ON THE "FREE-RIDER", CONDITIONAL PROBABILITIES, AND RECURSIVE GAMES

Heinz A. Schleicher

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Princeton University
Econometric Research Program
207 Dickinson Hall
Princeton, New Jersey

ABSTRACT

The problem of "free-riding", which is connected with the notion of a public good or externality, is treated as a question of tax compliance. It is shown that tax evasion is largely a problem of strategy. A Bayesian model is used to deduce an optimal strategy for the inspector faced with finding the tax evader. Individual auditing of a firm is described in terms of a recursive game model (inspector's game) from which one obtains minimax probabilities of detecting at least one violation. This last model is extended to comprise absolute banking secrecy and "obvious" violations.

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Princeton University

1. The Problem

Public goods and externalities involve the problem of "free loaders" or "free riders". By these notions it is meant that the public or collective character of a good like national defense, or in the case of an externality, the spillover effect(s), do not normally allow the exclusion of non-payers from the use of the particular public good, the charging of proper compensation to the externally benefited, or the paying of proper compensation to the externally inflicted person, group or company. It is generally agreed upon that the price mechanism provides a system which may tend to allocate resources optimally between private uses. Such a system has yet to be invented for the allocation of resources to public uses in the face of externalities which cannot be internalized.

Theoretically one can derive Pareto-optimal conditions for a mixed economy which produces private and public goods, given that the citizens reveal their true preferences for both private and public goods. Under even stronger assumptions, (for example, those permitting the construction of a social welfare function), one can determine the conditions for a welfare maximum[1] which

* I am indebted to Professor Oskar Morgenstern, Princeton University, and Dr. F. Sand, Mathematica, for their helpful comments on various parts of this paper and for their reading the final manuscript.

[1] P. A. Samuelson (1954, pp. 387-389 and 1955, pp. 350-356). R. A. Musgrave (1959, pp. 8-9, pp. 80-86).

represents the maximal "bliss point" of society. Apart from the impossibility of specifying a social welfare function, no one could devise an operational system, equivalent to the price system in a pure private goods economy, which would necessarily lead society in a mixed economy to its maximal "bliss point". The greatest obstacle for devising such a system is the problem of free riders.

Nevertheless, economists and politicians have had to tackle this problem and reach a realistic decision of how much of a society's resources should be devoted to private and public, and between public purposes. Economists disposed of the problem in two ways: (1) either it was assumed, again a strong assumption, that voters are revealing their true preferences; or (2) that the government imposed a tax system which leaves the citizen a choice only to vote for the quantity of a public good provided.

The first method is of no interest here, and it would go beyond the scope of this paper to discuss in detail the second, constitutional approach to the problem of compensation.[2] However, it seems necessary to state that the second approach disposes of the free rider problem only if it is assumed that voters do not exaggerate their true demand for the public good in question. Given a particular tax system which allocates the costs and which the taxpayer can expect to stay constant for some time, he may vote for more public park than he would if he had to pay a user charge according to the true marginal utility he derives from a particular size of the park. Because the cost of his share is fixed he will vote for an increase in the public park facilities as long as his marginal utility is positive. Even if taxpayers do not overstate their demand for public goods at a given tax price, the constitutional approach to tax institutions remains necessarily an inefficient one, if one takes into consideration

[2]For a survey of solutions see: J. M. Buchanan (1967, pp. 113-125).

the imperfect information, the inherent uncertainty of the social or political decision process, tax avoidance and tax evasion. Tax avoidance, a legal procedure, makes a tax system less general while everybody retains his vote for the public goods. Additionally, some citizens may take illegal action to modify the tax price they would have to pay under the general fixed tax system. Some estimates show that the amounts involved in tax evasion are considerable. Sample-audit techniques applied to a representative sample of tax returns for 1948, 1949, and 1950 showed that 10 percent of the amount voluntarily declared on filed tax returns was unpaid.[3] This still omits the amount of unpaid taxes on income which is never declared on tax returns. A second group of studies,[4] which compared income data from tax returns with data estimated by the Department of Commerce, showed a gap of 14 percent between estimated and reported income for the year 1946. This result was corroborated by later studies.

