

THE APPROXIMATION OF PERFECT COMPETITION BY A LARGE,
BUT FINITE, NUMBER OF TRADERS

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ABSTRACT

This paper uses the techniques developed by Debreu and Hildenbrand for representing sequences of economies by sequences of measures on a certain topological space to prove a property similar to upper semi-continuity of the correspondence \mathcal{C} which maps each economy into the set of allocations in the core of that economy. This result is then used to extend Scarf's proof of the nonemptiness of the core of certain finite economies to infinite economies with a finite number of different types of agent. It is also possible to use Scarf's result to prove the existence of a competitive equilibrium for a finite economy. Finally, the upper semi-continuity of \mathcal{C} is used to prove Hildenbrand's result that, loosely speaking, an allocation in the core of an approximately perfectly competitive economy is close to being a competitive allocation. It is shown how the Debreu-Scarf limit theorem on the core of an economy is a special case of this result.

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1. Introduction

Eighty years ago, Edgeworth [18] showed that if the number of traders in a bargaining situation were increased in a certain way, then any outcome of the bargaining which was not "competitive" would eventually be rejected by some group of the traders. This result, that the core of an economy shrinks to the set of competitive allocations as the number of traders increases to infinity, was revived by Shubik [30] and was substantially strengthened by Debreu and Scarf [15]. Aumann [2] and Vind [32] both saw that it was worthwhile to try to find a representation for the limit of such a sequence of economies with an increasing number of traders. They saw that this limit economy was best represented by a "nonatomic" measure space of traders.

It was then shown by Kannai [22] that the limiting process itself could be represented on such a measure space. Kannai's approach included the Debreu-Scarf result as a special case, but did not permit an adequate economic interpretation. Hildenbrand [21] has recently overcome this objection by demonstrating that any economy can be represented by choosing a particular measure on the Borel sets in a topological space consisting of the set of all possible "characteristics" of economic agents. The properties of a sequence of economies can then be studied by looking at a sequence of measures on this space.

This technique made the study of limits of economies much more tractable. However, Hildenbrand's limit result applied only to pure exchange economies,

required monotonicity of preferences and considered only allocations which assigned the same bundle to all traders of the same type. Although every allocation in the core possesses this characteristic when there is strict convexity of preferences and when there are equal numbers of each type of agent (see Theorem 2, page 241 in [15]), relaxation of either of these conditions permits allocations to be in the core which assign different bundles to agents of the same type.

This paper presents a model which overcomes these weaknesses. In sections 2, 3 and 4 of this paper, the activities of an economy are given a measure-theoretic description. This discussion follows Hildenbrand [21] closely. Section 5 describes a type of convergence for sequences of economies and Section 6 illustrates these concepts by applying them to the model of Debreu and Scarf [15] and to a similar model of Drèze, Gepts and Gabszewicz [16]. Section 7 presents the basic results of the paper on the continuity properties of the core correspondence and uses these results to demonstrate that Scarf's proof [29] of the nonemptiness of the core can be extended to infinite economies with a finite number of types of traders. In Section 8 it is demonstrated that an allocation in the core of an "approximately competitive" economy is approximately a Walras allocation. Section 9 discusses some of the shortcomings of this paper. Finally, there is an Appendix which gives an example of a finite economy where there is an allocation in the core which assigns different consumption plans to consumers of the same type. This example has a connection to the work of Drèze, Gepts and Gabszewicz [16].

2. Description of economic agents.

The commodity space S^1 is an N_0 -dimensional Euclidean space. The economic characteristics of any economic agent a consist of

- (i) his endowment of resources, $\omega(a)$;
- (ii) his preferences, which are represented by a subset $P(a)$ of $S \times S$ (that is, a prefers the vector x at least as much as the vector z if and only if $(x,z) \in P(a)$);
- (iii) his productive technology, which is represented by a subset $Y(a)$ of S .

A "pure producer" a can be included within this framework by giving him initial resources $\omega(a) = 0$ and giving him preferences $P(a) = \{ (0,0) \}$. Thus he is treated as a fictitious consumer. Similarly a "pure consumer" a would have no productive ability: $Y(a) = \{0\}$. The value deriving from letting economic agents be both consumers and producers is explained in some detail in [9], pages 14-17.

We shall assume that the characteristics $(\omega(a), P(a), Y(a))$ of any agent satisfy the following assumptions:

- (P.1) Feasibility of resource endowment: $(\omega(a), \omega(a)) \in P(a)$.
- (P.2) Reflexive preferences: $(x,y) \in P(a)$ implies that $(x,x) \in P(a)$ and $(y,y) \in P(a)$.
- (P.3) Continuity of preferences: $P(a)$ is closed in $S \times S$.

¹The zero element of S will be denoted 0 and the Euclidean norm on S is denoted $\| \cdot \|$. For any subset K of S and for any positive scalar η , $B_\eta(K)$ is the set $\{x \in S: \text{there exists } y \in K \text{ with } \|x-y\| < \eta\}$; that is, $B_\eta(K)$ is an " η -ball" around K .

$L(K)$ will denote the smallest affine manifold containing K ; $ri(K)$ will denote the interior of K with respect to the relative topology on $L(K)$; $int(K)$ will indicate the interior of K with respect to the usual topology on S and \bar{K} is the closure (in S) of K .

We shall define the addition of two subsets T and T' of S by $T + T' = \{x \in S: x = y + z \text{ for some } y \text{ in } T \text{ and } z \text{ in } T'\}$. Then for any z in S , $T + z$ means $T + \{z\}$. A correspondence Z from a set A to S is a mapping which assigns to each a in A a subset $Z(a)$ of S . If Z is a correspondence from A to S and if f is a function from A to S , then $Z + f$ is the correspondence mapping any point a in A into $Z(a) + f(a)$.

(Y.1) Possibility of not producing: $0 \in Y(a)$.

(Y.2) $Y(a)$ is a closed subset of S .

We remark that (P.1) means that $P(a)$ is not empty. (P.2) implies that the consumption possibilities set $X(a)$ for agent a can be represented

$$X(a) = \text{proj}_S P(a)$$

where proj_S is the mapping projecting $S \times S$ onto its first coordinate space.

Thus the description of $X(a)$ is implicit in the specification of $P(a)$.

Let \mathcal{P} be the set of nonempty, closed subsets P of $S \times S$ such that $(x,y) \in P$ implies $(x,x) \in P$ and $(y,y) \in P$. Let \mathcal{Y} be the set of closed subsets Y of S which contain 0 . Define the set

$$C = \{(\omega, P, Y) \in S \times \mathcal{P} \times \mathcal{Y} : (\omega, \omega) \in P\}.$$

C is the set of all possible characteristics of economic agents. If $I = [0,1]$ is the closed unit interval of the real line, let

$$A = C \times I \subset S \times \mathcal{P} \times \mathcal{Y} \times I.$$

A is interpreted as the set of all possible economic agents. If $a = (c,i) \in C \times I$, then the coordinate c of a gives a 's economic characteristics and the component i is an index serving to distinguish different agents of the same type.

We shall assume that the sets \mathcal{P} and \mathcal{Y} have the corresponding Hausdorff metric topologies² and that C and A have the corresponding relative topologies considered as subsets of $S \times \mathcal{P} \times \mathcal{Y}$ and, respectively, $S \times \mathcal{P} \times \mathcal{Y} \times I$ with the corresponding product (metric) topologies. (given that S and I have the usual metric topologies).³ \mathcal{A} will denote the Borel subsets of A (i.e. the

²For a discussion of the Hausdorff topology on the set of closed, nonempty subsets of a metric space, see [4] or [21].

³It is clear that C (resp. A) is a closed subset of $S \times \mathcal{P} \times \mathcal{Y}$ (resp. $S \times \mathcal{P} \times \mathcal{Y} \times I$).

smallest σ -field of subsets of A containing all the closed subsets of A). Elements of \mathcal{A} may be referred to as coalitions. This particular σ -field on A is chosen because of its compatibility with the topology on the set of probability measures on \mathcal{A} which is specified in Section 5 below.

Letting proj_L be a projection mapping with range space L , we define four functions on the space A :

$$\begin{aligned} \text{resources allocation} \quad \omega : a &\longmapsto \omega(a) = \text{proj}_S \text{proj}_C (a) \\ \text{preference correspondence} \quad P : a &\longmapsto P(a) = \text{proj}_C \text{proj}_C (a) \\ \text{consumption correspondence} \quad X : a &\longmapsto X(a) = \text{proj}_S P(a) \\ \text{production correspondence} \quad Y : a &\longmapsto Y(a) = \text{proj}_Y \text{proj}_C (a) . \end{aligned}$$

For any fixed value c in C , the images $\omega((c,i))$, $P((c,i))$, $X((c,i))$ and $Y((c,i))$ are constant as i ranges over I . Thus the proofs by Hildenbrand [21], pages 9-11 of the following statements remain valid:

Proposition 1 $\{(a,x,z) \in A \times S \times S : (x,z) \in P(a)\}$ is closed in $A \times S \times S$ (i.e., P has a closed graph) and P is L S C .⁴

Proposition 2 The consumption correspondence X and the production correspondence Y are L S C and have closed graphs. In particular, for each a in A , $X(a)$ and $Y(a)$ are closed.

⁴A correspondence Z from one topological space A to another, S , is lower semi-continuous (L S C) if for every open set G in S , the set $\{a \in A : Z(a) \cap G \neq \emptyset\}$ is open in A . Z is upper semi-continuous (U S C) if for any open set G , the set $\{a \in A : Z(a) \subset G\}$ is open in A . These concepts are developed in [4], for example.

Proposition 3 If p is a vector in S , then the functions $s_{p,X}$ and $s_{p,Y}$ defined by

$$s_{p,X}(a) = \sup p \cdot X(a), \quad a \in A$$

$$s_{p,Y}(a) = \sup p \cdot Y(a)$$

are continuous from A to $R \cup \{+\infty\}$.

We note that the function ω from A to S is continuous, also.

3. Specification of economies.

The space A represents all possible economic agents. An economy can be represented by a subset \mathcal{E} of A consisting of those agents comprising the economy. Such a representation is arbitrary to some degree. For example, if e is a one-to-one mapping from A to A such that $\text{proj}_C(e(a)) = \text{proj}_C(a)$, then an economy could just as well be represented by $e(\mathcal{E})$ as by \mathcal{E} . This simply amounts to the fact that the index i of an agent $a = (c,i)$ is arbitrary. However, it will be clear that the results obtained below do not depend on which representation is chosen.

For any finite economy \mathcal{E} , let $\chi_{\mathcal{E}}$ be the characteristic function of \mathcal{E} ; that is,

$$\chi_{\mathcal{E}}(a) = \begin{cases} 1 & a \in \mathcal{E} \\ 0 & a \notin \mathcal{E} \end{cases}.$$

⁵For any vector p in S , $p(\cdot)$ will denote the function whose value at x in S is $p \cdot x$, the inner product of p and x . If $K \subset S$, then

$$\sup p \cdot K = \sup \{p \cdot x : x \in K\}$$

$$\inf p \cdot K = - \sup (-p) \cdot K.$$

Define a probability measure¹ on the Borel sets \mathcal{A} by

$$\mu_{\mathcal{E}}^d(F) = \frac{1}{|\mathcal{E}|} \sum_{a \in F} \chi_{\mathcal{E}}(a), \quad F \in \mathcal{A},$$

where $|\mathcal{E}|$ is the number of elements of \mathcal{E} . Intuitively, $\mu_{\mathcal{E}}^d(F)$ is the proportion of the agents of economy \mathcal{E} which belong to coalition F .

If \mathcal{E} is a singleton $\{a\}$, then the corresponding measure $\mu_{\mathcal{E}}^d$ is known as a Dirac measure concentrated at a . We see that for an arbitrary finite \mathcal{E} , $\mu_{\mathcal{E}}^d$ is a convex combination of Dirac measures where each of the component Dirac measures $\mu_{\mathcal{C}}^d$ receives equal weight. Such a measure will be called a uniform discrete measure.

The reverse procedure is also possible: for any finite collection $\mu_{\{a_1\}}, \dots, \mu_{\{a_n\}}$ of Dirac measures concentrated at, respectively a_1, \dots, a_n , the measure μ defined by

$$\mu(F) = \frac{1}{n} \sum_{i=1}^n \mu_{\{a_i\}}(F)$$

is equal to the measure $\mu_{\mathcal{E}_\mu^d}^d$ where $\mathcal{E}_\mu^d = \{a_1, \dots, a_n\}$. In particular, $\mathcal{E}_\mu^d = \text{supp } \mu$.² Thus we have established a one-to-one relation between finite economies and uniform discrete measures.

¹In general, a measure μ is a real-valued function from \mathcal{A} to \mathbb{R} such that if $\{E_n\}$ is a countable collection of Borel sets satisfying $E_n \cap E_m = \emptyset$ if $n \neq m$, then $\mu(\bigcup E_n) = \sum \mu(E_n)$. A probability measure is a measure μ with values in $[0,1]$ and with $\mu(A) = 1$.

²For a general measure μ , the support of μ or $\text{supp } \mu$ is the smallest closed subset F of A such that $\mu(F) = 1$. In this paper we shall only consider measures for which $\text{supp } \mu$ exists. This is guaranteed by assumption (E.1) below.

We remark that for any probability measure μ on \mathcal{A} we can define the marginal of μ on \mathcal{C} , μ^m , by

$$\mu^m(E) = \mu(E \times I), \quad E \subset \mathcal{C} \text{ and } E \in \mathcal{A}.$$

For any $E \subset \mathcal{C}$, $\mu^m(E)$ is the proportion of the agents in the economy represented by μ which have the characteristics in E . μ^m is free of the arbitrary element mentioned earlier. Further, μ^m is (at least conceptually) empirically observable.

The initial resources of any coalition F are given by

$$(1) \quad |\mathcal{E}| \int_F \omega \, d\mu_{\mathcal{E}}^d = \begin{cases} \sum_{a \in \mathcal{E} \cap F} \omega(a) & \mathcal{E} \cap F \neq \emptyset \\ 0 & \mathcal{E} \cap F = \emptyset \end{cases}$$

Similarly, the consumption possibilities of F are

$$(2) \quad |\mathcal{E}| \int_F X \, d\mu_{\mathcal{E}}^d = \begin{cases} \sum_{a \in \mathcal{E} \cap F} X(a) & \mathcal{E} \cap F \neq \emptyset \\ \{0\} & \mathcal{E} \cap F = \emptyset \end{cases}$$

and the production possibilities for F are

$$(3) \quad |\mathcal{E}| \int_F Y \, d\mu_{\mathcal{E}}^d = \begin{cases} \sum_{a \in \mathcal{E} \cap F} Y(a) & \mathcal{E} \cap F \neq \emptyset \\ \{0\} & \mathcal{E} \cap F = \emptyset \end{cases}.$$

³For any correspondence $Z: A \rightarrow S$ (i.e. $Z(a)$ is a subset of S for each a in A) we follow Aumann [1] in defining for each F in \mathcal{A} :

where $\int_F Z \, d\mu = \{z \in S: \text{there exists } f \in \mathcal{L}_{Z, \mu} \text{ with } z = \int_F f \, d\mu\}$,

$$\mathcal{L}_{Z, \mu} = \{\mu\text{-integrable functions } f: A \rightarrow S \text{ such that } f(a) \in Z(a) \text{ } \mu\text{-almost everywhere}\}.$$

When μ is a discrete measure, this reduces to

$$\mathcal{L}_{Z, \mu} = \{\mathcal{A}\text{-measurable functions } f: A \rightarrow S \text{ such that } f(a) \in Z(a) \text{ for } a \in \text{supp } \mu\}.$$

assuming there are no interactions among the technologies available to separate producers.

The scale factor $|\mathcal{E}|$ on the left-hand sides of equations (1), (2) and (3) could be avoided by defining a new measure $\tilde{\mu}_{\mathcal{E}}^d$:

$$\tilde{\mu}_{\mathcal{E}}^d(F) = |\mathcal{E}| \mu_{\mathcal{E}}^d(F).$$

The resources, consumption possibilities and production possibilities of F would then be $\int_F \omega d\tilde{\mu}_{\mathcal{E}}^d$, $\int_F X d\tilde{\mu}_{\mathcal{E}}^d$ and $\int_F Y d\tilde{\mu}_{\mathcal{E}}^d$. This procedure is unsatisfactory, however, because we want to study limits of sequences of economies \mathcal{E}_n where $|\mathcal{E}_n| \rightarrow \infty$. In this case, the measures $\tilde{\mu}_{\mathcal{E}_n}^d$ would have no limit measure.

An alternative procedure is to work with "averages" for any economy $\mu_{\mathcal{E}}^d$:

$$\int_F \omega d\mu_{\mathcal{E}}^d, \int_F X d\mu_{\mathcal{E}}^d \text{ and } \int_F Y d\mu_{\mathcal{E}}^d.$$

The "total" quantities can always be recovered by multiplying these average quantities by the scalar $|\mathcal{E}| = |\text{supp } \mu_{\mathcal{E}}^d|$.

Another solution is to associate with \mathcal{E} a measure $\mu_{\mathcal{E}}^h$ which is the uniform discrete measure having support equal to the set $|\mathcal{E}| \cdot \mathcal{E}$; ⁴ that is,

$$\mu_{\mathcal{E}}^h(F) = \frac{1}{|\mathcal{E}|} \sum_{a \in F} \chi_{|\mathcal{E}| \cdot \mathcal{E}}(a).$$

Thus $\mu_{\mathcal{E}}^h\left(\frac{1}{|\mathcal{E}|} F\right) = \mu_{\mathcal{E}}^d(F)$. The total resources in economy \mathcal{E} of coalition F are then given by $\int_F \omega d\mu_{\mathcal{E}}^h$, the consumption possibilities by $\int_F X d\mu_{\mathcal{E}}^h$ and the production technology by $\int_F Y d\mu_{\mathcal{E}}^h$.

⁴ $|\mathcal{E}| \cdot \mathcal{E} = \{a \in C \times R: a = |\mathcal{E}| \cdot a' \text{ for some } a' \in \mathcal{E}\}$ is a subset of A if and only if for each a in \mathcal{E} , $\text{proj}_I a \leq \frac{1}{|\mathcal{E}|}$. We shall assume that such a representation is chosen for \mathcal{E} .

Conversely, to any uniform discrete measure μ there corresponds an economy $\mathcal{E}_\mu^h = \frac{\text{supp } \mu}{|\text{supp } \mu|}$. Thus we have established a second one-to-one correspondence between finite economies and uniform discrete measures. For any such measure μ , we shall call the economy \mathcal{E}_μ^d the economy directly represented by μ and the economy \mathcal{E}_μ^h will be called the economy homothetically represented by μ . The economic interpretations of these two representations will be described in Section 6 below. Suffice it to say now that the possibility of assigning two interpretations to results achieved for a measure μ substantially enhances the power of the methods developed here.

So far we have demonstrated that every uniform discrete measure has two economic interpretations. However, we want to study what happens when the size of an economy becomes arbitrarily large. To do this we must also consider measures which have infinite supports. Such measures have no direct interpretation as economies. They can only have significance in so far as they are approximated by finite economies. It is the aim of this paper to show that such "infinite economies" are useful in discovering results about finite economies.

We now list the assumptions which will be made at various times below on the characteristics of agents a :

(P.4) Convexity. $P(a)$ is convex. if $x \neq z$

(P.4') Strong Convexity. If $(x,z) \in P(a)$, and if $t \in (0,1)$, then $(tx + (1-t)z, z) \in P(a)$ but $(z, tx + (1-t)z) \notin P(a)$.

