an economy with preferences satisfying (P.1) - (P.4) and with a production correspondence Y which satisfies $Y(A) + Y(A) \subset Y(A)$, if α is a Pareto allocation then $\alpha \in \mathcal{E}^s(\alpha)$.²⁷

PROOF: Let α be a Pareto allocation and suppose there is a coalition E weakly blocking α . Then there is an allocation β such that $\beta \succ_E \alpha$ and $\beta(E) \in Y(E) + \alpha(E)$. Redefine β on $A \setminus E$ so that $\beta|_{A \setminus E} = \alpha|_{A \setminus E}$. Then $\beta \succ_A \alpha$.

$$\begin{aligned} \beta(A) - \omega(A) &= \beta(E) - \alpha(E) + \alpha(E) + \beta(A \setminus E) - \omega(A) \\ &= \beta(E) - \alpha(E) + \alpha(A) - \omega(A) \\ &\in Y(E) + Y(A) \subset Y(A). \end{aligned}$$

But this contradicts the assumption that α was a Pareto allocation.²⁸

THEOREM 4: In a nonatomic, regular convex economy which has a production correspondence satisfying $Y(A) + Y(A) \subset Y(A)$, if α is a locally non-satiating Pareto allocation, then $\alpha \in \mathcal{Q}(\alpha)$ and if the economy also satisfies (E.1) and (E.2), then $\alpha \in \mathcal{W}(\alpha)$.

Theorem 4 is an immediate corollary of the preceding Lemma and of Theorems 2 and 3.

The relations which we have just established among the various types of "efficient" allocations in an economy allow us to investigate the "stability" of these efficient allocations. An allocation α is stable if $\alpha \in \mathcal{E}(\alpha)$.

²⁷We remark that for this Lemma to be true, Y need not even satisfy (Y.1) - (Y.3). We only require that Y be monotone; that is, $E \subset F$ implies $Y(E) \subset Y(F)$. This Lemma and its use in the following Theorem are due to Hildenbrand [21].

²⁸This proof would be invalid if a Pareto allocation were defined to be any feasible allocation which could not be blocked by A . It is for this reason that weak blocking is used in the definition of Pareto allocations.

This means that if the allocation α were effected, then any later attempt by some coalition E to find another allocation β which it could produce (that is, $\beta(E) \in Y(E) + \alpha(E)$) and which it prefers (that is, $\beta \succ_E \alpha$) cannot succeed.

We shall first present an example to show that even the allocations in the core of an economy need not be stable. We suppose that the consumption possibilities set for each nonzero coalition is the closed, positive orthant of R^3 . There are three consumers $\{A_1, A_2, A_3\}$ in the economy with the following characteristics:

	<u>resources</u>		<u>utility functions</u>
$\omega(A_1)$	= (0,b,0)	$U_1((x,y,z))$	= x
$\omega(A_2)$	= (0,0,c)	$U_2((x,y,z))$	= y
$\omega(A_3)$	= (a,0,0)	$U_3((x,y,z))$	= z .

The parameters a, b, c are positive.

Define an allocation α :

$$\alpha(A_1) = (a, 0, 0), \quad \alpha(A_2) = (0, 0, c), \quad \alpha(A_3) = (0, b, 0).$$

It is clear that the only coalition which can weakly block α is $\{A_1, A_2, A_3\}$.²⁹

However, this coalition cannot block α . Thus

$$\alpha \in \mathcal{C}(\omega) \quad \text{but} \quad \alpha \notin \mathcal{C}^s(\omega).$$

Once the distribution of commodities specified by α is carried out, then $\{A_2, A_3\}$ can do better. Define a second allocation β :

²⁹ Thus α is not a Pareto allocation. Because this pure trade economy satisfies the hypothesis of the Lemma, an allocation can be unstable only if it is not a Pareto allocation.

$$\beta(A_1) = (a, 0, 0) , \quad \beta(A_2) = (0, b, 0) , \quad \beta(A_3) = (0, 0, c) .$$

β is better for $\{A_2, A_3\}$ than α and, given α , is feasible for $\{A_2, A_3\}$.

Thus $\alpha \notin \mathcal{E}(\alpha)$.

The economy specified in this example satisfies condition (E.2). It also satisfies condition (E.1) if the utility functions defined above are replaced by the following utility functions:

$$U_1'((x, y, z)) = x + \frac{1}{2}(y + z)$$

$$U_2'((x, y, z)) = y + \frac{1}{2}(x + z)$$

$$U_3'((x, y, z)) = z + \frac{1}{2}(x + y) .$$

If in this modified economy we have $a = b = c$, then we continue to find that

$$\alpha \in \mathcal{E}(\omega) \quad \text{but} \quad \alpha \notin \mathcal{E}^S(\alpha) \quad \text{and} \quad \alpha \notin \mathcal{E}(\alpha) .$$

In order to ensure that an allocation in $\mathcal{E}(\omega)$ is also stable, it appears necessary to make assumptions which ensure that $\mathcal{E}(\omega) = \mathcal{E}^S(\omega)$:

THEOREM 5: (i) In an economy satisfying (P.1) - (P.4) and $Y(A) + Y(A) \subset Y(A)$, all Pareto allocations and all allocations in $\mathcal{E}^S(\omega)$ are stable.

(ii) If the economy also satisfies (P.6), then all locally nonsatiating allocations in \mathcal{W} are stable.

(iii) If the economy is also nonatomic, regular, convex and satisfies (E.1) and (E.2), then locally nonsatiating allocations in $\mathcal{E}(\omega)$ are stable.

This Theorem is an immediate consequence of the preceding Lemma and Theorems. To ensure that every Edgeworth allocation in a finite economy is stable, it would appear necessary to have weak blocking be equivalent to blocking. This problem has been investigated in [7].

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Core, Competitive equilibrium, Nonatomic economy, Game Theory.						

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