These few considerations show that, even granted some strong assumptions, the problem of the free riders is generally unresolved. The purpose of this paper is to investigate the problem of tax evasion, i.e. of violating a given tax system or tax law which is, though somewhat loosely, connected with the free rider problem. Thus, given a tax system, determining how to minimize tax evasion would minimize free riding of a particular kind and hence, within the limited constitutional approach achieve a better allocation of resources.

However, three qualifications should be mentioned. First, if a tax system forces a citizen to pay a predetermined tax price without having much influence on the amount of the public good(s) provided, then a suboptimal

[3] H. M. Groves (1959, p. 38).

[4] H. M. Groves (1959, p. 39).

allocation of resources may result. For instance, the marginal utility of the amount of the public good may be lower than the tax price, and in this case the allocation of resources would be clearly inefficient even under a zero tax evasion. Moreover, the exclusion of a tax liability might only be a correction of resource allocation and not of free riding, although it was done deliberately by the citizen.

Second, under the constitutional approach to tax institutions where every citizen can vote for the optimal quantity of public goods, given a fixed tax price, the final outcome may be inefficient because of the deficiencies present in every political decision process. Therefore, abusive or fraudulent modification of the tax price by citizens may not be a problem of free riding.

Third, the question of tax incidence must be mentioned. The main subject of this paper will be federal and state income taxes. Nevertheless, income taxes may be shifted forward or backward. Hence, paying the tax and bearing the burden need not be the same, and minimizing tax evasion need not lead to a more efficient allocation of resources from the point of view of tax incidence. From these qualifications it follows that the scope of this paper remains rather limited, and the problem of free rider will not be more than touched upon.

2. Possible ways toward a solution.

Neoclassical economic theory is not able to deal with the observable fact of abusive or fraudulent tax behavior in a satisfactory way. The behavioral model of the utility-maximizing citizen or the tax-minimizing taxpayer is not applicable in the case of tax law violation. It is not even applicable in the case of tax avoidance - a legal procedure. Consider a typical citizen who earns a monthly salary. By filing his income tax return and claiming a tax refund for the preceding year, the taxpayer may overstate or even invent some questionable deduction in order to decrease his taxable income It is obvious that such behavior does not simply mean that he minimizes his taxable income, or that he maximizes his utility, for a minimum taxable income which at least partly disregards legal barriers would be a zero taxable income.

Therefore, if an individual disregards the given tax laws as constraints to a minimum taxable income, what does constitute a restriction under which a taxpayer tries to minimize his taxable income? The answer to this question inevitably leads one to the idea of strategic behavior. Assuming a citizen has the intention to violate some tax law with all the implied consequences, then he must fear that his violation will be detected. Such a detection is more probable the better organized is the federal or state tax administration and tax system; i.e., the more noticeable the presence of the revenue agents, the more perfect the audit procedures, the more brilliant the intelligence force. Also, the more the violator deviates by unlawful methods from his legal taxable income, the more he has to fear investigation by federal or state authorities.

These few considerations on tax fraud reveal two points. (1) In every case in which a violation of a tax law occurs and is detected, two parties are involved: the taxpayer or, violator, and the government whose representative may be called the inspector. Generally, the intentions of both parties are strictly opposed, assuming explicitly that the taxpayer consciously wants to violate tax law, and the inspector wants to detect every tax violation. (2) This situation suggests treating such a problem as a game of strategy - in this particular case as a constant-sum-game where the inspector and the violator have optimal strategies of detecting, or evading detection, respectively. The constant sum model implies that it suffices to evaluate the optimal strategy of only one of the players since the maximum gain one player can secure is exactly the loss which the loser suffers in the worst possible case. This minimax strategy may give the inspector the highest assured probability to discover a violation.[5] But this latter decision rule may also be based on a measure of relative suspiciousness caused by events claimed by the violator as tax deductible. By using a Bayesian approach[6] the inspector may select events for inspection on the basis of their evaluated relative suspiciousness, their costs, and expected gains in tax payment.

Both approaches will be used in this paper. First, a Bayesian model of relative suspiciousness of various events will be presented. Second, the problem is treated with the framework of an inspection game. At this stage, the model is too simple to be readily used for auditing purposes, but it shows one possible way of devising effective auditing procedures. In addition, the results point to various proposals for federal tax reform regarding the enforceability of a tax law.