(P.5) Local nonsatiation. For every (x,x) in $P(a)$ and for every neighborhood of (x,x) there exists (y,x) in that neighborhood such that $(y,x) \in P(a)$ and $(x,y) \notin P(a)$.

(P.6) Completeness. $(x,x) \in P(a)$ and $(y,y) \in P(a)$ imply $(x,y) \in P(a)$ or $(y,x) \in P(a)$.

(P.7) Transitivity. $(x,y) \in P(a)$ and $(y,z) \in P(a)$ implies $(x,z) \in P(a)$.

These assumptions can easily be restated in terms of more conventional notation. If $a \in A$ and if x and y are any vectors in $X(a)$, then

$x \underset{=a}{\geq} y$ means $(x,y) \in P(a)$,

$x \underset{=a}{=} y$ means $x \underset{=a}{\geq} y$ and $y \underset{=a}{\geq} x$,

$x \underset{>a}{\geq} y$ means $x \underset{=a}{\geq} y$ and not $y \underset{=a}{\geq} x$.

(P.4) implies that for agent a , the consumption possibilities set $X(a)$ is convex and for every z in $X(a)$, the set $\{x \in X(a): x \underset{=a}{\geq} z\}$ is convex.

(P.4), (P.6) and (P.7) imply that for every z in $X(a)$, $\{x \in X(a): x \underset{>a}{\geq} z\}$ is convex.

The following additional assumption on the production technology of agent a will be used below:

(Y.3) $Y(a)$ is convex.

In the sequel, when we say that some of the preceding assumptions hold for an economy μ , we shall mean that they hold for every a in $\text{supp } \mu$.⁵ It is convenient to state several additional assumptions as requirements to be met by economies rather than agents:

(E.1) There exists a compact set K in A such that $\mu(K) = 1$.

(E.2) Similarity of consumption technologies. The correspondence X factors $\text{supp } \mu$ finitely, linearly and measureably.⁶

⁵It can be shown that if the assumptions (P.4), (P.6) and (Y.3) hold for μ -almost every a , then they hold for every a in $\text{supp } \mu$.

⁶This means that there are n disjoint Borel sets A_i satisfying $\bigcup_{i=1}^n A_i = \text{supp } \mu$ and there exist n linear subspaces L_i of S^1 such that $a \in A_i$ implies that $L(X(a))$ equals L_i .

(E.3) Lower bound for consumption. There exists a vector b_μ in S such that a in $\text{supp } \mu$ implies $X(a) \subset \Omega + b_\mu$.⁷

(E.4) There exist two Borel set E_1 and E_2 such that $E_1 \cup E_2 = \text{supp } \mu$ and $a \in E_1$, implies $Y(a)$ is compact and there exists a convex cone $K \subset S$ such that $Y(a) = K$ for a in E_2 .

(E.5) There is a closed, convex, pointed⁸ cone K' such that $K' \cap \Omega = \{0\}$ and $Y(a) \subset K'$ for every a in $\text{supp } \mu$.

(E.6) $L(X(a)) = L(Y(a))$ for μ -almost every a .

(E.7) For μ -almost every a , $[\text{ri } X(a)] \cap [(Y(a) + \omega(a))] \neq \emptyset$.

(E.7') For μ -almost every a , $[\text{ri } X(a)] \cap [\text{ri } (Y(a) + \omega(a))] \neq \emptyset$.

(E.8) Similarity of preferences and positive initial endowments. There exists a nonempty subset D_μ of $\{1, \dots, N_0\}$ such that for every a in $\text{supp } \mu$,

(i) for every $\rho > 0$ and $k \in D_\mu$ and f in $\mathcal{L}_{X, \mu}$, $f(a) + k(\rho)$ ⁹
 $>_a f(a)$,

(ii) there exists $\rho > 0$ and $k \in D_\mu$ such that $\omega(a) - k(\rho) \in X(a)$.

Assumption (E.1) is satisfied for any finite (uniform discrete) economy μ .

Its economic significance in general is not clear. It means that μ is tight;

⁷ Ω is the closed positive orthant in S .

⁸ K' is pointed if $K' \cap (-K') = \{0\}$.

⁹ $k(\rho)$ is the vector in S all of whose coordinates equal 0 except for the k^{th} which equals ρ .

that is, for every $\rho > 0$ and $E \in \mathcal{A}$ there exists a compact subset K of E such that $\mu(E \setminus K) < \rho$. The set of probability measures on (A, \mathcal{A}) satisfying (E.1) is denoted by \mathbb{E} .

Assumption (E.2) means, loosely, that if one distinguishes among agents according to the directions in which they can alter their consumption vectors within their consumption possibility sets, then there are only a finite number of different types of agents. Although no two of the sets $X(a)$, $a \in A_i$, need be the same, nevertheless there is a subspace L_i of S such that $L_i = L(X(a))$. This means that if $x \in \text{ri}(X(a))$ and $z \in L_i$, then, for some $\rho > 0$, $x + \rho z \in X(a)$ also.

Assumption (E.3) means that for every a in $\text{supp } \mu$ and for every x in $X(a)$, $x \geq b_\mu$ where \geq is the usual, coordinate-wise vector order on S . This assumption has a natural economic interpretation (see [11], page 53).

Assumption (E.4) is a technical assumption which, together with (Y.1) - (Y.3) implies that for any coalition E ,

$$\int_E Y \, d\mu = \int_E^c Y \, d\mu$$

where

$$\int_E^c Y \, d\mu = \left\{ z \in S : z = \int_E f \, d\mu, \text{ some } f \in \mathcal{L}_{Y, \mu}^c \right\}$$

for

$$\mathcal{L}_{Y, \mu}^c = \{ f \in \mathcal{L}_{Y, \mu} : f \text{ is continuous} \} \quad \text{9a}$$

Assumption (E.5) is a strengthened version of the irreversibility and of the no-free-goods conditions; that is, for a in $\text{supp } \mu$ (E.5) implies that $Y(a) \cap \Omega = \{0\}$ (no positive outputs without some negative inputs) and

^{9a}This result is proven in Theorem 3 of [7] and is used in Theorem 2 below.

$Y(a) \cap (-Y(a)) = \{0\}$ (since K is pointed). This assumption, together with (E.3), is used in Theorem 3 below to conclude that the core of a certain type of economy is bounded.

Assumption (E.6) appears to be very strong because it suggests that there can be no intermediate goods which are used in production but which cannot be consumed. However, perhaps it is not unreasonable to permit the "consumption" of such goods together with other goods in the same proportions as they enter $Y(a)$. A steel ingot can be "consumed," i.e., stored, if it is combined with appropriate amounts of other commodities such as space and perhaps, transportation services. Of course, such "consumption" need not increase that agent's utility and may even decrease it.¹⁰

For an agent a in $\text{supp } \mu$ (outside a μ -null set) for which $Y(a) = \{0\}$, (E.7) and (E.7') are equivalent to

$$(4) \quad \omega(a) \in \text{ri}(X(a)).$$

In general, (4) is neither sufficient nor necessary for (E.7) or (E.7'). Assumptions (E.7') and (E.8) are essentially sufficient conditions for a μ -quasi-competitive allocation to be a μ -Walras allocation (see Section 4 below and see Section 4 in [13] and Theorem 3 in [9]).

4. The results of trade and production.

The outcome of the trade and production activities in an economy μ is a specification of what each agent in the economy receives to consume and a specification of what production plan each agent carries out. The distribution of consumption bundles will be represented by a μ -allocation which is a function f in $\mathcal{L}_{X,\mu}$. The requirement that $f(a) \in X(a)$ μ -almost everywhere is equivalent to requiring that the consumption plan allocated by f to μ -almost any

¹⁰This discussion suggests that the consumption and production activities of an agent might be better treated as joint rather than distinct. This idea has recently been suggested, for different reasons, by Lancaster [25] and Becker [3].

consumer in $\text{supp } \mu$ be feasible for that consumer. The requirement that f be μ -integrable simply means that it be possible to sum the consumption plans of any coalition of consumers.¹

Given a measure μ with finite support and given a μ -allocation f , any agent a in $\text{supp } \mu$, which is the economy directly represented by μ , receives the consumption plan²

$$f(a) = |\text{supp } \mu|^{-1} \cdot \int_{\{a\}} f \, d\mu.$$

and any coalition F receives

$$|\text{supp } \mu|^{-1} \cdot \int_F f \, d\mu.$$

Under the homothetic interpretation, any agent in $|\text{supp } \mu|^{-1} \text{supp } \mu$ receives

$$\mu(\{a\}) \cdot f(a) = \int_{\{a\}} f \, d\mu$$

and any coalition F receives

$$\int_F f \, d\mu.$$

The distribution of production among the agents of an economy μ is specified by a μ -production assignment h which is a function in $\mathcal{L}_{Y,\mu}$. For a uniform discrete measure μ and for any coalition F , $\int_F h \, d\mu$ is the average (over the number of agents in $\text{supp } \mu$) production plan for coalition F in the economy directly represented by μ and is the total production plan for F in the economy homothetically represented by μ .

The only μ -allocations f of interest are those for which

$$\int_A f \, d\mu \in \int_A (\omega + Y) \, d\mu; \quad ^3$$

¹The requirement of integrability may be interpreted as an accountant's restriction.

²We recall that $|\text{supp } \mu|$ is the cardinality of the set $\text{supp } \mu$.

³ $\omega + Y$ is the correspondence mapping the point a into the set $\omega(a) + Y(a)$.

that is, the commodity vector $\int_A f \, d\mu$ of consumption for a whole economy⁴ must belong to the set $\int_A (\omega + Y) \, d\mu$ of commodity vectors which the economy can produce from its resources $\int_A \omega \, d\mu$. It is useful to generalize this notion of feasibility slightly by defining for any coalition F :

$$\mathcal{F}_\mu(F) = \{f \in \mathcal{L}_{X,\mu} : \int_F f \, d\mu \in \int_F (\omega + Y) \, d\mu\} .$$

$\mathcal{F}_\mu(F)$ is the set of allocations which coalition F can attain from its own resources $\int_F \omega \, d\mu$ and technology $\int_F Y \, d\mu$ independently of what coalition $A \setminus F$ ⁵ does.

We shall specify criteria for choosing among the feasible outcomes for an economy μ by considering three types of efficiency: competitive efficiency, Pareto efficiency and core efficiency. The idea of competitive efficiency is based on the use of prices in various ways by the agents of the economy. Given a price vector p in S , an agent a in A can secure a profit arbitrarily close to the number in $R \cup \{+\infty\}$ defined by

$$\pi(p,a) = \sup p \cdot Y(a) .$$

A function π , called the profit function, will assign to any element (p,a) of $S \times A$ the value $\pi(p,a)$. Similarly, a wealth function⁶ from $S \times A$ to $R \cup \{+\infty\}$ may be defined by

$$w(p,a) = p \cdot \omega(a) + \pi(p,a).^7$$

⁴This sentence is written with the economy homothetically represented by μ in mind. The corresponding statement for the alternative interpretation is obvious.

⁵ $A \setminus E = \{a \in A : a \notin E\}$.

⁶We use the term wealth rather than income because we envisage the commodity space to include all commodities available at any time in the agent's lifetime.

⁷The functions $\pi(p, \cdot)$ and $w(p, \cdot)$ are continuous by Proposition 3 above.

We remark again that the profit (resp., wealth) of any agent a is $\pi(p,a) = |\text{supp } \mu| \cdot \mu(\{a\}) \cdot \pi(p,a)$ (resp. $|\text{supp } \mu| \cdot \mu(\{a\}) \cdot w(p,a)$) under the direct interpretation of μ and is $\mu(\{a\}) \cdot \pi(p,a)$ (resp., $\mu(\{a\}) \cdot w(p,a)$) under the homothetic interpretation of μ .

The budget correspondence β from $S \times A$ to S is defined by

$$\beta(p,a) = \{x \in X(a) : p \cdot x \leq w(p,a)\} .$$

The demand correspondence, β^0 from $S \times A$ to S is defined by

$$\beta^0(p,a) = \{x \in \beta(p,a) : \text{there exists no } z \in \beta(p,a) \text{ with } z \succ_a x\} .$$

$\beta^0(p,a)$ is the set of feasible consumption plans which "maximize a 's utility" subject to his budget constraint.

A μ -allocation f is a competitive or Walras allocation for μ if there exists nonzero p in S and a μ -production assignment h such that

$$\int_A f \, d\mu = \int_A (\omega + h) \, d\mu \text{ and for } \mu\text{-almost every } a \text{ in } A :$$

$$(i) \quad f(a) \in \beta^0(p,a) ,$$

$$(ii) \quad p \cdot h(a) = \pi(p,a) .$$

The mathematical techniques which are currently available to find conditions under which some given allocation is a Walras allocation consist of first exhibiting conditions under which the allocation satisfies a slightly weaker type of competitive efficiency and then of finding conditions under which an allocation satisfying this weaker type of competitive efficiency is in fact a Walras allocation. This two stage procedure was formalized by Debreu [13] who defined the following type of competitive efficiency: An allocation f for μ is quasi-competitive for μ if there exists a nonzero p in S and a μ -production assignment h with

$$\int_A f \, d\mu = \int_A (\omega + h) \, d\mu \text{ and for } \mu\text{-almost every agent } a :$$

- (i) $f(a) \in \beta(p, a)$,
- (ii) Either $f(a) \in \beta^0(p, a)$
or else $w(p, a) = \inf p \cdot X(a)$,
- (iii) $p \cdot h(a) = \pi(p, a)$.

The technique for demonstrating that a quasi-competitive allocation f is, in fact, a Walras allocation is to show that p can be chosen so that either $w(p, a) = \inf p \cdot X(a)$ occurs only for a μ -null set of a 's or else that $f(a)$ is maximal for \geq_a in $\beta(p, a)$ for μ -almost every a in A even if $w(p, a) = \inf p \cdot X(a)$ for a nonnull collection of agents a .

We remark that the reason for calling such allocations competitive is that such an allocation can be achieved if the appropriate price system is announced and if each agent in the economy accepts those prices as fixed in determining his consumption and production plans. It is possible to define a type of efficiency which is freer of institutional connotations than competitive efficiency. The idea underlying this concept is that an allocation f may be "inefficient" because a coalition of economic agents may be able to achieve for itself with its own resources and technology an allocation g which its members prefer to f . More precisely, we say that a μ -nonnull coalition E in \mathcal{A} blocks for μ a μ -allocation f if there exists another μ -allocation g such that

- (i) $g(a) >_a f(a)$ for μ -almost every a in E
- (ii) $\int_E g \, d\mu \in \int_E (\omega + Y) \, d\mu$.

It is convenient also to define a weaker type of blocking: a μ -nonnull coalition E in \mathcal{A} weakly blocks for μ a μ -allocation f if there exists a μ -allocation g satisfying condition (ii) above and satisfying:

(i') there exist two subcoalitions E_1 and E_2 of E such that

$$\begin{aligned} E \setminus (E_1 \cup E_2) & \text{ is } \mu\text{-null,} \\ g(a) & \underset{a}{>} f(a) \quad \text{for } a \text{ in } E_1 \text{ and } \mu(E_1) > 0, \\ g(a) & \underset{a}{=} f(a) \quad \text{for } a \text{ in } E_2. \end{aligned}$$

It is clear that if E blocks an allocation f for μ then E also weakly blocks f .⁸

An allocation f in $\mathcal{F}_\mu(A)$ is a Pareto allocation for μ if the coalition A cannot weakly block f for μ . An allocation f in $\mathcal{F}_\mu(A)$ is in the core for μ or, equivalently, is an Edgeworth allocation for μ if no coalition can block f for μ . Finally, an allocation f in $\mathcal{F}_\mu(A)$ is in the strong core for μ if no coalition can weakly block f for μ .

An atom of an economy μ is a coalition E satisfying

- (i) $\mu(E) > 0$
- (ii) if $F \subset E$, then $\mu(F) = 0$ or $\mu(F) = \mu(E)$.

It can be shown that if E is an atom for an economy μ satisfying (E.1), then there exists $a \in E$ such that $\mu(\{a\}) = \mu(E)$ and $\mu(E \setminus \{a\}) = 0$.⁹

Thus if μ has an atom then there exists an individual agent a with $\mu(\{a\}) > 0$ and

$\mathcal{F}_\mu(\{a\}) \neq \emptyset$ and hence a has blocking power. Thus only in nonatomic economies μ (i.e. measures μ with no atoms) do individual agents have no influence on the choice of outcomes of trade and production. It is for this reason that "perfect competition" only makes sense for nonatomic economies.

Henceforth the term "perfectly competitive" will be used interchangeably with the terms "nonatomic" and "diffuse."¹⁰

⁸We have defined blocking for all elements of \mathcal{A} . It is clear, however, that only subcoalitions of $\text{supp } \mu$ are actually significant.

⁹See Exercise 9a), page 63 in Bourbaki [6].

¹⁰ μ is diffuse if $\mu(\{a\}) = 0$ for every $a \in A$. By the previous footnote, a measure μ satisfying (E.1) is nonatomic if and only if μ is diffuse.

We shall adopt the following notation:

- $\mathcal{W}(\mu)$ = set of Walras allocations of μ ,
- $\mathcal{Q}(\mu)$ = set of quasi-competitive allocations for μ ,
- $\mathcal{P}(\mu)$ = set of Pareto allocations for μ ,
- $\mathcal{C}(\mu)$ = the core for μ ,
- $\mathcal{C}^s(\mu)$ = the strong core for μ .

We note that each of these symbols can be used to define a correspondence from \mathbf{E} to the set of measurable, S -valued functions on (A, \mathcal{A}) ; that is, the symbol \mathcal{C} , for example, will represent the correspondence which maps the element μ of \mathbf{E} into $\mathcal{C}(\mu)$. The basic purpose of this paper is to study the continuity properties of this correspondence. The next section explores the problem of finding a suitable topology for \mathbf{E} .

5. The convergence of a sequence of economies.

The aim of this Section is to describe a "natural" topology on the set of economies. This is equivalent to having a criterion for determining when two distinct economies are close to each other. It is intuitively appealing to consider two economies μ_1 and μ_2 to be close if the proportions in which their members ($\text{supp } \mu_1$ and $\text{supp } \mu_2$) are distributed in the subcoalitions of A are similar, that is, if $\mu_1(E)$ is close to $\mu_2(E)$ for each E in \mathcal{A} . We would then say that a sequence $\{\mu_n\}_{n=1}^{\infty}$ converged to μ if

$$(1) \quad \mu_n(E) \rightarrow \mu(E), \quad \text{for each } E \text{ in } \mathcal{A}.$$

Suppose λ is a uniform discrete measure with support equal to $\{a_1, \dots, a_n\}$ and suppose for each $i=1, \dots, n$, a_{im} is an element of A satisfying $a_{im} \neq a_i$ but $a_{im} \rightarrow a_i$ as $m \rightarrow \infty$. If λ_m is the uniform discrete measure on

$\{a_{im}, i=1, \dots, n\}$, then condition (1) is not satisfied by $\lambda, \lambda_m, m=1, 2, \dots$ and for $E = \{a_i\}$, some $i=1, \dots, m$. This suggests that condition (1) is too restrictive.

The preceding remarks suggest that we want to make use of the topology already on A in defining a topology on \mathbb{E} or on \mathcal{M}_P , the set of probability measures on (A, \mathcal{Q}) . This is accomplished by the weak topology which has a rich mathematical structure already developed and which yields a type of convergence weaker than that of (1). Convergence in the weak topology will be denoted by

$$\mu_n \Rightarrow \mu$$

and is characterized by

$$(2) \quad \mu_n(Q) \rightarrow \mu(Q) \text{ for every } \mu\text{-boundaryless coalition } Q.^1$$

The significance of (2) will be demonstrated by showing that it is satisfied by the economies $\lambda, \lambda_m, m=1, 2, \dots$ described immediately below (1). It is clear that for any $\delta > 0$ there exists M_δ such that $m \geq M_\delta$ implies that for every $a_i \in \text{supp } \mu$, there exists a_{mi} within δ of a_i . If $\delta \leq \frac{1}{2} \min_{i \neq j} d(a_i, a_j)$,^{1a} then each δ neighborhood of an element of $\text{supp } \lambda$ contains exactly one element of $\text{supp } \lambda_m$ for $m \geq M_\delta$. Suppose Q is a λ -boundaryless subset of A and $\lambda(Q) > 0$. Then

$$[\text{supp } \lambda] \cap Q \subset \text{int}(Q).$$

Hence there exists $\delta > 0$ satisfying $\delta \leq \frac{1}{2} \min_{i \neq j} d(a_i, a_j)$ and satisfying

$$B_\delta(a) \subset Q \text{ for every } a \in [\text{supp } \lambda] \cap Q.$$

¹The set Q is μ -boundaryless if $\mu(\bar{Q} \cap (A \setminus Q)) = 0$ where \bar{Q} is the closure of Q and $A \setminus Q = \{a \in A: a \notin Q\}$. The set $\bar{Q} \cap A \setminus Q$ is the boundary of Q .

^{1a} $d(a_i, a_j)$ is the distance between a_i and a_j with respect to the metric on A .

But then $\lambda_m(Q) = \lambda(Q)$ for $m \geq M_\delta$. To prove that $\lambda(Q) = 0$ implies that eventually $\lambda_m(Q) = 0$ is equally easy.

The remainder of this Section is devoted to a discussion of the mathematical characteristics of weak convergence. The next section will present economic examples which illustrate its usefulness.

The set \mathcal{M}_P of probability measures on A is a subset of the set of continuous linear functionals on the space $C(A)^2$ which consists of the continuous bounded real-valued functions on A together with the topology of uniform convergence. Thus \mathcal{M}_P can be given the (relative) weak* topology which will henceforth be called the weak topology.³ A sequence $\{\mu_n\}$ converges to μ in this topology if and only if

$$(3) \quad \int_A f d\mu_n \rightarrow \int_A f d\mu \quad \text{for every } f \in C(A).$$

It can be verified that (2) and (3) are equivalent to each other and to each of the following conditions:⁴

$$(4) \quad \int_Q f d\mu_n \rightarrow \int_Q f d\mu \quad \text{for every } \mu\text{-boundaryless Borel set } Q \text{ and for every } f \text{ in } C(A).$$

(5) For any measurable function f continuous except on a μ -null set:

$$\mu_n(\{a \in A: f(a) \leq \alpha\}) \rightarrow \mu(\{a \in A: f(a) \leq \alpha\})$$

for every real number α at which the function defined by

²For example, see Theorem IV, 6.2, page 262 in [17].

³ \mathcal{M}_P is a complete metrizable space if A is complete and separable and \mathcal{M}_P is compact if A is compact. (Proofs of these statements can be found in Varadarajan [31]). In general, A is not compact or separable. However we shall often consider the relative topology on the set of probability measures whose supports are contained in the compact support of some given measure.

⁴Proofs of the equivalence of the conditions can be found in [5], except for (4) which is easily derived from (2) and (5).

$$D_{\mu}(\alpha) = \mu(\{a \in A: f(a) \leq \alpha\})$$

is continuous.

$$(6) \quad \limsup \mu_n(F) \leq \mu(F) \quad \text{for every closed } F \subset A \quad \text{and} \quad \mu_n(A) \rightarrow \mu(A).$$

$$(7) \quad \liminf \mu_n(G) \geq \mu(G) \quad \text{for every open } G \subset A \quad \text{and} \quad \mu_n(A) \rightarrow \mu(A).$$

It will be useful later to remark that if $\{\mu_n\}$ is a sequence of economies for which $\text{supp } \mu_n \subset \text{supp } \mu$ for every n , then $\mu_n \Rightarrow \mu$ is equivalent to:

$$(2') \quad \mu_n(Q) \rightarrow \mu(Q) \quad \text{for every Borel subset } Q \text{ of } \text{supp } \mu \text{ which is } \mu\text{-boundaryless in } \text{supp } \mu.$$

$$(3') \quad \int_A f \, d\mu_n \rightarrow \int_A f \, d\mu \quad \text{for every } f \in C(\text{supp } \mu).^5$$

The uniform discrete measures are dense in \mathbb{E} . In fact, for every μ in \mathbb{E} there exists a sequence $\{\mu_n\}_{n=1}^{\infty}$ of uniform discrete measures converging weakly to μ and such that for every n ,

$$\text{supp } \mu_n \subset \text{supp } \mu_{n+1} \subset \text{supp } \mu.^6$$

This means that every economy in \mathbb{E} is accessible by realizable (i.e. finite) economies through this type of convergence. Thus for this type of convergence, \mathbb{E} is not too large. A partial converse is also clear: if there exists a compact set K in A and a sequence of measures μ_n with $\text{supp } \mu_n \subset K$ for every n and if $\mu_n \Rightarrow$ some μ in \mathcal{M}_P , then $\mu \in \mathbb{E}$. Thus \mathbb{E} is "large enough".

If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of uniform discrete measures for which there exists a nonatomic measure μ with compact support such that

⁵See Varadarajan [31].

⁶See Theorem IV.1, page 35 in [21].

$$\begin{aligned} \text{supp } \mu_n \subset \text{supp } \mu & \quad \text{for every } n, \\ \mu_n \Rightarrow \mu & \end{aligned}$$

then the sequence $\{\mu_n\}$ will be called asymptotically perfectly competitive.⁷

If $\{\mu_n\}_{n=1}^{\infty}$ is asymptotically perfectly competitive with limit μ , then (6) implies that $\mu_n(\{a\}) \rightarrow \mu(\{a\}) = 0$ for every a in $\text{supp } \mu$. In fact, it is not difficult to show that this convergence is uniform on $\text{supp } \mu$. Thus $\frac{1}{\mu_n(\{a\})}$, $a \in \text{supp } \mu_n$ which is the number of agents in economy μ_n , becomes arbitrarily large. Further, since the initial resources function ω is continuous on the compact set $\text{supp } \mu$, it is also bounded and hence the resources $\mu_n(\{a\}) \cdot \omega(a)$ of any agent in the economy represented homothetically by μ_n approach zero uniformly. For any agent a in the economy represented directly by μ_n , his characteristics $(\omega(a), X(a), P(a), Y(a))$ remain unchanged as n increases. However, the total resources in economy μ_n are

$$\begin{aligned} & |\text{supp } \mu_n| \int_A \omega d\mu_n \rightarrow \infty \quad \text{if } \int_A \omega d\mu \neq 0, \\ \text{since } & |\text{supp } \mu_n| \rightarrow \infty \quad \text{and} \quad \int_A \omega d\mu_n \rightarrow \int_A \omega d\mu. \end{aligned}$$

Thus, unless $\int_A \omega d\mu = 0$, the size of any individual's initial resources relative to the resources of the whole economy approaches zero for any asymptotically perfectly competitive sequence of economies.

Let $\{\mu_n\}$ be an asymptotically perfectly competitive sequence satisfying $\text{supp } \mu_n \subset \text{supp } \mu_m$ for $m \geq n$. Let $\mathcal{E}_n^h = \frac{1}{|\text{supp } \mu_n|} \text{supp } \mu_n$ be the economy homothetically represented by μ_n . If $a \in \mathcal{E}_n^h$ then $\frac{|\text{supp } \mu_n|}{|\text{supp } \mu_m|} a \in \mathcal{E}_m^h$.

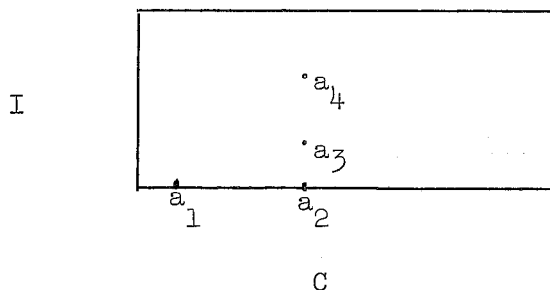
⁷Those for whom the term "perfectly competitive" has too many behavioral or normative connotations might prefer the name "asymptotically diffuse".

This makes explicit the speed with which the individual agents shrink to zero in the economies homothetically represented by the measures μ_n .

6. An example: repeated division or replication of a finite economy.

In this Section we shall demonstrate how the procedure devised by Debreu and Scarf [15] for replicating an economy and the analogous procedure devised by Drèze, Gepts and Gabszewicz [16] for dividing an economy can be interpreted as simple examples of an asymptotically perfectly competitive sequence of economies.

Let $\mathcal{E}_1 = \{a_1, \dots, a_m\} \subset A$ be a finite economy and suppose $a_i = (c_i, j_i)$ for $i = 1, \dots, m$ with $j_i < 1$ for every i and with $j_{i+1} > j_i$ holding for any i for which $c_{i+1} = c_i$. This can be represented schematically:



One replication of this economy gives an economy with two agents of the same type for each agent in \mathcal{E}_1 . This can be represented by the economy:

$$\mathcal{E}_2 = \{a_{ij}; \quad i=1, \dots, m, \quad j=1, 2\}$$

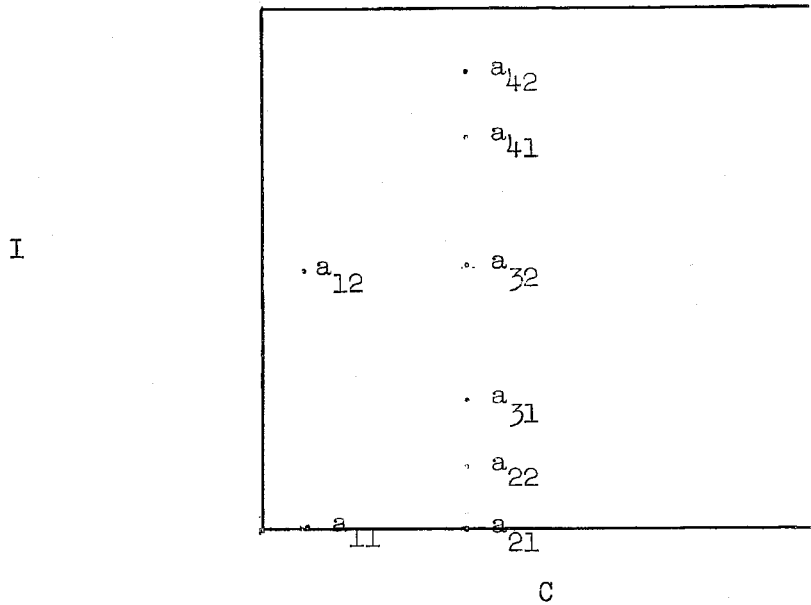
where

$$(1) \quad a_{ij} = (c_i, j_i + \frac{(j-1)}{4} (j'_{i+1} - j_i))$$

where

$$(2) \quad j'_{i+1} = \begin{cases} j_{i+1} & \text{if } c_{i+1} = c_i \\ 1 & \text{otherwise} \end{cases} .$$

This can be represented in the following diagram



In general, the economy gotten by replicating \mathcal{E}_1 $r-1$ times can be represented:

$$\mathcal{E}_r = \{a_{ij}; \quad i=1, \dots, m, \quad j=1, \dots, r\}$$

where

$$(3) \quad a_{ij} = (c_{i,j_i} + \frac{(j-1)}{2r} (j'_{i+1} - j_i)) .$$

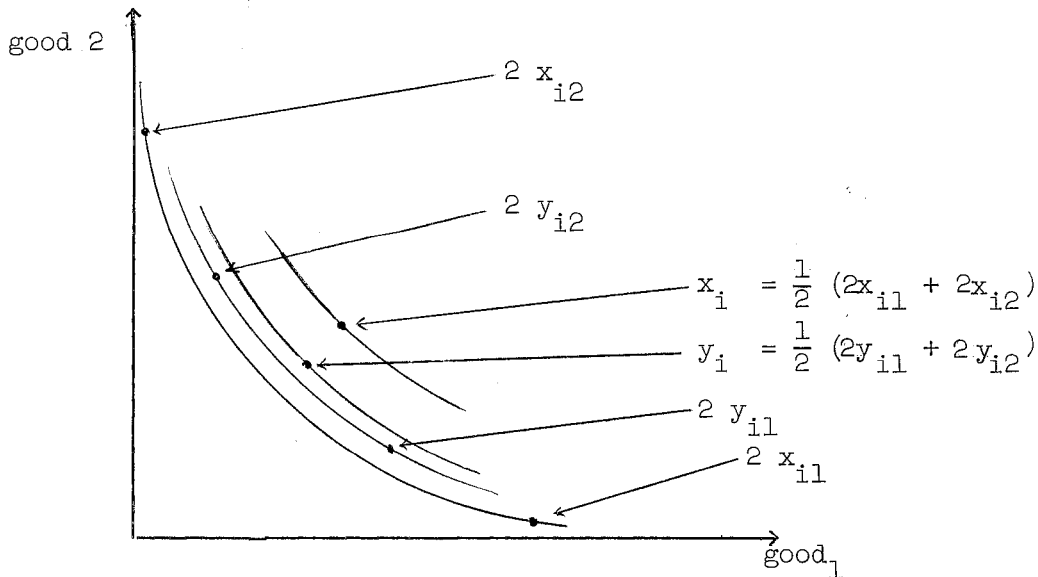
Let v_r , $r=1, \dots$ be the uniform measure with support equal to \mathcal{E}_r .

\mathcal{E}_r is the economy directly represented by v_r . The economy \mathcal{E}_r^h homothetically represented by v_r is the economy gotten by dividing economy \mathcal{E}_1^h $r-1$ times. Each agent $\tilde{a}_i = \frac{1}{m} a_i$ in \mathcal{E}_1^h is replaced in \mathcal{E}_r^h by r agents \tilde{a}_{ij} , $j=1, \dots, r$ where $\tilde{a}_{ij} = \frac{1}{r m} a_{ij}$ where a_{ij} is defined in (3). These agents are characterized by:

$$\begin{aligned} \omega(\tilde{a}_{ij}) &= \frac{1}{r} \quad \omega(\tilde{a}_i) \\ X(\tilde{a}_{ij}) &= \frac{1}{r} \quad X(\tilde{a}_i) \\ P(\tilde{a}_{ij}) &= \frac{1}{r} \quad P(\tilde{a}_i) \\ Y(\tilde{a}_{ij}) &= \frac{1}{r} \quad Y(\tilde{a}_i) . \end{aligned}$$

\mathcal{E}_r^h represents the economic situation which arises when each agent \tilde{a}_i in \mathcal{E}_1^h delegates authority to r separate "brokers" to trade and produce for him.¹ He gives each of these brokers \tilde{a}_{ij} the resources $\frac{1}{r} \omega(\tilde{a}_i)$, preferences $\frac{1}{r} P(\tilde{a}_i)$ and production technology $\frac{1}{r} Y(\tilde{a}_i)$. If agent \tilde{a}_{ij} chooses a consumption plan x_{ij} and a production vector y_{ij} , then agent \tilde{a}_i consumes $x_i \equiv \sum_{j=1}^r x_{ij}$ and produces $y_i \equiv \sum_{j=1}^r y_{ij}$. Because $r x_{ij} \in X(\tilde{a}_i)$ if $x_{ij} \in X(\tilde{a}_{ij})$ (resp., $r y_{ij} \in Y(\tilde{a}_i)$ if $y_{ij} \in Y(\tilde{a}_{ij})$) and because $X(\tilde{a}_i)$ (resp., $Y(\tilde{a}_i)$) is convex, then $x_i = \frac{1}{r} \sum_{j=1}^r r x_{ij} \in X(\tilde{a}_i)$ (resp., $y_i = \frac{1}{r} \sum_{j=1}^r r y_{ij} \in Y(\tilde{a}_i)$).

This method of aggregating the consumption plans of the agents \tilde{a}_{ij} , $j=1, \dots, r$ to get a consumption plan for \tilde{a}_i may not "preserve preferences". For example, there may be two sets of consumption vectors $\{x_{ij}, j=1, \dots, r\}$ and $\{y_{ij}, j=1, \dots, r\}$ such that $y_{ij} \succ_{\tilde{a}_{ij}} x_{ij}$, $j=1, \dots, r$ but $\sum_{j=1}^r x_{ij} \succ_{\tilde{a}_i} \sum_{j=1}^r y_{ij}$. This possibility is illustrated in the following figure:



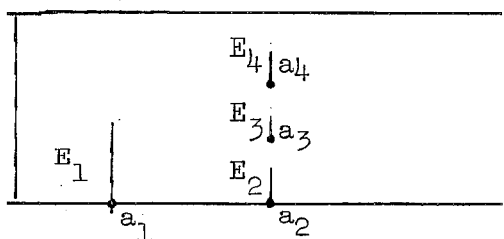
¹This interpretation of \mathcal{E}_r^h is due to Drèze, Gepts and Gabszewicz [16].

There is one case when this possibility cannot occur; namely, when $x_{ij} = x_{ik}$ for $j \neq k$ and for all $i=1, \dots, m$. This is true when assumptions (P.1) - (P.4') hold and when $\{x_{ij}, i=1, \dots, m, j=1, \dots, r\}$ corresponds to an allocation in the core of \mathcal{E}_r^h .²

We have presented two possible interpretations of the sequence of economies $\{v_r, r=1, 2, \dots\}$ specified above. We shall now demonstrate that this sequence is asymptotically perfectly competitive. For $i=1, 2, \dots, m$, let E_i be the closed subinterval of the interval $[(c_i, 0), (c_i, 1)] = \{c_i\} \times I$ defined by

$$(4) \quad E_i = \begin{cases} [(c_i, j_i), (c_i, \frac{j_{i+1} + j_i}{2})] & \text{if } c_{i+1} = c_i \\ [(c_i, j_i), (c_i, \frac{1+j_i}{2})] & \text{otherwise.} \end{cases}$$

These intervals are always nondegenerate. They are represented in the following figure



²For a proof of this, see Theorem 2, page 241 in [15]. The collection $\{x_{ij}, i=1, \dots, m, j=1, \dots, r\}$ "corresponds" to an allocation in the core of \mathcal{E}_r^h when there exists an allocation f in $\mathcal{C}(v_r)$ such that $x_{ij} = v(a_{ij}) f(a_{ij})$ for all i and j . For a discussion of cases where $x_{ij} \neq x_{ik}$ for j and k unequal, see the Appendix of this paper.

Let ν be the diffuse measure with support equal to $\bigcup_{i=1}^m E_i$ and with a uniform distribution of the mass $\frac{1}{m}$ on E_i (with respect to one-dimensional Lebesgue measure). To show that $\{\nu_r\}_{r=1}^{\infty}$ is asymptotically perfectly competitive it suffices to show that $\nu_r \Rightarrow \nu$. Thus, if f is any bounded, continuous real-valued function on A , then we must show that $\int_A f d\nu_r \rightarrow \int_A f d\nu$. For this it suffices to show that $\int_{E_i} f d\nu_r \rightarrow \int_{E_i} f d\nu$, $i=1, \dots, m$. But this follows from the uniform continuity of f on E_i and the definition of the measures ν_r , $r=1, 2, \dots$.