3. Further classification of the issue.

After stating the problem and ways toward a possible solution, a few conceptually important distinctions have to be introduced. Six different situations have to be distinguished:

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[5]Models of games of this type have been analyzed by M. Dresher (1962), H. W. Kuhn (1966), M. Maschler (1966, 1967). For a general treatment of games of strategy see J. Von Neumann, and O. Morgenstern (1953).

[6]C.f. M. Davis and H. W. Kuhn (1966).

1. The taxpayer has perfect or imperfect information on positive tax laws which are relevant for him.

2. The taxpayer pursues willfully an honest or dishonest conduct in his tax compliance.

3. The taxpayer has either a problem of interpretation of a particular tax law, paragraph or legal term, or he has no such problem.

Moreover, interesting combinations of these different assumptions such as the following are possible. It is, of course, always assumed that the inspector has perfect information and no problem of interpretation regarding the tax law. Suppose that perfect information prevails, no problem of interpretation exists, and that all taxpayers are fully compliant. Then estimated and actual tax payments are exactly equal. This is the unrealistic case which corresponds to the perfect competitive market in price theory: no uncertainty exists and everyone reveals his true preferences.

Next, suppose that there is imperfect knowledge of existing tax laws, no problem of interpretation, and honest behavior. The imperfect information can lead to actual tax payments which either over-report or under-report the true tax liability. The first possibility leads to a problem of fiscal equity which has to be dealt with if this aim is to be taken seriously. Given a tax system which precisely expresses the individual or group utilities for a given public expenditure program, and given that the actual tax payments exceed the legal tax liabilities of one or several citizens, this implies inefficiencies from an economic point of view which are in no way different from those inefficiencies which result from plain tax avoidance.

Also of considerable interest is the second possibility of tax payments which are too low. Underlying this may be unintended abuse, but whether the under-reporting of tax liabilities is a consequence of imperfect knowledge or deliberate non-compliance is of no importance here. The aim is to show possible ways of enforcing the legally stipulated tax payments. Hence there is no need for a different treatment of the above case and the model where perfect information, no problem of interpretation, and willful abusive or fraudulent behavior are present. Again, from a moral or legal point of view, both cases are fundamentally different, but they are the same from the point of view of collecting the legally stipulated tax payments.

Imperfect information can occur together with deliberately abusive or fraudulent behavior. The result may be too high, too low or correct total tax payments. The latter is possible because the incorrect behavior may be neutralized by the existence of imperfect information. Correct total tax payment, however, need not prevent detecting the sources of error. There may be errors which it is probably not feasible to have segregated or estimated by the federal or state revenue agents. "Some taxpayers will be more or less successful in concealing information necessary to the correct determination of tax liability."[7] However, "errors that experienced Bureau of Internal Revenue investigators would find if all of the returns of tax-payers were audited"[8] can be detected, provided that some particular methods or strategies are used by the inspectors. Again, from the point of view of the free-rider, the two cases of overpayment or underpayment of taxes are of particular interest.

[7] M. Farioletti (1949, p. 150).

[8] Ibidem.

Finally, the problem of interpretation has to be considered. In various areas of expenses, such as entertainment deductions for business, a clearcut distinction between a deductible and a nondeductible expense allowance is almost impossible. An example is the amendment to the Public Debt Limit and Rate Extension Bill of 1960. The amendment sought to first disallow entertainment expenses, other than expenses paid or incurred for food or beverages; second, to limit gift expense deductions with respect to any donee to $10.00 per year; and third, to prohibit deductions for dues or initiation fees in social, atheletic, or sporting clubs or organizations. This amendment which was later dropped by the House and Senate conferees shows clearly the difficulties involved. In discussing this amendment on the Senate Floor, Senator Clark stated that "legitimate advertising expenses cannot be considered gifts under any reasonable construction of the term." On the House floor, however, the chairman of the Ways and Means Committee observed that "it is difficult to devise a statutory distinction between gifts that are just plain gifts and gifts that are advertising."[9]

An almost natural consequence of such imprecise legal definitions is the over-extension of deductions. Granted for this such "weak points" have to be given particular interest in audit controls. Under certain assumptions, such cases do not cause greater difficulties than over-extensions of deductions resulting from imperfect information or dishonest conduct. These assumptions have to be specified in the next section.