The Debreu-Scarff sequence of economies $\{\mathcal{E}_r, r=1, 2, \dots\}$ satisfies the following property: the ratio of the number of agents of some type c_i to the total number of agents in the economy remains constant as r increases. The Debreu-Scarff procedure can be generalized slightly by defining a sequence of economies which does not satisfy this property. Given \mathcal{E}_1 as above, define

\mathcal{E}'_r , $r=2, 3, \dots$ by

$$\mathcal{E}'_r = \{a_{ij}; i=1, \dots, m, j=1, \dots, k_{i,r}\}$$

where

$$a_{ij} = (c_i, j_i + \frac{(j-1)}{2 k_{i,r}} (j'_{i+1} - j_i))$$

where j'_{i+1} is defined in (2). $k_{i,r}$ is the number of agents of type i in

\mathcal{E}'_r .

Let ν'_r be the uniform discrete measure having a support equal to \mathcal{E}'_r .

It is easily seen that the conditions

$$(5) \quad \min \{k_{i,r} : i=1, \dots, m\} \rightarrow \infty \text{ as } r \rightarrow \infty$$

$$(6) \quad \text{for each } i=1, \dots, m, \quad \frac{k_{i,r}}{\sum_{j=1}^m k_{j,r}} \rightarrow \text{some limit as } r \rightarrow \infty$$

are necessary and sufficient for $\{v_r^i\}$ to be asymptotically perfectly competitive.

The limit economy, in this case, is the diffuse measure v' which uniformly distributes the mass $\lim_{r \rightarrow \infty} \frac{k_{i,r}}{\sum_{j=1}^m k_{j,r}}$ on the set E_i defined in (4).

Condition (5) means that each agent of type c_i contributes a decreasing share of the resources of all the agents of that type. Condition (6) means that the proportion $\frac{k_{i,r}}{\sum_{j=1}^m k_{j,r}}$ of agents of type i in economy \mathcal{E}'_r approaches some limit.

We remark that when $k_{i,r} \neq k_{j,r}$ for some i and j , then an allocation $\{x_{ij}; i=1, \dots, m, j=1, \dots, k_{ir}\}$ in the core of \mathcal{E}'_r need not satisfy

$$x_{ij} = x_{ik} \quad \text{for all } j \text{ and } k.$$

In particular, if an allocation $\{x_i; i=1, \dots, m\}$ for \mathcal{E}_1 is defined by $x_i = \frac{1}{k_{i,r}} \sum_{j=1}^{k_{i,r}} x_{ij}$, then $\{x_i, i=1, \dots, m\}$ may not be in the core of \mathcal{E}_1 .³ This fact would seem to limit the applicability of the delegation-procedure interpretation of a sequence of economies.

³An example where this occurs is given in the Appendix.

7. Upper semi-continuity of the correspondence \mathcal{C} .

This section is the heart of this paper. It demonstrates that under certain conditions the correspondence \mathcal{C} satisfies a property similar to upper semi-continuity (Theorem 2). This result is used to give a simple extension to an infinite economy of Scarf's proof [28] of the nonemptiness of the core.

The proof of Theorem 2 is based on several preliminary results which we now consider.

LEMMA 1: If h is a μ -integrable function from A to S and if φ is a closed- and convex-valued, measurable¹ correspondence from A to S satisfying

$$h(a) \in \text{ri}(\varphi(a)), \quad a \in A,$$

then for any Borel E ,

$$\int_E h \, d\mu \in \text{ri}\left(\int_E \varphi \, d\mu\right)$$

and, if h is continuous and if φ is L S C, then

$$\int_E h \, d\mu \in \text{ri}\left(\int_E^c \varphi \, d\mu\right)^2.$$

PROOF: It suffices to consider only the case where $0 \in \varphi(a)$ everywhere, because if the lemma were true for $\varphi-h$, then it would be true for φ . Because $0 \in \int_E \varphi \, d\mu$, the smallest linear manifold L containing $\int_E \varphi \, d\mu$ is a subspace of S . The condition $\int_E h \, d\mu \notin \text{ri}\int_E \varphi \, d\mu$ would imply that there were a

¹A correspondence $\varphi: A \rightarrow S$ is measurable if its graph $\{(a,x) \in A \times S: x \in \varphi(a)\}$ is measurable with respect to the product σ -field on $A \times S$. In this Lemma, as in the rest of the paper, it is only the behavior of h and φ on the set $\text{supp } \mu$ that is important. Because $\text{supp } \mu$ is compact, it is also separable and hence the product σ -field on $(\text{supp } \mu) \times S$ coincides with the Borel σ -field. Thus the measurability requirement could be weakened to the assumption that $\{(a,x) \in (\text{supp } \mu) \times S: x \in \varphi(a)\}$ be a Borel subset of $(\text{supp } \mu) \times S$.

² $\int_E^c \varphi \, d\mu = \left\{ \int_E f \, d\mu: f \in \mathcal{L}_{\varphi, \mu} \text{ and } f \text{ is continuous} \right\}.$

hyperplane in L which contained $\int_E h \, d\mu$ and which supported $\int_E \varphi \, d\mu$ since $\int_E \varphi \, d\mu$ is convex.³ Thus it suffices to show that if p is a nonzero vector in L , then $p \cdot \int_E h \, d\mu < \sup p \cdot \int_E \varphi \, d\mu$.

Suppose $p \cdot \int_E h \, d\mu = \sup p \cdot \int_E \varphi \, d\mu$. Now the set

$$F = \{ a \in E: p \cdot h(a) < \sup p \cdot \varphi(a) \}$$

is μ -measureable⁴ and the correspondence ψ mapping a into $\{x \in \varphi(a): p \cdot x > p \cdot h(a)\}$ is nonempty on the set F . There exists a μ -null set N such that the correspondence $\psi: F \setminus N \rightarrow S$ is measureable.⁵ Hence by the Measureable Choice Theorem,⁶ there exists a μ -measureable function g such that $p \cdot g(a) > p \cdot h(a)$ almost everywhere on F . If $\mu(F) > 0$, this would contradict $p \cdot \int_E h \, d\mu = \sup p \cdot \int_E \varphi \, d\mu$. Hence

$$(1) \quad p \cdot h(a) = \sup p \cdot \varphi(a) \quad \text{almost everywhere on } E.$$

Relation (1), together with

$$h(a) \in \text{ri } \varphi(a) \quad \text{almost everywhere,}$$

implies that $\varphi(a) \subset H(a)$ almost everywhere on E where

$$H(a) = \{x \in S: p \cdot x = \sup p \cdot \varphi(a)\}.$$

³This is just a restatement of Minkowski's separating hyperplane theorem. This theorem is discussed on page 25 of [11] and is proven in [19] and [4]. This result will be used in the sequel without citing this reference.

⁴ F is " μ -measureable" means $F \in \mathcal{A}_\mu$, the completion of \mathcal{A} with respect to μ . The measureability of F derives from the measureability of the function σ mapping an element a of A into the point $\sup p \cdot \varphi(a)$ of $R \cup \{+\infty\}$. This is demonstrated in Lemma 1 in Appendix II.

⁵See Lemma 2 in Appendix II.

⁶The Measureable Choice Theorem is stated in Appendix II.

But $0 \in \varphi(a) \subset H(a)$ means

$$H(a) = p^{-1}(0)$$

almost everywhere on E . Thus $H \equiv p^{-1}(0)$ is a linear subspace of S containing almost every $\varphi(a)$. This implies that $\int_E \varphi \, d\mu \subset H \cap L$, which is impossible since $H \cap L$ is a proper subspace of L and L is the smallest subspace containing $\int_E \varphi \, d\mu$. This contradiction shows that $p \cdot \int_E h \, d\mu < \sup p \cdot \int_E \varphi \, d\mu$.

To show that $p \cdot \int_E h \, d\mu < \sup p \cdot \int_E^c \varphi \, d\mu$ when h is continuous and φ is L S C, it suffices to show that if $p \cdot \int_E h \, d\mu = \sup p \cdot \int_E^c \varphi \, d\mu$, then relation (1) holds. To show that (1) holds, suppose $\mu(F) > 0$. Because μ is tight, there exists a closed subset F' of F such that $\mu(F') > 0$.⁷ The correspondence ψ is non-empty-valued on F' and is L S C. By the Continuous Selection Principle,⁸ there exists a continuous function $f: F' \rightarrow S$ such that $f(a) \in \psi(a)$ everywhere on F' . f can be extended on A to yield a continuous function satisfying $f(a) \in \psi'(a)$ everywhere,⁹ where $\psi'(a) = \{x \in \varphi(a): p \cdot x \geq p \cdot h(a)\}$ since ψ' is L S C. But then $p \cdot \int_E f \, d\mu > p \cdot \int_E h \, d\mu$ which would contradict $p \cdot \int_E h \, d\mu = \sup p \cdot \int_E^c \varphi \, d\mu$.

LEMMA 2: If G is a convex open subset of S , then $G = \text{int } \bar{G}$.

PROOF: The relation $G \subset \text{int } \bar{G}$ is clear. Conversely, suppose $x \in (\text{int } \bar{G}) \setminus G$. Then there exists a nonzero vector p in S such that $p \cdot x \geq \sup p \cdot G$. Thus $p \cdot x \geq \sup p \cdot \bar{G}$. But $x \in \text{int } \bar{G}$ and $p \neq 0$ imply that $p \cdot x < \sup p \cdot \bar{G}$. This contradiction establishes that $(\text{int } \bar{G}) \setminus G$ is empty.

⁷See pages 12-13.

⁸See Theorem 3.2", page 367 in [27].

⁹This is by Proposition 1.4, page 363 of [27] which generalizes Tietze Extension Theorem. Professor D. Edward Smallwood asserts that the Tietze Extension Theorem is otiose.

LEMMA 3: If G is convex and open and if G is a dense subset of K , then $G = \text{int } K$.

PROOF: By exercise G , page 57 in [24],

$$\begin{aligned} \text{int } K &\subset \text{int} \overline{G \cap \text{int} K} \\ &= \text{int } \bar{G} \\ &= G \end{aligned}$$

where the last equality follows from the preceding lemma.

We now look at the continuity properties of preferences. The correspondence $P: a \longmapsto P(a) = \{(x,y) \in S \times S: x \succeq_a y\}$ is closed and LSC (Proposition 1, Section 2). This implies that the compact-valued correspondence

$$P_N(a) = P(a) \cap \{(x,y) \in S \times S: |x|/|y| \leq N\}$$

is a continuous correspondence for every integer N . If f is a continuous allocation, then the correspondence mapping a into the set $\{(x,y) \in S \times S: y = f(a)\}$ is continuous and hence for any compact $K \subset A$ and for sufficiently large N , the correspondence $\bar{\Phi}_{f,N}: K \longmapsto S$ defined below is continuous:

$$\begin{aligned} \bar{\Phi}_{f,N}(a) &= \{x \in X(a): x \succeq_a f(a), |x| \leq N\} \\ &= \text{proj} [P_N(a) \cap \{(x,y): y = f(a)\}] \end{aligned}$$

where $\text{proj}[\cdot]$ is the mapping projecting $S \times S$ onto its first factor space.

It is helpful to introduce the following additional notation. If f is a μ -allocation for some economy μ , then

$$\varphi_f(a) = \{x \in X(a): x \succ_a f(a)\} \quad \text{for } a \text{ in } A;$$

$$\bar{\varphi}_f(a) = \text{closure of } \varphi_f(a)$$

$$= \{x \in X(a): x \succeq_a f(a)\} \quad \text{by (P.5) and (P.7) for } a \text{ in } \text{supp } \mu;$$

$$P_{f,\mu}(E) = \int_E \varphi_f d\mu \quad \text{for } E \text{ in } \mathcal{a}.^{10}$$

¹⁰The set $P_{f,\mu}(A)$ is closely related to the Scitovsky indifference curve through $\int_A f d\mu$ and determined by f (see [11] page 97) and has been used extensively by [32], [8] and [9].

THEOREM 1: In an economy μ satisfying (P.1) - (P.7) and (E.1) and (E.7), a continuous allocation f can be blocked only if there is a continuous allocation g and a nonnull coalition E such that

$$\int_E g \, d\mu \in \int_E (\omega + Y) \, d\mu ,$$

$$g(a) \succ_a f(a) \quad \text{for every } a \text{ in } \text{supp } \mu ,$$

$$g(a) \in \text{ri}(X(a)) \quad \text{for every } a \text{ in } \text{supp } \mu .$$

PROOF: Because f can be blocked, there exists a coalition E which is nonnull and a vector x in $[P_{f,\mu}(E)] \cap [\int_E (\omega + Y) \, d\mu]$. By (E.7) the correspondence mapping any element a of $\text{supp } \mu$ into the set $[\text{ri}(X(a))] \cap [\omega(a) + Y(a)]$ is nonempty-valued off some μ -null subset of $\text{supp } \mu$. This correspondence is clearly also L S C and hence by the Continuous Selection Principle and Lemma 1, there exists a vector z in $[\text{ri}(\int_E X \, d\mu)] \cap [\int_E (\omega + Y) \, d\mu]$. By Theorem 3 in [8], there exists y in the open line segment (x,z) such that $y \in P_{f,\mu}(E)$. Because $\int_E (\omega + Y) \, d\mu$ is convex, $y \in \int_E (\omega + Y) \, d\mu$. It is also easy to show that

$$(2) \quad y \in \text{ri}(\int_E X \, d\mu) .$$

By (2), there exist a finite number k of elements $\{x_n\}_{n=1}^k$ of $\text{ri}(\int_E X \, d\mu)$ such that their convex hull, $\langle x_n \rangle_{n=1}^k$, is a neighborhood of y in the subspace $L(\int_E X \, d\mu)$ of S .

For each n , there exists, by Theorem 3 in [8],

$$x'_n \in (y, x_n) \cap P_{f,\mu}(E) .$$

But $\langle x'_n \rangle_{n=1}^k$ is also a neighborhood of y in $L(\int_E X \, d\mu)$ and is contained in $P_{f,\mu}(E)$, which is convex by (P.4). Since $L(P_{f,\mu}(E)) \subset L(\int_E X \, d\mu)$, this demonstrates that

$$(3) \quad y \in \text{ri}(P_{f,\mu}(E)) .$$

We now make a similar argument to show that for any a satisfying (P.3) and (P.6),

$$(4) \quad \text{ri}(\varphi_f(a)) \subset \text{ri}(X(a)).$$

Let $u \in \varphi_f(a)$ and $v \in \text{ri}(X(a))$. By (P.3) and (P.6), $\varphi_f(a)$ is relatively open in $X(a)$ so there exists $w \in (u,v) \cap \varphi_f(a)$. Since $v \in \text{ri}(X(a))$, then $w \in \text{ri}(X(a))$ also. Hence there exist x_1, \dots, x_k all in $X(a)$ such that $\langle x_n \rangle_{n=1}^k$ is a neighborhood of w in $X(a)$. Because $\varphi_f(a)$ is open in $X(a)$, there exist $x'_n \in (w, x_n) \cap \varphi_f(a)$. But $\langle x'_n \rangle_{n=1}^k$ is also a neighborhood of w in $X(a)$. This implies $L(X(a)) \subset L(\varphi_f(a))$. The opposite inclusion is obvious so $L(X(a)) = L(\varphi_f(a))$ so (4) is valid.

Because of (3) and (4), the proof of Theorem 1 will be complete if we can demonstrate that

$$(5) \quad \text{ri}\left(\int_E \varphi_f d\mu\right) = \int_E^c \text{ri} \varphi_f d\mu$$

where $\text{ri} \varphi_f$ represents the correspondence mapping the point a into $\text{ri}(\varphi_f(a))$. Equality (5) derives from the following sequence of relations:

$$(6) \quad \text{ri}\left(\int_E \varphi_f d\mu\right) = \text{ri}\left(\int_E \bar{\varphi}_f d\mu\right)$$

$$(7) \quad = \text{ri}\left(\bigcup_{N=1}^{\infty} \int_E \bar{\varphi}_{fN} d\mu\right)$$

$$(8) \quad = \text{ri}\left(\bigcup_{N=1}^{\infty} \int_E^c \bar{\varphi}_{fN} d\mu\right)$$

$$(9) \quad = \text{ri}\left(\int_E^c \bar{\varphi}_f d\mu\right)$$

$$(10) \quad = \int_E^c \text{ri} \varphi_f d\mu .$$

Equality (6): Because $\int_E \phi_f d\mu \subset \int_E \bar{\phi}_f d\mu$, we have $ri \int_E \phi_f d\mu \subset ri \int_E \bar{\phi}_f d\mu$. Conversely, let $h \in \mathcal{L}_{\bar{\phi}_f, \mu}^c$. Define, for an arbitrary $\eta > 0$ and for any a in A :

$$\psi(a) = \{x \in X(a): x \geq_a h(a) \text{ and } |x - h(a)| < \eta\}$$

By (P.5), $\psi(a)$ is nonempty and ψ is clearly measurable since h is.¹¹ By the Measurable Choice Theorem, there is a function j in $\mathcal{L}_{\psi, \mu}$. But $|\int_E j d\mu - \int_E h d\mu| < \eta$ and $\int_E j d\mu \in \int_E \phi_f d\mu$. Thus $\int_E \phi_f d\mu$ is a dense subset of $\int_E \bar{\phi}_f d\mu$. Because preferences are convex, (P.4), $\int_E \phi_f d\mu$ is convex. Thus $ri(\int_E \phi_f d\mu)$ is an open, convex, dense subset of $ri(\int_E \bar{\phi}_f d\mu)$. By Lemma 3, these two sets are equal.

Equality (7): This equality is established in the same way as (6) by using the fact that $\bigcup_{N=1}^{\infty} \int_E \bar{\phi}_{fN} d\mu$ is convex and dense in $\int_E \bar{\phi}_f d\mu$.

Equality (8): This is immediate from the fact that for sufficiently large N , $\int_E \bar{\phi}_{fN} d\mu = \int_E^c \bar{\phi}_{fN} d\mu$ by Theorem 3 in [7] since by the remarks preceding Theorem 1, $\bar{\phi}_{fN}$ is a compact- and convex-valued continuous correspondence on $\text{supp } \mu$.

Equality (9): The compactness of $\text{supp } \mu$ implies that

$$\int_E^c \bar{\phi}_f d\mu = \bigcup_{N=1}^{\infty} \int_E^c \bar{\phi}_{fN} d\mu.$$

Equality (10): This demonstration is similar to that given for (6) above. We shall show first that $\int_E^c ri \phi_f d\mu$ is dense in $\int_E^c \bar{\phi}_f d\mu$. Suppose h belongs to $\mathcal{L}_{\bar{\phi}_f, \mu}^c$ and for any $\eta > 0$ define

¹¹See Lemma 2 in [21].

$$\psi'(a) = \text{ri}\{x \in X(a): x >_a h(a) \text{ and } |x - h(a)| < \eta\}$$

for a in A . Then $\psi'(a)$ is nonempty by (P.5) and is LSC since h is continuous and by (P.3).

By the Continuous Selection Principle¹² there exists a continuous function g in $\mathcal{L}_{\psi'}$. By (P.7) and by the argument establishing (4) (i.e., that $L(\varphi_f(a)) = L(X(a)) = L(\{x \in X(a): x >_a h(a) \text{ and } |x - h(a)| < \eta\})$, we note that $g(a) \in \text{ri } \varphi_f(a)$ everywhere on $\text{supp } \mu$. Thus

$$\int_E g \, d\mu \in \int_E^c \text{ri } \varphi_f \, d\mu .$$

Furthermore, $|\int_E g \, d\mu - \int_E h \, d\mu| < \eta$ so $\int_E^c \text{ri } \varphi_f \, d\mu$ is dense in $\int_E^c \text{ri } \bar{\varphi}_f \, d\mu$. The convexity of $\int_E^c \text{ri } \varphi_f \, d\mu$ is clear from (P.4). Thus

$$\begin{aligned} \int_E^c \text{ri } \varphi_f \, d\mu &= \int_E^c \text{ri } \bar{\varphi}_f \, d\mu \\ &\subset \text{ri} \left(\int_E^c \bar{\varphi}_f \, d\mu \right) && \text{by Lemma 1} \\ &= \text{ri} \left(\int_E^c \varphi_f \, d\mu \right) && \text{by Lemma 3} \\ &\subset \int_E^c \text{ri } \varphi_f \, d\mu . \end{aligned}$$

Thus $\int_E^c \text{ri } \varphi_f \, d\mu = \text{ri} \int_E^c \bar{\varphi}_f \, d\mu$.

LEMMA 4: Given a Borel subset E of A and an allocation g satisfying $g(a) \in \text{ri}(X(a))$ everywhere and given $\rho > 0$, then there exists $\eta > 0$ such that $z \in B_{\eta} \left(\int_E g \, d\mu \right) \cap L \left(\int_E X \, d\mu \right)$ implies there exists an allocation h such that $\int_E h \, d\mu = z$ and $|g(a) - h(a)| < \rho$ everywhere.

¹²See Theorem 3.1 page 368 in [27].

PROOF: Define

$$\psi(a) = B_{\rho}(g(a)) \cap \text{ri}(X(a)) .$$

$\psi(a)$ is nonempty and $\text{ri}(\psi(a)) = \psi(a)$. By Lemma 1,

$$\int_{\mathbf{E}} \psi \, d\mu \subset \text{ri} \left(\int_{\mathbf{E}} \bar{\psi} \, d\mu \right) .$$

Because $\int_{\mathbf{E}} \psi \, d\mu$ is a convex, dense subset of $\int_{\mathbf{E}} \bar{\psi} \, d\mu$, we have

$$\begin{aligned} \text{ri} \left(\int_{\mathbf{E}} \bar{\psi} \, d\mu \right) &= \text{ri} \left(\int_{\mathbf{E}} \psi \, d\mu \right) \quad \text{by Lemma 3,} \\ &\subset \int_{\mathbf{E}} \psi \, d\mu . \end{aligned}$$

Thus $\int_{\mathbf{E}} \psi \, d\mu = \text{ri} \left(\int_{\mathbf{E}} \psi \, d\mu \right)$. It is easy to see that the smallest affine manifold containing $\int_{\mathbf{E}} \psi \, d\mu$ is also the smallest affine manifold containing $\int_{\mathbf{E}} X \, d\mu$. Thus $\int_{\mathbf{E}} \psi \, d\mu$ is open in $L \left(\int_{\mathbf{E}} X \, d\mu \right)$.

LEMMA 5: Let ψ be a measurable correspondence from A to S such that $\mathcal{L}_{\psi, \mu}$ is uniformly integrable with respect to μ . Then for every $\delta > 0$, there exists $\rho > 0$ such that $\mu(E \Delta F) < \rho$ implies

$$\int_{\mathbf{E}} \psi \, d\mu \subset B_{\delta} \left(\int_{\mathbf{F}} \psi \, d\mu \right) .$$

This Lemma implies that if $\int_{\mathbf{E}} \psi \, d\mu$ is closed for every Borel set E , then this defines a uniformly continuous mapping from the Borel subsets of A , with the μ -metric topology, to the nonempty, closed subsets of S , with the Hausdorff metric topology.

PROOF: Note that for every pair E, F of Borel sets:

$$\int_{\mathbf{E}} \psi \, d\mu \subset \int_{\mathbf{F}} \psi \, d\mu + \int_{\mathbf{E} \setminus \mathbf{F}} \psi \, d\mu - \int_{\mathbf{F} \setminus \mathbf{E}} \psi \, d\mu .$$

Thus if $\int_{E \setminus F} \psi d\mu \in B_{\delta \sqrt{2}}(0) \supset \int_{F \setminus E} \psi d\mu$, then

$$\int_E \psi d\mu \in B_{\delta} \left(\int_F \psi d\mu \right).$$

But $\int_{\psi, \mu}$ uniformly integrable implies that for any $\delta > 0$ there exists $\rho > 0$ such that $\mu(H) < \rho$ implies

$$\int_H \psi d\mu \in B_{\delta \sqrt{2}}(0).$$

(See Proposition II.5.2, page 50 in [28].)

LEMMA 6: Under the hypothesis of Lemma 4 and if μ satisfies (E.2)

then there exists $\delta > 0$ such that $F \in B_{\delta}(E)^{13}$ and $z \in B_{\delta} \left(\int_F g d\mu \right) \cap L \left(\int_F X d\mu \right)$ imply that there exists an allocation h such that

$$\int_F h d\mu = z \quad \text{and} \quad |g(a) - h(a)| \leq \rho \quad \text{everywhere.}$$

PROOF: Let $\{A_i\}$ be the measurable partition of A and $\{L_i\}_{i=1}^n$ be the subspaces of S such that $L(X(a)) = L_i$ for a in A_i by (E.2). Assume that the Lemma has been proven for each of the correspondences $X_i = X|_{A_i}$. Then there exist $\delta_i > 0$ such that if $F \in B_{\delta_i}(E \cap A_i) \cap \mathcal{A}_i$, where \mathcal{A}_i is the collection of Borel subsets of A_i , and if $z \in B_{\delta_i} \left(\int_F g d\mu \right) \cap L_i$, then there exists an allocation h with $\int_F h d\mu = z$ and $|g(a) - h(a)| \leq \rho$ everywhere on A_i .

Let $\delta' = \min \{\delta_i\}_{i=1}^n > 0$ and choose δ , in $(0, \delta')$ so that $y \in [B_{\delta}(0)] \cap [\Sigma L_i]$ implies there exist y_i in L_i with $\Sigma y_i = y$ and $|y_i| < \delta'$. If $F \in B_{\delta}(E)$, then $\mu((F \cap A_i) \Delta (E \cap A_i)) \leq \mu(F \Delta E) \leq \delta$ so $F \cap A_i \in B_{\delta_i}(E \cap A_i) \cap \mathcal{A}_i$. If $z \in B_{\delta} \left(\int_F g d\mu \right) \cap L \left(\int_F X d\mu \right)$, then there exist y_i in L_i satisfying

¹³ $B_{\delta}(E) = \{F: F \text{ is a Borel subset of } A \text{ and } \mu(F \Delta E) < \delta\}$.

$$\sum y_i = z - \int_F g \, d\mu \quad \text{and} \quad |y_i| < \delta' .$$

Let $z_i = y_i + \int_{F \cap A_i} g \, d\mu$. Then $z_i \in B_\delta(\int_{F \cap A_i} g \, d\mu) \cap L_i$. By the assumption made above, there exist allocations h_i with the specified properties. Define an allocation h :

$$h(a) = h_i(a) \quad \text{if} \quad a \in A_i .$$

Then clearly $z = \int_F h \, d\mu$ and $|h(a) - g(a)| \leq \rho$ everywhere.

This demonstrates that it suffices to prove the Lemma for the case where $L(X(a))$ is constant on A . In fact, to simplify notation we shall assume that $L(X(a)) = S$ everywhere.

Let ψ be the correspondence defined in the proof of Lemma 4 and let $\eta > 0$ be the scalar whose existence is asserted by Lemma 4. It suffices to find a neighborhood $U(E)$ of E such that $F \in U(E)$ implies

$$(11) \quad B_{\eta/4}(\int_F g \, d\mu) \subset \int_F \bar{\psi} \, d\mu .$$

Claim: Relation (11) is implied by

$$(12) \quad B_{3\eta/4}(\int_F g \, d\mu) \subset B_{\eta/2}(\int_F \bar{\psi} \, d\mu) .$$

We shall show that there is a neighborhood $U(E)$ such that for F in $U(E)$ relation (12) holds and we shall then show that the Claim is correct.

Lemma 5 and the uniform integrability of $\mathcal{L}_{\bar{\psi}, \mu}$ imply that there exists a neighborhood $U'(E)$ such that $F \in U'(E)$ implies

$$\int_E \psi \, d\mu \subset B_{\eta/2}(\int_F \bar{\psi} \, d\mu) .$$

and hence by Lemma 4

$$(13) \quad B_\eta(\int_E g \, d\mu) \subset B_{\eta/2}(\int_F \bar{\psi} \, d\mu) .$$

There is a neighborhood $U''(E)$ such that $F \in U''(E)$ implies

$$\left| \int_{F \Delta E} g \, d\mu \right| < \eta/4.$$

Choose any F in $U(E) = U'(E) \cap U''(E)$ and choose any y in $B_{3\eta/4}(\int_F g \, d\mu)$. Then $y \in B_{\eta}(\int_E g \, d\mu)$ and so by (13) $y \in B_{\eta/2}(\int_F \bar{\psi} \, d\mu)$ which establishes (12).

Suppose relation (12) holds but that (11) does not; that is, there exists y in $B_{\eta/4}(\int_F g \, d\mu)$ and $y \notin K = \int_F \bar{\psi} \, d\mu$. K is convex and closed since $\bar{\psi}$ is convex, closed and integrably bounded [1]. Hence there exists a vector p in S with $p \cdot p = 1$ (that is, $|p| = 1$) and with

$$p \cdot y > \sup p \cdot K.$$

But $y + \eta/2p \in B_{3\eta/4}(\int_F g \, d\mu) \subset B_{\eta/2}(K)$.

On the other hand,

$$p \cdot (y + \eta/2p) = p \cdot y + \eta/2 > \eta/2 + \sup p \cdot K$$

and $\sup p \cdot B_{\eta/2}(K) = \eta/2 + \sup p \cdot K$. Thus $y + \eta/2p \notin B_{\eta/2}(K)$. This contradiction establishes that (12) implies (11).

MAIN LEMMA: Under the assumption that $\mu_n \Rightarrow \mu$, for every n , $\text{supp } \mu_n$ is contained in the compact set $\text{supp } \mu$, and that μ satisfies (E.2), if E is a Borel subset of A , if $\rho > 0$ and if g is a continuous allocation satisfying

$$g(a) \in \text{ri}(X(a)) \quad \text{everywhere}$$

then there exists $\delta > 0$ such that for any μ -boundaryless Borel set Q with $\mu(E \Delta Q) < \delta$ there exists N such that if $n \geq N$ then $z \in L(\int_Q X \, d\mu_n)$ and $|z - \int_Q g \, d\mu_n| < \delta$ imply there exists an allocation h with $z = \int_Q h \, d\mu_n$ and $|h(a) - g(a)| \leq \rho$ everywhere.

PROOF: Let ψ be the correspondence defined in the proof of Lemma 4. By Lemma 6, there exists $\delta' > 0$ such that if Q is a Borel set in $B_{\delta',/4}(E)$, then

$$(14) \quad B_{\delta'}(\int_Q g \, d\mu) \cap L(\int_Q X \, d\mu) \subset \int_Q \bar{\psi} \, d\mu .$$

Choose any such Q which is also μ -boundaryless. Choose N' so that $n \geq N'$ implies

$$|\int_Q g \, d\mu_n - \int_Q g \, d\mu| < \delta'/4 .$$

Then $\text{supp } \mu_n \subset \text{supp } \mu$ implies $L(\int_Q X \, d\mu_n) \subset L(\int_Q X \, d\mu)$ and hence

$$(15) \quad B_{3\delta',/4}(\int_Q g \, d\mu_n) \cap L(\int_Q X \, d\mu_n) \subset B_{\delta'}(\int_Q g \, d\mu) \cap L(\int_Q X \, d\mu)$$

Claim: There exists N'' such that $n \geq N''$ implies

$$(16) \quad \int_Q \bar{\psi} \, d\mu \subset B_{\delta',/2}(\int_Q \bar{\psi} \, d\mu_n) .$$

Suppose the Claim is valid. Then (14), (15) and (16) imply

$$\begin{aligned} B_{3\delta',/4}(\int_Q g \, d\mu_n) \cap L(\int_Q X \, d\mu_n) &\subset B_{\delta'}(\int_Q g \, d\mu) \cap L(\int_Q X \, d\mu) \\ &\subset \int_Q \bar{\psi} \, d\mu \\ &\subset B_{\delta',/2}(\int_Q \bar{\psi} \, d\mu_n) . \end{aligned}$$

Thus

$$(17) \quad B_{3\delta',/4}(\int_Q g \, d\mu_n) \cap L(\int_Q X \, d\mu_n) \subset B_{\delta',/2}(\int_Q \bar{\psi} \, d\mu_n) .$$

The same reasoning used to deduce the inclusion (11) from (12) in the proof of Lemma 6 allows us to conclude:

$$B_{\delta',/4}(\int_Q g \, d\mu_n) \cap L(\int_Q X \, d\mu_n) \subset \int_Q \bar{\psi} \, d\mu_n .$$

Letting $\delta = \delta'/4$ and $N = \max \{N', N''\}$ we have the desired result.

Suppose the Claim were false. Then there exists a sequence of functions f_n in $\mathcal{L}_{\bar{\psi}, \mu}$ such that for every h in $\mathcal{L}_{\bar{\psi}, \mu_n}$

$$\left| \int_Q f_n d\mu - \int_Q h d\mu_n \right| \geq \delta'/2 .$$

Since $\mathcal{L}_{\bar{\psi}, \mu}$ is uniformly integrable and is convex and closed in $L^1(\mu)$ and hence is weakly closed (that is, with respect to $\sigma(L^1(\mu), L^\infty(\mu))$), then $\mathcal{L}_{\bar{\psi}, \mu}$ is weakly compact¹⁴ and so there is a subsequence f'_n converging weakly to some f in $\mathcal{L}_{\bar{\psi}, \mu}$. In particular, $\int_Q f'_n d\mu \longrightarrow \int_Q f d\mu$.

If h is any continuous function in $\mathcal{L}_{\bar{\psi}, \mu}$, then $h \in \mathcal{L}_{\bar{\psi}, \mu_n}$ and hence

$$\left| \int_Q f d\mu - \int_Q h d\mu \right| = \lim_n \left| \int_Q f'_n d\mu - \int_Q h d\mu_n \right| \geq \delta'/2 .$$

But by Theorem 3 in [7] there exists a continuous function $h \in \mathcal{L}_{\bar{\psi}, \mu}$ such that $\int_Q h d\mu = \int_Q f d\mu$. This contradiction establishes the Claim so the proof of the Main Lemma is complete.

THEOREM 2: For economies $\{\mu; \mu_n, n=1, 2, \dots\}$ satisfying (P.1) - (P.7), (Y.1) - (Y.3) and (E.1) - (E.2), (E.4), (E.6) - (E.7), if $\mu_n \Rightarrow \mu$, if $\text{supp } \mu_n \subset \text{supp } \mu$ for every n , if $f_n \in \mathcal{E}(\mu_n)$ and if f is a continuous allocation feasible for μ such that for every $\rho > 0$ there exists N such that $n \geq N$ implies

$$|f(a) - f_n(a)| \leq \rho, \quad a \in \text{supp } \mu_n,$$

then $f \in \mathcal{E}(\mu)$.

PROOF: Suppose some μ -nonnull coalition E blocks f . Then by Theorem 1 there exists a continuous μ -allocation g satisfying

¹⁴See Proposition IV. 2.3, page 118 in [28].

$$(18) \quad g(a) \in \text{ri}(X(a)) \quad \text{everywhere on } \text{supp } \mu ,$$

$$(19) \quad g(a) \underset{a}{>} f(a) \quad \text{everywhere on } \text{supp } \mu ,$$

$$(20) \quad \int_{\mathbb{E}} g \, d\mu \in \int_{\mathbb{E}} (\omega + Y) \, d\mu .$$

By assumptions (Y.1) - (Y.3) and (E.4), $\int_{\mathbb{E}} Y \, d\mu = \int_{\mathbb{E}}^c Y \, d\mu$ (see Theorem 3 and its proof in [7]). This means that $\int_{\mathbb{E}} (\omega + Y) \, d\mu = \int_{\mathbb{E}}^c (\omega + Y) \, d\mu$ since ω is a continuous function. Thus (20) implies that there exists a continuous function h on A such that

$$(21) \quad \int_{\mathbb{E}} h \, d\mu = \int_{\mathbb{E}} g \, d\mu$$

$$(22) \quad h(a) \in [L(X(a))] \cap [\omega(a) + Y(a)] \quad \text{by (E.6).}$$

By a slight generalization of Lemma 1, page 42 in [21], there exists $\rho > 0$ such that for every a in $\text{supp } \mu$ we have

$$(23) \quad [B_{\rho}(g(a))] \cap X(a) \underset{a}{>} [B_{\rho}(f(a))] \cap X(a) .$$

On the basis of the objects \mathbb{E} , g , and ρ specified above, choose $\delta > 0$ by the Main Lemma. Using Lemma 2 on page 44 in [21], choose a μ -boundaryless Borel set Q so that the following two conditions are met:

$$(24) \quad \mu(Q \triangle \mathbb{E}) < \min\{\delta, \frac{\mu(\mathbb{E})}{2}\} ,$$

$$(25) \quad \left| \int_Q g \, d\mu - \int_Q h \, d\mu \right| \leq \left| \int_Q g \, d\mu - \int_{\mathbb{E}} g \, d\mu \right| + \left| \int_{\mathbb{E}} h \, d\mu - \int_Q h \, d\mu \right| < \frac{\delta}{2} ,$$

where the latter inequality makes use of (21) above.

Condition (24), together with $\mu_n \Rightarrow \mu$, implies there exists N' so $n \geq N'$ ensures that

$$(26) \quad \mu_n(Q) > 0 .$$

Choose another integer N to be at least as large as N' and as large as the integer specified by the Main Lemma and so that if $n \geq N$, then

$$(27) \quad \left| \int_Q g \, d\mu - \int_Q g \, d\mu_n \right| < \frac{\delta}{4} \quad \text{and} \quad \left| \int_Q h \, d\mu - \int_Q h \, d\mu_n \right| < \frac{\delta}{4} .$$

Inequalities (25) and (27) imply that for $n \geq N$

$$\left| \int_Q g \, d\mu_n - \int_Q h \, d\mu_n \right| < \delta$$

and relation (22) implies that

$$\int_Q h \, d\mu_n \in L\left(\int_Q X \, d\mu_n\right) .$$

Hence by the Main Lemma there exists, for each $n \geq N$, a μ_n -allocation g_n with

$$(28) \quad \int_Q g_n \, d\mu_n = \int_Q h \, d\mu_n \in \int_Q (\omega + Y) \, d\mu_n$$

and

$$(29) \quad |g_n(a) - g(a)| < \rho \quad \text{everywhere.}$$

Condition (28) means that g_n is attainable by coalition Q in economy μ_n and

(23) and (29) imply that

$$g_n(a) \succ_a [B_\rho(f(a))] \cap [X(a)], \quad a \in \text{supp } \mu_n .$$

But for sufficiently large n we have by hypothesis the fact that

$$f_n(a) \in [B_\rho(f(a))] \cap [X(a)], \quad a \in \text{supp } \mu_n .$$

Thus f_n can be blocked by the μ_n -nonnull coalition Q using the allocation g_n in economy μ_n . This contradiction implies that $f \in \mathcal{E}(\mu)$.