These few considerations of the underlying causes of over-reporting or under-reporting of tax liabilities have shown that dishonest conduct of the taxpayer is only one of several sources of non-compliance. This has to be kept in mind when the simplifying notion of the violator is used in the following section.

[9] D. A. Lindsay (1961, p. 246). See also D. T. Smith (1961, pp. 68-75).

4. The Necessary Assumptions and Their Implications.

The models for auditing federal and state taxes which will be presented are essentially of a stochastic nature. This implies certain properties which have to be described and at least intuitively checked for their empirical content. For the models to apply it is necessary to assume that certain events occur which put different indicators in various states. These events, indicators, and states normally vary with different taxes and groups of income. That no particular tax or group of income is completely void of non-tax compliance becomes evident from several studies done.[10] The models will therefore have to be adapted for different taxes and different groups of income.

Presently we are interested in the administration of an income tax. Let one particular kind of income be rental income, particularly residential rent. Deliberate tax evasion, as far as tax returns are filed at all, can result in two ways; either in under-reporting of gross rent; or as an exaggeration of expenses. Both the reporting of gross rent and the reporting of expenses can be seen as events. If the stated gross rental income is correct, the event is true; if it is too high or too low, the event is false, and in other words, a violation has occurred.

It is difficult to devise indicators and find out the states of these indicators, given that the event reported rental income was true or false. Two ways are possible. (1) For office-auditing, one may check property-tax records and look for additional income sources of the landlord in question. There may be other possibilities which can reveal plausible probabilities for rental income actually received but not reported at all or else under-rated. One possibility might be that expresnes which exceed a certain common ratio between gross rental

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[10] H. M. Groves (1959, p. 37). R. Haig and C. Shoup (1952, Chap. V). For some of the continental experience in tax-compliance see: G. Schmölders (1960, pp. 113-128).

income and expenses imply an under-reporting of gross rental income. (2) For field-auditing, plausible estimates may be available in most cases. But it is important to note that this method has to consider every particular case. Stochastic models are not applicable. Indicators and their mutually exclusive states are perhaps easier to determine, given the event, i.e., expenses. Expenses in the case of residential rent consist of a given number of items such as: repairs, maintenance, furniture depreciation, equipment depreciation, telephone expenses, taxes, water etc. In all these cases there exist plausible ratios between gross rent, total and particular expenses.[11] For instance, there is the belief among real estate men that repairs and maintenance require on the average one month's gross rental.[12] From this information one may conclude the following: the indicator, repairs and maintenance, is in state 2 (= $\frac{1}{12}$ of gross rent) with a high probability, maybe 70 percent, and in states 1 and 3 with low equal probabilities, maybe 15 percent, respectively. These are the possible states of the indicator: repairs and maintenance, given the event expenses is true. If a violation prevails, the distribution of the random variable, state of indicator repairs and maintenance, is probably different: the stated expenses within particular item are either too high, or too low. Three possible states are again most likely, but given a violation, states 1 and 3 may occur with a probability of 40 percent each, and state 2 with a probability of 20 percent.

Besides the events, indicators, and their states, the Bayesian model implies the knowledge or "intensity of belief" that an event is suspicious at all. It is relatively easy to deal with this requirement. In fact, present auditing techniques of the Internal Revenue Service have developed methods which seem to deal rather effectively with the problem of detecting suspicious events.

[11] H. M. Groves (1958, pp. 294-297).

[12] Ibidem.

It is important to nitice that having a suspicion about an event is only the first step toward efficient auditing. The second step then has to deal with the relative suspiciousness, which is extensively treated in the following sections. The third and final step would be to audit the selected returns.

The Bayesian model which will be presented in the following sections relies crucially on some a priori probability that a violation has occurred and on the relative frequencies or probability distributions of states of indicators. It is assumed that it is possible to determine such ratios or probabilities for many- if not all - of the relevant items which appear on a federal or state income tax return. Before justifying this belief, a word must be said about taxpayers who willfully do not file returns or who do, but under-report incomes.

It has already been mentioned in the previous section that not all errors pervading the internal revenue are likely to be found by a particular Audit Control Program or by the proposed models. All the methods mentioned are devised to acceptably administer a tax with a limited staff of skilled personnel. This implies primarily correspondence, office-audit, and to a lesser extent, field work at least as far as mass taxes are concerned. However, the cases where no returns are filed, or returns are filed but particular income items remain unmentioned, do require some amount of field work in order to be detected at all. As far as this is necessary, the proposed procedures work only partially or not at all.