COROLLARY: For any economy μ satisfying (P.1) - (P.7), (Y.1) - (Y.3) and (E.1) - (E.2), (E.4), (E.6) - (E.7), the set $\mathcal{E}^c(\mu)$ of continuous μ -allocation in $\mathcal{E}(\mu)$ is closed under uniform convergence on $\text{supp } \mu$.

This Corollary is immediate from Theorem 2 and from the fact that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions converging uniformly on $\text{supp } \mu$ to some function f , then f is also continuous on $\text{supp } \mu$.

We remark that if μ is a nonatomic economy satisfying (P.4'), (E.3), (E.7') and (E.8) with $\mathcal{D}_\mu = \{1, \dots, N_0\}$,¹⁵ then $\mathcal{E}(\mu) = \mathcal{E}^c(\mu)$. This is because if $f \in \mathcal{E}(\mu)$, then the above assumptions imply $f \in \mathcal{W}(\mu)$ and the associated price vector p satisfies the condition:

$$(30) \quad w(p, a) > \inf p \cdot X(a)$$

everywhere. (See [13] or [9], page 28.)

Condition (E.8) also implies that $p \gg 0$ and hence by (E.3), the sets $\beta(p, a)$ are uniformly bounded for a in $\text{supp } \mu$. But then $\beta^0(p, a)$ is $\cup S C$ (see Proposition 4 in Debreu [10]). Now (P.4') implies $\beta^0(p, a) = \{f(a)\}$ for every a in $\text{supp } \mu$ and hence f is continuous on $\text{supp } \mu$.¹⁶

To illustrate the power of Theorem 2, we shall use it to extend Scarf's proof of the nonemptiness of the core to an infinite economy μ with a finite number of types of traders. By this we mean that there is a finite subset $\{c_i\}_{i=1}^n$ of C such that $\text{supp } \mu \subset \bigcup_{i=1}^n \{c_i\} \times I$. We say that μ has a rational marginal distribution if each of the numbers $\mu(\{c_i\} \times I)$, $i=1, \dots, n$ is rational; that is, μ^m is rational-valued.

THEOREM 3: If μ is an economy satisfying (P.1) - (P.7), (P.4'), (Y.1) - (Y.3) and (E.1) - (E.7) and if μ has a finite number of types of agents, a finite number of atoms and a rational marginal distribution, then $\mathcal{E}(\mu)$ is not empty.

¹⁵We recall that N_0 is the dimension of the commodity space S . This means that all commodities are "desirable".

¹⁶See footnote 9 above.

PROOF: There exist a purely atomic measure μ_a and a disjoint nonatomic measure μ_d such that $\mu = \mu_a + \mu_d$. More explicitly, μ_a is a uniform discrete measure and μ_d is a diffuse measure such that

$$\text{supp } \mu_a \cap \text{supp } \mu_d = \emptyset .$$

Since μ^m and μ_a^m are rational valued, so is μ_d^m . Let $\{c_i\}_{i=1}^n$ be the types of agents in μ_d and let $p_i, i=1, \dots, n$ and q be positive integers such that $\mu_d(\{c_i\} \times I) = \frac{p_i}{q}$. Without loss of generality, we may assume that for each $i=1, \dots, n$ there are p_i closed, /nondegenerate intervals $E_{ij}, j=1, \dots, p_i$, in $\{c_i\} \times I$ and such that $\text{supp } \mu_d = \bigcup_{i,j} E_{ij}$ and such that μ_d distributes the mass $\frac{1}{q}$ uniformly on the interval E_{ij} with respect to (one-dimensional) Lebesgue measure on E_{ij} . Let a_{ij} be the lower (for the I-coordinate of a_{ij}) end point of E_{ij} . The economy $\{a_{ij}, i=1, \dots, n, j=1, \dots, p_i\}$ will be designated by \mathcal{E}_1 . Let ν_r be the measure directly representing the r^{th} -replication of \mathcal{E}_1 , as explained in Section 6. Finally, let

$$\hat{\nu}_r = \mu_a + \mu(\text{supp } \mu_d) \cdot \nu_r .$$

The sequence $\hat{\nu}_r$ converges weakly to μ and satisfies $\text{supp } \hat{\nu}_r \subset \text{supp } \hat{\nu}_{r+1} \subset \text{supp } \mu$ for every r .

Because of (P.4'), any allocation in the core of $\hat{\nu}_r$ is constant $\hat{\nu}_r$ -almost everywhere on each set E_{ij} ¹⁷ and hence, by an argument similar to that made following Lemma 7 below, this allocation can be represented by a continuous function on A . Furthermore, $e^c(\hat{\nu}_r)$ is a bounded subset of S^A by (Y.1) - (Y.3) and (E.3) - (E.5).¹⁸ By the preceding Corollary, $e^c(\hat{\nu}_r)$ is closed with respect

¹⁷See Theorem 2, page 241 in [15].

¹⁸See Proposition 2, page 77 in [11]. These assumptions on consumption and production technologies imply that $\int_A X d\mu \subset \Omega + b, \int_A Y d\mu$ is closed and convex, $[\int_A Y d\mu] \cap \Omega = \{0\}$ and $[\int_A Y d\mu] \cap [-\int_A Y d\mu] = \{0\}$.

to uniform convergence on $\text{supp } \hat{v}_r$. But since every element of $\mathcal{E}^c(\hat{v}_r)$ is constant on each E_{ij} , then uniform convergence on $\text{supp } \hat{v}_r$ coincides with pointwise convergence. Thus each set $\mathcal{E}^c(\hat{v}_r)$ is bounded and closed and hence compact in the product topology on S^A .

By Scarf's result [29], each $\mathcal{E}^c(\hat{v}_r)$ is not empty. It is clear that $\mathcal{E}^c(\hat{v}_{r'}) \subset \mathcal{E}^c(\hat{v}_r)$ when $r' > r$. Thus the compactness of the sets $\mathcal{E}^c(\hat{v}_r)$ implies that $\bigcap_{r=1}^{\infty} \mathcal{E}^c(\hat{v}_r)$ is not empty. By Theorem 2, $\bigcap_{r=1}^{\infty} \mathcal{E}^c(\hat{v}_r) \subset \mathcal{E}^c(\mu)$ so $\mathcal{E}(\mu)$ is not empty.

8. The price implications of approximately perfect competition.

We have identified the notion of approximate perfect competition with the concept of an asymptotically competitive sequence of economies. This section demonstrates that just as every Edgeworth allocation in a perfectly competitive economy is a Walras allocation, so is the core of an approximately competitive economy equal to the set of approximately Walras allocations. As an introduction to this result, we first make the simpler assertion:

THEOREM 4: If $\{\mu_n\}$ is an asymptotically competitive sequence of economies with limit μ , if each of these economies satisfies (P.1) - (P.7), (Y.1) - (Y.3), (E.1) - (E.2), (E.4), (E.6) - (E.7), if f_n is in $\mathcal{E}(\mu_n)$ and if f_n converges uniformly on $\text{supp } \mu_n$ to a continuous μ -allocation f ,¹ then $f \in \mathcal{Q}(\mu)$. If μ also satisfies (E.7') and (E.8), then $f \in \mathcal{W}(\mu)$.

This theorem is an immediate consequence of Theorem 2 and of the usual result characterizing allocations in the core of a nonatomic economy by prices.²

¹By this we mean that f_n converges to f in the way specified in the statement of Theorem 2.

²For example, see Theorems 2 and 3 in [9]. Similar results can be found in [2], [20], [21] and [32].

We now use Theorem 4, together with the proof of Theorem 3, to derive a theorem giving conditions for the existence of a Walras allocation for a finite economy. Even though these conditions are overly strong when compared with other existence proofs (see [13], for example), this result is interesting because it illustrates how the study of infinite economies can be used to get information about finite economies. It is also the first time that the nonemptiness of the core of a finite economy has been used to demonstrate the existence of a Walras allocation for that economy.

COROLLARY: If μ is a finite economy satisfying (P.1) - (P.7), (P.4'), (Y.1) - (Y.3) and (E.1) - (E.7), then $\mathcal{Q}(\mu)$ is not empty and if (E.7') and (E.8) also hold, then $\mathcal{W}(\mu)$ is not empty.

PROOF: Let \mathcal{E}_1 be the economy directly represented by the uniform, discrete measure μ , let $\{v_r\}$ be the Debreu-Scarff sequence of replications of \mathcal{E}_1 and let v be the limit of the asymptotically diffuse sequence $\{v_r\}$. The proof of Theorem 3 demonstrated that each of the sets $\mathcal{C}^c(v_r)$ is a compact, nonempty subset of S^A (with the product topology). Further, these sets are nested. Consequently, there exists f in $\bigcap_{r=1}^{\infty} \mathcal{C}^c(v_r)$. By Theorem 4, $f \in \mathcal{Q}(v)$ and, if (E.7') and (E.8) also hold for μ , then they hold for v and $f \in \mathcal{W}(v)$. It remains to show that if $f \in \mathcal{Q}(v)$ (resp., $f \in \mathcal{W}(v)$), then $f \in \mathcal{Q}(\mu)$ (resp., $f \in \mathcal{W}(\mu)$). This is accomplished by Lemma 7.

LEMMA 7: If $\{v_r, r = 1, 2, \dots\}$ is a Debreu-Scarff sequence with limit v satisfying (P.1) - (P.3) and (Y.1) - (Y.3), then $f \in \mathcal{Q}(v)$ implies $f \in \mathcal{Q}(v_1)$.

PROOF: Because $f \in \mathcal{Q}(v)$, there exists $h' \in \mathcal{L}_{Y,v}^f$ such that $\int_A f \, dv = \int_A (\omega + h') \, dv$ and there exists a non-zero price vector p such that for v -almost every a :

- (1) either $f(a) \in \beta^0(p,a)$ or $f(a) \in \beta(p,a)$ and $w(p,a) = \inf p \cdot X(a)$,
- (2) $p \cdot h'(a) = \pi(p,a)$.

We shall show that there exists a production assignment h in \mathcal{L}_{Y,v_1}^f such that

$$(3) \quad \int_A f \, dv_1 = \int_A (\omega + h) \, dv_1$$

and that, for v_1 -almost every a , statements (1) and (2) above are valid if h' is replaced by h . This will suffice to show $f \in \mathcal{Q}(v_1)$.

Let $E_i, i=1, \dots, m$ be the closed, connected subsets of A which were defined on page 29 and which satisfy

$$\bigcup_{i=1}^m E_i = \text{supp } v.$$

Because f and ω are constant on each set E_i and because $v(E_i) = \frac{1}{m} = v_1(E_i)$, we have

$$\int_A (f - \omega) \, dv_1 = \int_A (f - \omega) \, dv.$$

Thus to show (3), it suffices to find h in \mathcal{L}_{Y,v_1}^f such that h is constant on each E_i and $\int_{E_i} h \, dv = \int_{E_i} h' \, dv$, because this will mean that $\int_A h \, dv_1 = \int_A h' \, dv$. The existence of h is immediate from

LEMMA 8: If Z is a correspondence on a space E such that for every a in E , $Z(a) = T$ for some closed convex subset T of S and if μ is a nonnegative measure on E , then

$$\int_E Z \, d\mu = \mu(E) \cdot T.$$

PROOF: If $g \in \mathcal{L}_{Z, \mu}$, then there exist simple measurable functions $g_n \in \mathcal{L}_{Z, \mu}$ such that $\int_E g_n d\mu \rightarrow \int_E g d\mu$. Then for some vectors a_{in} in T and some subsets F_{in} of E we have

$$g_n = \sum_{i=1}^{k_n} a_{in} \chi_{F_{in}}.$$

Define

$$a_n = \frac{1}{\mu(E)} \sum_{i=1}^{k_n} \mu(F_{in}) a_{in}.$$

$a_n \in T$ because T is convex. Further

$$a_n = \frac{1}{\mu(E)} \int_E g_n d\mu \rightarrow \frac{1}{\mu(E)} \int_E g d\mu \equiv a.$$

Then $a \in T$ since T is closed and

$$\int_E g d\mu = \mu(E) \cdot a.$$

Thus $\int_E Z d\mu = \mu(E) \cdot T$.

To complete the proof of Lemma 7, we note that (1) holds for ν -almost every a in each E_i . The constancy on E_i of the functions mapping a into, respectively, $f(a)$, $\beta^0(p, a)$, $\beta(p, a)$, $w(p, a)$ and $\inf p \cdot X(a)$ implies that if (1) does not hold for some a in E_i , then it does not hold for any a in E_i . This is impossible since $\nu(E_i) > 0$. Thus (1) holds for every a in $\text{supp } \nu_1$,

i.e., ν_1 -almost everywhere. To show that (2) holds ν_1 -almost everywhere when h' is replaced by h , we note that (2) implies

$$p \cdot \int_A h d\nu = p \cdot \int_A h' d\nu = \sup p \cdot \int_A (Y + \omega) d\nu.$$

This means that the set

$$\{a \in A: p \cdot h(a) < \pi(p, a)\}$$

is ν -null so (2) holds ν -almost everywhere when h' is replaced by h . To show that it in fact holds ν_1 -almost everywhere, an argument analogous to that made in the preceding paragraph using the constancy of h and $\pi(p, \cdot)$ on E_i can be made.

This completes the proof of Lemma 7 and of the Corollary to Theorem 4.

We now consider the Edgeworth-Debreu-Scarff limit theorem which is the prototype for a more general limit result later. Given an economy $\mathcal{E}_1 = \{a_1, \dots, a_m\}$, we defined in Section 6 an economy $\mathcal{E}_r = \{a_{ij}; i=1, \dots, m, j=1, \dots, r\}$ consisting of r "replicas" of \mathcal{E}_1 . The economy $\mathcal{E}_r^h = \{\tilde{a}_{ij}, i=1, \dots, m, j=1, \dots, r\}$ where $\tilde{a}_{ij} = \frac{1}{rm} a_{ij}$ was also defined. Finally, an economy $\frac{1}{r} \mathcal{E}_1$ can be defined as $\{\tilde{\tilde{a}}_{ij}; i=1, \dots, m, j=1, \dots, r\}$ where $\tilde{\tilde{a}}_{ij} = \frac{1}{r} a_{ij}$. An allocation for \mathcal{E}_r (respectively, for \mathcal{E}_r^h or $\frac{1}{r} \mathcal{E}_1$) is a collection $x_{(r)} = \{x_{ij}; i=1, \dots, m, j=1, \dots, r\}$ (respectively, $\tilde{x}_{(r)} = \{\tilde{x}_{ij}; i=1, \dots, m, j=1, \dots, r\}$ or $\tilde{\tilde{x}}_{(r)} = \{\tilde{\tilde{x}}_{ij}; i=1, \dots, m, j=1, \dots, r\}$) of commodity vectors such that for every i, j , $x_{ij} \in X(a_{ij})$ (respectively, $\tilde{x}_{ij} \in X(\tilde{a}_{ij})$ or $\tilde{\tilde{x}}_{ij} \in X(\tilde{\tilde{a}}_{ij})$).

Define for any integer $r \geq 1$:

$$C_r^1 = \{x_{(1)} \in X(a_1) \times \dots \times X(a_m): \text{if } x_{(r)} = \{x_{ij} = x_i, i=1, \dots, m, j=1, \dots, r\}, \text{ then } x_{(r)} \text{ is in the core of } \mathcal{E}_r\},$$

$$C_r^2 = \{x_{(1)} \in X(a_1) \times \dots \times X(a_m): \text{if } \tilde{x}_{(r)} = \{x_{ij} = \frac{1}{rm} x_i, i=1, \dots, m, j=1, \dots, r\} \text{ then } \tilde{x}_{(r)} \text{ is in the core of } \mathcal{E}_r^h\},$$

$$C_r^3 = \{x_{(1)} \in X(a_1) \times \dots \times X(a_m): \text{if } \tilde{\tilde{x}}_{(r)} = \{x_{ij} = \frac{1}{r} x_i, i=1, \dots, m, j=1, \dots, r\} \text{ then } \tilde{\tilde{x}}_{(r)} \text{ is in the core of } \frac{1}{r} \mathcal{E}_1\}.$$

Thus C_r^1 is the set of allocations $x_{(1)} = (x_1, \dots, x_m)$ for \mathcal{E}_1 such that if each of the r traders a_{ij} , $j=1, \dots, r$ of the same type as a_i received x_i in economy \mathcal{E}_r , then the result would be in the core for \mathcal{E}_r . C_r^3 is the set of allocations $x_{(1)} = (x_1, \dots, x_m)$ for \mathcal{E}_1 such that if each trader a_i in \mathcal{E}_1 delegated trading authority to r "brokers" $\tilde{a}_{ij} = \frac{1}{r} a_{ij}$, $j=1, \dots, r$, then the allocation $\tilde{\tilde{x}}_{(r)}$ defined by $\tilde{\tilde{x}}_{ij} = \frac{1}{r} x_i$ would be in the core of the resulting economy. Finally, C_r^2 is the set of allocations $x_{(1)} = (x_1, \dots, x_m)$ for which $\tilde{x}_{(r)}$ defined by $\tilde{x}_{ij} = \frac{1}{rm} x_i$ is in the core of \mathcal{E}_r^h .

Edgeworth [18] and Debreu and Scarf [15] have found conditions under which if $x_{(1)} \in \prod_{r=1}^{\infty} C_r^1$, then $x_{(1)}$ is a Walras allocation for \mathcal{E}_1 . Drèze, Gepts and Gabszewicz [16] have found conditions under which if $x_{(1)} \in \prod_{r=1}^{\infty} C_r^3$, then $x_{(1)}$ is a Walras allocation for \mathcal{E}_1 . However,

- (a) $x_{(r)}$ is in the core of $\mathcal{E}_r \iff$ there exists $f \in \mathcal{C}(v_r)$ with $x_{ij} = f(a_{ij})$,
- (b) $\tilde{x}_{(r)}$ is in the core of $\mathcal{E}_r^h \iff$ there exists $\tilde{f} \in \mathcal{C}(v_r)$ with $\tilde{x}_{ij} = \frac{1}{rm} \tilde{f}(a_{ij})$,
- (c) $\tilde{\tilde{x}}_{(r)}$ is in the core of $\frac{1}{r} \mathcal{E}_1 \iff$ there exists $\tilde{\tilde{f}} \in \mathcal{C}(v_r)$ with $\tilde{\tilde{x}}_{ij} = \frac{1}{r} \tilde{\tilde{f}}(a_{ij})$.

But then

$$\begin{aligned} x_{(r)} \text{ is in the core of } \mathcal{E}_r &\iff \tilde{x}_{(r)} = \frac{1}{rm} x_{(r)} \text{ is in the core of } \mathcal{E}_r^h \\ &\iff \tilde{\tilde{x}}_{(r)} = \frac{1}{r} x_{(r)} \text{ is in the core of } \frac{1}{r} \mathcal{E}_1. \end{aligned}$$

Hence for every r , $C_r^1 = C_r^2 = C_r^3$. In particular, the Edgeworth-Debreu-Scarf result and the Drèze - Gepts - Gabszewicz result are special cases of

THEOREM 5: Given an economy $\mathcal{E}_1 = \{a_1, \dots, a_m\}$ satisfying (P.1) - (P.7), (Y.1) - (Y.3), (E.6) - (E.7), if $x_{(1)} \in \prod_{r=1}^{\infty} C_r^2$, then $x_{(1)}$ is quasi-competitive for \mathcal{E}_1 . If \mathcal{E}_1 also satisfies (E.7') and (E.8), then $x_{(1)}$ is a Walras allocation.

PROOF: It has been shown in Section 6 that the replication and division procedures can both be represented by the asymptotically perfectly competitive sequence of economies v_r with limit v . In order to use Theorem 4 above, we want to find allocations f_n and f which correspond in some way to $x_{(1)} = \{x_1, \dots, x_m\}$. We do this by defining a modified consumption correspondence X' on A :

$$X'(a) = \begin{cases} \{x_i\} & \text{if } a \in E_i^3, \text{ any } i=1, \dots, m \\ X(a) & a \notin \bigcup_{i=1}^m E_i. \end{cases}$$

The lower semicontinuity of X' follows easily from the following simply proven Lemma:

LEMMA 9: If X is a LSC correspondence from A to S , if E is a closed subset of A and if there exists a common element x in $X(a)$ for every a in E , then the correspondence X' defined by

$$X'(a) = \begin{cases} X(a) & a \notin E \\ \{x\} & a \in E \end{cases}$$

is also lower semicontinuous.

By the Continuous Selection Theorem,⁴ there exists a continuous function f on A such that $f(a) \in X'(a)$ for every a . Because $f(a) = x_i$ for a in E_i and $x_{(1)} \in \prod_{r=1}^{\infty} C_r^2$, then $f \in \mathcal{C}(v_r)$ for every r . Hence we can conclude that $f \in \mathcal{Q}(v)$ if we can show that the hypothesis of Theorem 4 is satisfied. We have explicitly assumed all the conditions of Theorem 4 except for (E.1), (E.2) and (E.4). It is clear that (E.1) and (E.2) are satisfied by any economy having only a finite number of different types of trader. Condition (E.4) need not be satisfied by v . However, it is easily seen by Lemmas 8 and 9 that for any Borel set F ,

$$\int_F Y \, dv = \int_F^c Y \, dv$$

whenever v is an economy with a finite number of types of traders each of which satisfies (Y.2) and (Y.3). Since (E.4) was assumed for Theorems 2 and 4 only to ensure this equality, we may apply Theorem 4 to conclude that $f \in \mathcal{Q}(v)$. It now

³The sets E_i were defined on page 29 in Section 6.

⁴This is Theorem 3.2", page 367 in [27].

remains to show that, in fact, $f \in \mathcal{Q}(v_1)$ and that, consequently, $x_{(1)}$ is quasi-competitive for \mathcal{E}_1 . This conclusion is immediate from Lemma 7.

To study approximately Walras allocations, we first need a Lemma which generalizes the argument of the last two paragraphs of the proof of Lemma 7:

LEMMA 10: Under conditions (P.1) - (P.7), (Y.1) - (Y.3) (E.1) and (E.4), if f is a continuous, quasi-competitive allocation with respect to prices p for an economy μ , then there exists a continuous production assignment h for μ and that $\int_A f \, d\mu = \int_A (\omega + h) \, d\mu$ and such that for every a in $\text{supp } \mu$:

$$(4) \quad f(a) \in \beta(p, a)$$

$$(5) \quad \text{either } w(p, a) = \inf p \cdot X(a) \quad \text{or } f(a) \in \beta^0(p, a)$$

$$(6) \quad p \cdot h(a) = \pi(p, a).$$

Remark: The essential feature of this Lemma is the assertion that for a continuous allocation f , conditions (4) - (6) hold everywhere on $\text{supp } \mu$ and not just μ -almost everywhere.

PROOF: To prove that (4) holds everywhere on $\text{supp } \mu$, we note that the set

$$\{a \in \text{supp } \mu: p \cdot f(a) \leq w(p, a)\}$$

is closed in A (since f and $w(p, \cdot)$ are continuous) and has μ -measure one. Consequently, this set equals $\text{supp } \mu$.

To show that condition (5) holds everywhere on $\text{supp } \mu$, we shall show that if $w(p, a_0) > \inf p \cdot X(a_0)$ for some a_0 in $\text{supp } \mu$, then $f(a_0) \in \beta^0(p, a_0)$. Now $w(p, a_0) > \inf p \cdot X(a_0)$ means that there does not exist a sequence of elements a_n in $\text{supp } \mu$ converging to a_0 and satisfying $w(p, a_n) = \inf p \cdot X(a_n)$, for every n . This is because the mapping taking the point a' into $w(p, a') - \inf p \cdot X(a')$ is continuous on A (Proposition 3 on page 6 above). Hence the set

$$F = \{a' \in A: w(p, a') - \inf p \cdot X(a') = 0\}$$

is closed. Thus the existence of such a sequence $\{a_n\}$ would imply $w(p, a_0) = \inf p \cdot X(a_0)$ which contradicts our hypothesis.

On the other hand, every neighborhood U of a_0 contains a point a' in $\text{supp } \mu$ such that either $w(p, a') = \inf p \cdot X(a')$ or $f(a') \in \beta^0(p, a')$, because if there were a neighborhood U containing no such a' , then $f \in \mathcal{L}(\mu)$ would imply $\mu(U) = 0$. This would contradict the fact that $a_0 \in \text{supp } \mu$. In conclusion, there exists a sequence of elements a_n of $\text{supp } \mu$ converging to a_0 and satisfying $f(a_n) \in \beta^0(p, a_n)$.

We want to show that for any x in $\beta(p, a_0)$, the relation $x \leq_{a_0} f(a_0)$ holds. Now $w(p, a_0) > \inf p \cdot X(a_0)$ implies that $\beta(p, \cdot)$ is LSC at a_0 (see the second part of the proof of (3) in [10]). Hence there exists x_n in $\beta(p, a_n)$ such that $x_n \rightarrow x$, where $\{a_n\}_{n=1}^{\infty}$ is the sequence described in the preceding paragraph. But

$$(a_n, x_n, f(a_n)) \rightarrow (a_0, x, f(a_0))$$

and $x_n \leq_{a_n} f(a_n)$ for every n so $x \leq_{a_0} f(a_0)$ since preferences are continuous (Proposition 1, page 5 above). Thus $w(p, a_0) > \inf p \cdot X(a_0)$ implies that $f(a_0) \in \beta^0(p, a_0)$.

To demonstrate the existence of h , we note that (Y.1)-(Y.3) and (E.4) imply $\int_A Y d\mu = \int_A^C Y d\mu$ (Theorem 3 in [7]). Hence there exists a continuous function h in $\mathcal{L}_{Y, \mu}^{\mathcal{P}}$ such that

$$\int_A f d\mu = \int_A (w+h) d\mu .$$

The argument made in the last paragraph of the proof of Lemma 7 demonstrates that condition (6) holds for μ -almost every a in $\text{supp } \mu$. The continuity of h and $\pi(p, \cdot)$ implies that in fact (6) holds for every a in $\text{supp } \mu$ by means of an argument analogous to that used to demonstrate (4).

For any $\epsilon > 0$, a λ -allocation g is ϵ -quasi-competitive for λ if there exists a nonzero price vector p and a λ -production assignment h such that for λ -almost every a :

$$(7) \quad \text{either } w(p,a) = \inf p \cdot X(a) \text{ or } g(a) \text{ is within } \epsilon \text{ of the set } \beta^0(p,a),$$

$$(8) \quad \left| \int_A g \, d\lambda - \int_A (\omega+h) \, d\lambda \right| \leq \epsilon,$$

$$(9) \quad p \cdot h(a) = \pi(p,a).$$

A function g is an ϵ -Walras allocation for λ if g is ϵ -quasi-competitive for λ and if (7) is supplemented by

$$(10) \quad g(a) \text{ is within } \epsilon \text{ of } \beta^0(p,a) \text{ for } \lambda\text{-almost every } a.$$

These concepts are related to the idea of asymptotic perfect competition by the next Theorem which was first proven by Hildenbrand [21] under more restrictive assumptions:

THEOREM 6: (Hildenbrand) Given an asymptotically perfectly competitive sequence of economies $\{\mu_1, \mu_2, \dots\}$ each of which, together with their limit μ , satisfies (P.1) - (P.7), (Y.1) - (Y.3), (E.1) - (E.2), (E.4), (E.6) - (E.7) and given a sequence of μ_n -allocations f_n where f_n belongs to $\mathcal{E}(\mu_n)$ and $\{f_n\}_{n=1}^{\infty}$ converges to some continuous function uniformly on $\text{supp } \mu_n$, then for any $\epsilon > 0$ there exists N such that $n \geq N$ implies f_n is ϵ -quasi-competitive for μ_n . If μ also satisfies (E.7') and (E.8) then f_n is an ϵ -Walras allocation for μ_n .

PROOF: Let f be the continuous function which is the uniform limit on $\text{supp } \mu_n$ of the allocations f_n . We recall that this means that for any $\epsilon > 0$ there exists N so $n \geq N$ implies:

$$|f_n(a) - f(a)| \leq \epsilon \quad \text{for every } a \in \text{supp } \mu_n.$$

By Theorem 4, f is in $\mathcal{L}(\mu)$. By Lemma 10, relations (4) - (6) hold everywhere on $\text{supp } \mu$ and hence hold everywhere on each $\text{supp } \mu_n$ since $\text{supp } \mu_n \subset \text{supp } \mu$.

In particular, we see that condition (9) holds μ_n -almost everywhere for every n .

To show that condition (7) holds μ_n -almost everywhere, we observe that (5) holds everywhere on $\text{supp } \mu_n$ and that for sufficiently large n , $f_n(a)$ is within ϵ of $f(a)$ everywhere on $\text{supp } \mu_n$. To show that condition (8) is satisfied, we note that for sufficiently large n

$$|f_n(a) - f(a)| \leq \frac{\epsilon}{3}, \quad a \text{ in } \text{supp } \mu_n$$

and hence

$$(11) \quad \left| \int_A f_n d\mu_n - \int_A f d\mu_n \right| \leq \frac{\epsilon}{3}.$$

For sufficiently large n we also have

$$(12) \quad \left| \int_A f d\mu_n - \int_A f d\mu \right| \leq \frac{\epsilon}{3}$$

since f is continuous and $\mu_n \Rightarrow \mu$. Finally, for large enough n we also have

$$(13) \quad \left| \int_A (\omega+h) d\mu - \int_A (\omega+h) d\mu_n \right| \leq \frac{\epsilon}{3}$$

since $\omega+h$ is continuous and $\mu_n \Rightarrow \mu$. Combining (11)-(13) together with the relation $\int_A f d\mu = \int_A (\omega+h) d\mu$ we conclude that for all n large enough (8) is valid, if $\lambda = \mu_n$.

The value of Theorem 6 is that it can be applied to less restrictive types of convergence of economies than that contained in the Edgeworth-Debreu-Scarf model. We shall demonstrate this by considering a generalized E-D-S sequence $\{v_n^i\}$ of asymptotically competitive economies as defined in Section 6 above. An example is given in the Appendix which demonstrates that even when (P.4') is valid, an allocation in $\mathcal{C}(v_n^i)$ need not allocate the same bundle to traders of the same type. However, the following result is valid.

THEOREM 7: Given a generalized E-D-S sequence of economies $\{v'_n\}$ each of which, together with the limit v' satisfies (P.1) - (P.7), (P.4'), (Y.1)-(Y.3) and (E.1) - (E.2), (E.4), (E.6)-(E.8) and (E.7') if f_n is an allocation in $\mathcal{E}(v'_n)$ for each n and if $\{f_n\}_{n=1}^{\infty}$ converges to some continuous function f uniformly on $\text{supp } v'_n$, then for every $\epsilon > 0$ there exists N such that $n \geq N$ implies $|f_n(a_0) - f_n(a_1)| \leq \epsilon$ for any two agents a_0 and a_1 of the same type in $\text{supp } v'_n$.

PROOF: This result is a direct consequence of Theorem 4 and Lemma 10 or of the proof of Theorem 6 by noting that if a_0 and a_1 are two agents of the same type, then $\beta^0(p, a_0) = \beta^0(p, a_1)$ where p is the price vector associated with the allocation f which is Walras for v' by Theorem 4. By Lemma 10, $f(a) \in \beta^0(p, a)$ for every a in $\text{supp } v'$ and by (P.4'), $\beta^0(p, a)$ consists of one point for every a in $\text{supp } v'_n$ for every n . Hence

$$\{f(a_0)\} = \beta^0(p, a_0) = \beta^0(p, a_1) = \{f(a_1)\}$$

if a_0 and a_1 are both in $\bigcup_{n=1}^{\infty} \text{supp } v'_n$. The conclusion of the Theorem is then immediate from the definition of uniform convergence on $\text{supp } v'_n$.

9. Conclusions

The main result of this paper is to find conditions under which the core correspondence is almost upper semi-continuous; namely, such that if $\mu_n \Rightarrow \mu$, $\text{supp } \mu_n \subset \text{supp } \mu$ and if $f_n \rightarrow f$ "uniformly on $\text{supp } \mu_n$ " and $f_n \in \mathcal{E}(\mu_n)$, then $f \in \mathcal{E}(\mu)$.¹ A corollary of this result is that under certain conditions, the set $\mathcal{E}^c(\mu)$ of continuous allocations in $\mathcal{E}(\mu)$ is closed with respect to the topology of uniform convergence on $\text{supp } \mu$. In particular, the core of any such finite economy is closed. This result is employed to find conditions under which, if the core of a finite economy is nonempty, then there exists a Walras allocation for that economy. This result is also used to extend Scarf's proof of the non-emptiness of the core to an infinite economy with a finite number of different types of agent.

The quasi-upper semi-continuity of \mathcal{E} is used to prove a result due to Hildenbrand [21]: if μ_n is a finite economy "close enough" to a perfectly competitive economy μ and if f_n is an allocation in $\mathcal{E}(\mu_n)$ close enough to an allocation in $\mathcal{E}^c(\mu)$, then f_n is approximately a Walras allocation for μ_n . This generalizes the Edgeworth-Debreu-Scarf limit theorem and can be used to study less regular sequences of economies, such as the "generalized E-D-S sequence" defined in Section 6.

In Section 6 it was suggested that it would be useful to have a concept of perfectly competitive behavior for finite economies which was not based on the institution of prices. This concept would be used to study trading in a mixed economy where some agents behave as perfect competitors and some do not. Drèze,

¹Of course the term upper semi-continuity gives an inaccurate description of this property of \mathcal{E} because it has not been shown that the conditions on the sequence $\{f_n\}_{n=1}^{\infty}$ correspond to convergence with respect to some topology on S^A . In the rest of this section we shall adopt the awkward, but convenient, name of quasi-upper-semi-continuity.

Gepts and Gabszewicz have suggested that an agent he called perfectly competitive if he is willing to delegate his trading to an arbitrarily large number of brokers. It is not clear whether this definition has any operational meaning. It also encounters certain logical difficulties which are discussed in Appendix 1. Nevertheless, the concept has some intuitive appeal, perhaps because our intuition is based on the few results which have so far been established in this field.

One way to study the usefulness of the Drèze-Gepts-Gabszewicz definition of perfect competition is suggested in Appendix 1 and derives from the original work in [16]. Another approach might be to search for conditions under which if some of the traders are perfectly competitive, then the result would be close to a Walras allocation. For example, one might place restrictions on how different the characteristics of those agents not delegating their trading could be from the characteristics of those agents delegating their trading. It is believed that Theorem 2 would be useful in studying this problem, since the limit economy μ need not be nonatomic. However, it would be necessary to relax the condition that

$$(1) \quad \text{supp } \mu_n \subset \text{supp } \mu \quad \text{for } n=1,2,\dots$$

To this end, we remark that the theorem as stated remains true if condition (1)

is replaced by the assumption that there exists a compact subset K of A such every a in K satisfies (P.6) and that $\text{supp } \mu \subset K$ and $\text{supp } \mu_n \subset K$ for $n=1,2,\dots$ and such that for any other economy λ , if $\text{supp } \lambda \subset K$, then $L(\int_E X d\lambda) \subset L(\int_E X d\mu)$ for any coalition E .

To verify that Theorem 2 remains true under this alteration it is only necessary to remark that:

- (i) the allocation g in the statement of Theorem 1 can, by Theorem 3 in [7], be chosen so $g(a) \succ_a f(a)$ for every a in K and $g(a) \in \text{ri}(X(a))$ for every a in K ;

(ii) the Main Lemma of Section 7 remains valid under the new assumptions.

This revised version of Theorem 2 may be of interest for other purposes.

Theorem 2 relies on the use of continuous allocations. For a finite economy, any allocation can be represented by a continuous function on A (see Lemma 9). However, there may exist an interesting sequence $\{\mu_n\}_{n=1}^{\infty}$ of economies with limit μ and a sequence of allocations f_n in $\mathcal{E}(\mu_n)$ with a limit f which is not continuous. It would be helpful to know how restrictive we are being when we only consider sequences $\{f_n\}_{n=1}^{\infty}$ with a continuous limit. We also want to know whether the assumptions which imply $\mathcal{E}^c(\mu)$ is closed also imply that $\mathcal{E}(\mu)$ is closed.

An assumption which appears to be unduly strong for Theorem 2 is (P.4), convexity of preferences. We remark that Theorem 5 is known to be true without it (see [22]). The essential step in dropping (P.4) appears to be to find conditions on a correspondence φ so that $\int_{\mathbb{E}}^c \varphi d\mu$ is convex even if φ is not convex-valued. Of course, the condition that μ be nonatomic is sufficient to show that $\int_{\mathbb{E}} \varphi d\mu$ is convex.² However, this is easily seen not to be a sufficient condition for $\int_{\mathbb{E}}^c \varphi d\mu$ to be convex. This convexity is used in Theorem 1 of Section 7 and in Theorem 3 in [7].

²This result is due to H. Richter and may be found in [1].

Appendix 1. Examples of economies with unequal numbers of traders of the same type.¹

Debreu and Scarf [15] have shown that for an economy $\mathcal{E}_r = \{a_{ij}; i=1, \dots, m, j=1, \dots, r\}$ gotten by replicating $\mathcal{E}_1 = \{a_1, \dots, a_m\}$ r times and satisfying (P.2), (P.4'), (P.6), (P.7), an allocation $x_{(r)} = \{x_{ij}; i=1, \dots, m, j=1, \dots, r\}$ is in the core of \mathcal{E}_r only if $x_{ij} = x_{ik}$ for all i, j and k . Debreu and Scarf also showed that if (P.4') is weakened to (P.4) and if $x_{(r)}$ is in the core of \mathcal{E}_r , then $x_{ij} = a_{ij} x_{ik}$ and if $x'_{(r)}$ is defined by

$$x'_{ik} = \frac{1}{r} \sum_{j=1}^r x_{ij}, \quad k = 1, \dots, r$$

then $x'_{(r)}$ is also in the core of \mathcal{E}_r (These allocations form the "strict core".)

This symmetry of the allocations in the core of \mathcal{E}_r disappears if there are "concavities" in preferences. This is easily seen if one constructs an Edgeworth box for the trading situation which arises when there are two commodities ($S = R^2$) and two identical traders with indifference curves which are concave to the origin. This symmetry also fails when there are unequal numbers of traders of the same type. We shall now illustrate this with a simple example.

We consider an economy where there are three traders: a_{11} , a_{12} and a_2 . Traders a_{11} and a_{12} are of the same type. The commodity space is R^2 with the first commodity called the "y-commodity" and the second the "z-commodity". The consumption possibilities set for each trader is the nonnegative orthant and the production possibilities set for each trader is $\{0\}$. Further, the preferences of every trader are given by the utility function²

$$U(y,z) = (y+1)(z+1), \quad (y,z) \in R^2, \quad (y,z) \geq 0.$$

¹The examples of this Appendix are due, in part, to Alan Kirman.

²We let $U(y,z) = (y+1)(z+1)$ rather than $U(y,z) = yz$ so that preferences are strictly convex on all of the set $\{(y,z) \in R^2: (y,z) \geq 0\}$.

The two types of trader differ only in their initial resources:

$$\omega(a_{11}) = \omega(a_{12}) = (7,1)$$

$$\omega(a_2) = (2,14)$$

We shall now show that the allocation x defined by

$$\begin{aligned} x_{11} &= (3,3) & x_2 &= (9,9) \\ x_{12} &= (4,4) \end{aligned}$$

is in the core.

First note that

$$(1) \quad U(x_{12}) > U(x_{11}) = 16 \geq 16 = U(\omega(a_{11})) = U(\omega(a_{12}))$$

and

$$(2) \quad U(x_2) = 100 > 45 = U(\omega(a_2)).$$

Thus x is individually rational; that is, none of the coalitions $\{a_{11}\}$, $\{a_{12}\}$ or $\{a_2\}$ can block x .

Inequalities (1) together with the strict convexity of preferences also imply that $\{a_{11}, a_{12}\}$ cannot block x . Suppose that an allocation x' existed such that

$$U(x'_{11}) > U(x_{11}) \quad \text{and} \quad U(x'_{12}) \geq U(x_{12})$$

or

$$U(x'_{11}) \geq U(x_{11}) \quad \text{and} \quad U(x'_{12}) > U(x_{12})$$

and

$$x'_{11} + x'_{12} = \omega(a_{11}) + \omega(a_{12}) = 2\omega(a_{11}).$$

Then the strict convexity of preferences would imply

$$U(\omega(a_{11})) = U\left(\frac{1}{2}(x'_{11} + x'_{12})\right) > U(x_{11})$$

which contradicts (1).

We now show that the coalition $\{a_{11}, a_2\}$ cannot block x .³ Let p be the vector $(1,1)$ in R^2 . p is interpreted as a price vector. Then the value of the resources of $\{a_{11}, a_2\}$ is given by the dot product:

$$(3) \quad p \cdot (\omega(a_{11}) + \omega(a_2)) = 24$$

and the value of the bundle allocated by x to $\{a_{11}, a_2\}$ is

$$p \cdot (x_{11} + x_2) = 24 .$$

Because the slope of the indifference curve through x_{11} (resp., through x_2) is minus one, if x' is any other allocation satisfying either

$$U(x'_{11}) > U(x_{11}) \quad \text{and} \quad U(x'_2) \geq U(x_2)$$

or

$$U(x'_{11}) \geq U(x_{11}) \quad \text{and} \quad U(x'_2) > U(x_2)$$

then

$$p \cdot (x'_{11} + x'_2) > p \cdot (x_{11} + x_2) = 24 .$$

But then relation (3) implies that we cannot have $x'_{11} + x'_2 = \omega(a_{11}) + \omega(a_2)$.

Thus $\{a_{11}, a_2\}$ cannot block x .

Because $p \cdot (x_{12} + x_2) > p \cdot (x_{11} + x_2) = p \cdot (\omega(a_{12}) + \omega(a_2))$, the above argument also demonstrates that $\{a_{12}, a_2\}$ cannot block x . Finally, because

$$p \cdot (x_{11} + x_{12} + x_2) = p \cdot (\omega(a_{11}) + \omega(a_{12}) + \omega(a_2)) ,$$

an analogous argument demonstrates that $\{a_{11}, a_{12}, a_2\}$ cannot block x .

The preceding example exhibited an allocation x in the core for which $x_{11} \neq x_{12}$. However, if x' is defined by

$$(4) \quad x'_{11} = x'_{12} = \frac{1}{2}(x_{11} + x_{12}) \quad x'_2 = x_2 ,$$

³The type of argument made in this paragraph derives from a comparable example presented by H. W. Kuhn in lectures on the theory of international trade.

then it is clear that x' also belongs to the core. We now give an example of a three person economy $\{a_{11}, a_{12}, a_2\}$ for which there exists an allocation x in the core where $x_{11} \neq x_{12}$ and where x' defined by (4) is not in the core. We shall find x by specifying a utility function for a_{11} and a_{12} such that there exist commodity vectors x_{11} and x_{12} with a common marginal rate of substitution different from the marginal rate of substitution at $\frac{1}{2}(x_{11} + x_{12})$. This means that for any given set of prices, the corresponding income-expansion curve should not be a straight line in the commodity space.

The example is gotten by again letting the consumption possibilities sets be the nonnegative orthant of R^2 and by assuming there are no production possibilities. Traders a_{11} and a_{12} have the common utility function⁴

$$U_{11}(y,z) = (y+5)^2 + 40 \log(z+1)$$

and trader a_2 has the utility function

$$U_2(y,z) = (y+1)(z+1)^{10}$$

For a_{11} and a_{12} the marginal rate of substitution at (y,z) , i.e. the negative of the slope of the indifference curve through (y,z) , is

$$MRS_{11}(y,z) = \frac{(y+5)(z+1)}{20}$$

For a_2 we have

$$MRS_2(y,z) = \frac{(z+1)}{10(y+1)}$$

Suppose resources are distributed so that

$$\omega(a_{11}) = \omega(a_{12}) = (0, 19 \frac{1}{2})$$

$$\omega(a_2) = (27, \frac{3}{7})$$

and that an allocation x is defined by

⁴This function might be given an intuitive interpretation by calling the z-good bread and the y-good cake. Examination of the income expansion curve reveals that the z-good is inferior.

$$\begin{aligned}x_{11} &= \left(9, \frac{3}{7}\right) \\x_{12} &= (15, 0) \quad x_2 = (3, 39)\end{aligned}$$

To verify that x is individually rational, we note that

$$U_{11}(x_{11}) \geq 196 + 40(.3) = 208$$

and

$$U_{11}(x_{12}) = 400$$

whereas

$$U_{11}(\omega(a_{11})) = U_{11}(\omega(a_{12})) \leq 25 + 40(3.1) = 149$$

Thus

$$U_{11}(x_{12}) > U_{11}(x_{11}) > U_{11}(\omega(a_{11})) = U_{11}(\omega(a_{12}))$$

Similarly,

$$U_2(x_2) = 4(40)^{10} > 28\left(\frac{10}{7}\right)^{10} = U_2(\omega(a_2))$$

Thus x is individually rational and, since preferences are strictly convex, x cannot be blocked by $\{a_{11}, a_{12}\}$.

Since

$$MRS_{11}(x_{11}) = MRS_{12}(x_{12}) = MRS_2(x_2) = 1,$$

we again choose a price vector $p = (1, 1)$ and note that the commodity bundle received by a_{11} and a_2 together has a greater value than their combined resources:

$$\begin{aligned}p \cdot (x_{11} + x_2) &= 51 \frac{3}{7} \\ &> 46 \frac{13}{14} = p \cdot (\omega_{11} + \omega_2)\end{aligned}$$

Thus $\{a_{11}, a_2\}$ cannot block x . Similarly, $p \cdot (x_{12} + x_2) > p \cdot (\omega_{11} + \omega_2)$ and $p \cdot (x_{11} + x_{12} + x_2) = p \cdot (\omega_{11} + \omega_{12} + \omega_2)$ so neither $\{a_{12}, a_2\}$ nor $\{a_{11}, a_{12}, a_2\}$ can block x .

Thus x is in the core of $\{a_{11}, a_{12}, a_2\}$. To show that x' defined by (4) above is not in the core, we only need to remark that

$$\begin{aligned} \text{MRS}_{11}(x'_{11}) &= \frac{(12+5)\sqrt[3]{14+1}}{20} \\ &= \frac{17}{20} \cdot \frac{17}{14} > 1 \end{aligned}$$

so that $\text{MRS}_{11}(x'_{11}) \neq \text{MRS}_2(x'_2)$.

This example is of interest for the delegation-of-trading model due to Drèze, Gepts and Gabszewicz [16] and explained in Sections 6 and 8 of this paper. Suppose we are given the economy $\mathcal{E}_1 = \{a_1, a_2\}$ where a_2 coincides with the trader of the same name in the preceding example. Trader a_1 is characterized by the preferences

$$U_1(y, z) = \left(\frac{y}{2} + 5\right)^2 + 40 \log\left(\frac{z}{2} + 1\right)$$

and resources

$$\omega(a_1) = (0, 39).$$

Thus, in the notation of Section 6, the traders a_{11} and a_{12} of the preceding example satisfy

$$a_{11} = a_{12} = \frac{1}{2} a_1.$$

Let x and x' be the allocations for $\{a_{11}, a_{12}, a_2\}$ specified in the above example and define an allocation z for \mathcal{E}_1 by

$$z = (x_{11} + x_{12}, x_2).$$

Then z is an allocation for \mathcal{E}_1 which could result if a_1 delegated trading authority to a_{11} and a_{12} . But z is not in the core of \mathcal{E}_1 since

$$\text{MRS}_1(x_{11} + x_{12}) \neq \text{MRS}_{11}(x'_{11}) \neq \text{MRS}_2(x'_2).$$

In this situation we see that it is not rational for a_1 to delegate his trading to a_{11} and a_{12} unless a_2 does likewise, because a_1 may be able to do better for himself (and for a_2) by bargaining directly with a_2 .

The model depicting the delegation of trading authority was devised to permit the definition of perfectly competitive behavior in a finite economy not possessing the institution of prices. An agent is perfectly competitive if he is willing to delegate his trading to an arbitrarily large number of "brokers". The preceding example illustrates that perfectly competitive behavior may not be rational for one trader unless all other traders behave the same way. Of course, this conclusion is part of the conventional wisdom of economics. A more important remark, however, is that a_1 and a_2 may have an inducement to continue trading after $\{a_{11}, a_{12}, a_2\}$ have finished. Thus we cannot hope to simplify the bargaining problem confronting a_1 and a_2 by assuming that a_1 acts as a perfect competitor.

There are situations in which this difficulty does not arise. For example, if a_1 had preferences for which all income expansion paths were straight lines in R^2 , then it would not be possible to construct a counterexample of the type above. However, the remarks on page 28 suggest that the condition that all expansion paths be straight lines may nevertheless, not be a sufficient condition to guarantee that the above difficulty does not arise. It would be interesting to know whether such a counterexample existed.

A different approach has been adopted by Drèze, Gepts and Gabszewicz [16]. They have generalized the result of Debreu and Scarf mentioned at the start of this paper to show that if \mathcal{E}_1 is an economy consisting of l traders of each of m types of trader, if the consumption possibilities set for each of these traders is the nonnegative orthant of R^n , if the preferences of the traders satisfy (P.2), (P.4'), (P.6), (P.7) and if (Y.1) is valid for every trader, then all traders of the same type receive the same commodity vector as long as the number M of traders who do not delegate their trading authority satisfies:

$$(5) \quad 0 \leq M \leq \ell$$

and as long as

$$(6) \quad 1 < \ell. \quad ^5$$

This result can be made more explicit by supposing

$$\mathcal{E}_1 = \{a_1, \dots, a_{m\ell}\}$$

and that these indices are chosen so that a_1, \dots, a_{M_0} are willing to delegate their trading where $M_0 = m\ell - M$. Suppose $x_{(r)} = \{x_{ij}, i=1, \dots, M_0, j=1, \dots, r\} \cup \{x_i: i = M_0 + 1, \dots, m\ell\}$ is in the core of

$$\mathcal{E}_r = \{a_{11}, \dots, a_{1r}, \dots, a_{M_0 1}, \dots, a_{M_0 r}, a_{M_0+1}, \dots, a_{m\ell}\}.$$

Then $x_{ij} = x_{ik}$ for every $i \leq M_0$ and all j and k .

Furthermore, if $x_{(1)}$ is defined by

$$x_i = r x_{ij} \quad i=1, \dots, M_0$$

then $x_{(1)}$ is in the core of \mathcal{E}_1 and $x_i = x_j$ if a_i and a_j are of the same type. Thus those who do not delegate their trading do no better than those who do. We remark that inequalities (5) and (6) seem to guarantee that the characteristics of one type of agent are not monopolized in such a way by those traders who choose not to delegate their trading that these traders can discriminate between those traders who delegate and those who do not. It would be interesting to know whether these inequalities could be weakened without adding compensating restrictions on the characteristics of the agents represented in \mathcal{E}_1 . This appears to be a promising area for further research.

⁵This is Lemma 4, page 16 in [16].

Appendix 2. Measureability Problems

In this Appendix, we shall state the Measureable Choice Theorem and shall prove two lemmas on the measureability of certain functions encountered in the proof of Lemma 1 in Section 7 of this paper.

We remark that a topological space W is a Polish space if it is separable and if it can be metrized in such a way that it is complete. We also recall that a correspondence ϕ from a measureable space (T, \mathcal{J}) to a measureable space (W, \mathcal{W}) is itself measureable if its graph is measureable in $T \times W$ with respect to the product σ -field. Whenever a space W is a topological space, then the associated σ -field \mathcal{W} is the collection of Borel subsets of W . If μ is a measure on (T, \mathcal{J}) , then \mathcal{J}_μ is the completion of \mathcal{J} with respect to μ . A function f from T to W is μ -measureable if it is measureable with respect to \mathcal{J}_μ and \mathcal{W} .

Measureable Choice Theorem. Let (T, μ) be a σ -finite measure space, let W be a Polish space, let ϕ be a nonempty-valued, measureable correspondence from T to W . Then there exists a measureable function $f: T \rightarrow W$ such that $f(t) \in \phi(t)$ for μ -almost every t in T .

This Theorem is proven by Aumann in [34] under somewhat weaker assumptions on the space W .

We now suppose that A is the metric space defined in Section 2 of this paper, \mathcal{a} is the set of Borel subsets, μ is a probability measure on (A, \mathcal{a}) and S is a finite-dimensional, real vector space. $R \cup \{+\infty\}$, the real line with the point $+\infty$ added, has a topology consisting of the open sets of R and sets of the form $\{x \in R \cup \{+\infty\}: M < x \leq +\infty\}$ (i.e. neighborhoods of $+\infty$). The σ -field on $R \cup \{+\infty\}$ is the corresponding collection of Borel sets.

If K is any compact subset of A , then of course K is Polish. Furthermore, a correspondence φ from K to S is measurable only if its graph is, in fact, a Borel set in $K \times S$ (see footnote 1 in Section 7, page 31). These remarks are useful in the proof of

LEMMA 1: If K is compact in A , if $\varphi: K \rightarrow S$ is measurable and if $p \in S$, then the function $\sigma: K \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma(a) = \sup p \cdot \varphi(a)$$

is μ -measurable.

PROOF: Consider the function $\sigma_n: A \rightarrow \mathbb{R}$ defined by

$$\sigma_n(a) = -n \chi_{[\sigma \leq -n]}(a) + \sum_{k=-n2^{n+1}}^{n2^n} \frac{k-1}{2^n} \chi_{[\frac{k-1}{2^n} < \sigma \leq \frac{k}{2^n}]}(a) + n \chi_{[n < \sigma \leq +\infty]}(a)$$

where

$$[\sigma \leq j] = \{a \in K: \sigma(a) \leq j\}$$

$$[j_1 < \sigma \leq j_2] = \{a \in K: j_1 < \sigma(a) \leq j_2\}$$

and where $\chi_{[\sigma \leq j]}(\cdot)$ is the characteristic function of the set $[\sigma \leq j]$.

Now for any a in K and for any n , $n \geq 1$,

$$\begin{aligned} |\sigma_n(a) - \sigma(a)| &\leq \frac{1}{2^n} && \text{if } |\sigma(a)| \leq n \\ |\sigma_n(a)| &= n && \text{if } |\sigma(a)| > n \end{aligned}$$

¹Of course, it is immediate from the definition of the topology on $\mathbb{R} \cup \{+\infty\}$, that to show σ is measurable, it suffices to show that for any real j and k , the sets $[j < \sigma < k]$ and $[j < \sigma \leq +\infty]$ are measurable. However, it seems more difficult to show $[j < \sigma < k]$ is measurable directly than to use the above argument approximating σ by σ_n so that we only have to show that $[j < \sigma \leq +\infty]$ is measurable.

so $\sigma_n \rightarrow \sigma$ pointwise on K . Thus to show σ is μ -measurable, it suffices to show each σ_n is μ -measurable. For this, it is enough that for any two j_1 and j_2 in $R \cup \{+\infty\}$, the sets $[\sigma \leq j_2]$ and $[j_1 < \sigma \leq j_2]$ are μ -measurable. Finally, for this it suffices to show that for any real j , $[j < \sigma \leq +\infty]$ is μ -measurable.¹

Now

$$[j < \sigma \leq +\infty] = \text{proj}_A \{f(G_\varphi) \cap [K \times (j, +\infty)]\}$$

where G_φ is the graph of φ :

$$G_\varphi = \{(a, x) \in K \times S: x \in \varphi(a)\}$$

so G_φ is Borel, and where f is the continuous mapping of the Polish space $K \times S$ into the Polish space $K \times R$ defined by

$$f((a, x)) = (a, p \cdot x).$$

Hence

$$f(G_\varphi) = \{(a, z) \in K \times R: z = p \cdot x, \text{ some } x \in \varphi(a)\}.$$

These conditions on $K \times S$, $K \times R$, G_φ , and f imply that $f(G_\varphi)$ is analytic in $K \times R$ (see Theorem T 13 (c), page 35 in [26]). Then $f(G_\varphi) \cap [K \times (j, +\infty)]$ is also analytic and hence $[j < \sigma \leq +\infty]$ is analytic in K . But then this set is μ -measurable (see III, 24, page 44 in [26]).

LEMMA 2: If φ is a measurable correspondence from A to S , if p is a vector in S and if h is a measurable function from A to S satisfying $h(a) \in \varphi(a)$ for μ -almost every a , then there exists a μ -null set N such that the correspondence ψ defined on $A \setminus N$ by $\psi(a) = \{x \in \varphi(a): p \cdot x > p \cdot h(a)\}$ has a measurable graph with respect to the product σ -field on $A \times S$.

PROOF: Since h is measurable, there exists a sequence $\{h_n\}$ of simple measurable functions converging almost everywhere to h . But then there exists a μ -null set N such that if $A' = A \setminus N$, then $\{(a,x) \in A' \times S: p \cdot x > p \cdot h(a)\} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \{(a,x) \in A' \times S: p \cdot x \geq p \cdot h_n(a) + \frac{1}{m}\}$.

Each of the sets on the right hand side of this equality is measurable since each h_n is simple and measurable. Thus the correspondence ψ defined on $A \setminus N$ has the measurable graph

$$\{(a,x) \in A' \times S: x \in \varphi(a)\} \cap \{(a,x) \in A' \times S: p \cdot x > p \cdot h(a)\}.$$

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