This need not be true for a second class of tax-administration problems: namely, those returns that are based largely on information that only the taxpayer himself can disclose. This group of returns includes farm, business and professional income. In the case of farm income, information at the source could be extended to include major payments of gross income to farmers, and auditing guided by expense ratios and periodic net-worth data could prove to be effective.[13] As far as typical expense-ratios, i.e., events and their probable states of indicators, can be determined, no problems exist regarding office auditing.

Apparently small business and professional returns present much the same problem to a lesser degree. Estimates of typical gross incomes and plausible expense ratios may be made.

Thus, having disposed of the mass of individual tax returns in a most efficient way, the remaining resources can be allocated efficiently to auditing of professional income and big business. Here fairly regular field-auditing will be the rule. Nevertheless, there will remain some resource constraint, and hence more efficient and less efficient approaches are to be discerned. It is here that a general rule of behavior for the inspector will be proposed which guarantees him the highest probability of detecting non-compliance, given that some violation actually has occurred.

Knowing about events, indicators and states of indicators, the crucial question is: how can the relevant events, the important indicators, and their most probable states in case of non-compliance and compliance be determined? Research in this area has developed along two lines. One applied by the Internal Revenue Service to the tax returns of various years used the sample-audit technique. A more or less similar method was applied to particular area studies. The second method, employed in certain government and National Bureau of Economic Research Studies, attempted a reconciliation of national income figures and aggregate tax-return data.[14] The aggregate income figures were available from estimates of aggregate income by source, prepared by the department of commerce,

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[13] H. M. Groves (1959, p. 53, 1958, pp. 297-201).

[14] H. M. Groves (1959, p. 38).

office of Business Economics, and the National Income Division. The size distribution was to be estimated by field surveys covering representative samples of families and single individuals, and by reports on workers with wage credits to the Bureau of Old Age and Survivors Insurance filed by employers. These data could be compared with the incomes reported in Federal income tax returns.[15]

Both methods lead to some common results. Among them is the general finding that tax returns account for very different proportions of the several income sources, e.g. extremely good coverage of wages and salaries and relatively poor coverage of farm and entrepreneurial income, interest and rent. While the sample-audit technique is a method to determine the sources of non-compliance, or events, and their changing of basic ratios and percentages, (i.e., indicators and their probable states), the comparison of income estimates with income reported for tax purposes can render data on income classes which show either high degrees of compliance or high percentages of non-compliance. Both methods give hints on the *a priori* suspiciousness of events.

To summarize the major assumptions of this section: Each item on a federal or state income tax return can be regarded as a true or false event. Depending on whether the event is true or false, one or several indicators are in mutually exclusive states. This implies that each indicator represents a probability distribution depending upon whether the event is true or false. Some of these probability distributions may be degenerate, being either 0 or 1. This being the case with arithmetic mistakes or incorrectly reported incomes, such as wages, salaries and dividends, it does not cause any difficulty for the proposed procedure. Furthermore, indicators and their probable states are most likely to differ between different classes of income according to its source, and between

[15] S. F. Goldsmith (1951, pp. 266-377).

income brackets in the same class. Taking these assumptions for granted, it will be shown in the following section that by using a Bayesian Model the relatively most suspicious tax returns for each event and each income group can be singled out by computerized methods for further investigation.

5. Auditing of a Mass Tax Based on a Bayesian Model.

5.1 Definitions and Assumptions

Suppose there are n events. Some of these events may be true events; others may imply some form of tax evasion. Let there further be k indicators. Each indicator may be in any one of a number of discrete states depending on the particular event to which the indicator responds. The i^{th} indicator indexed by j_i can occupy any one of m_i states. The state of the system of indicators corresponding to a particular event is specified by the vector $S = (j_1, j_2, j_3, \ldots, j_k)$, i.e. indicator 1 is in state j_1, indicator 2 is in state j_2, etc. Further, let $p_i(j)$ be the probability that indicator i is in state j, given that the antecedent event was a violation, and let $q_i(j)$ be the probability that indicator i is in state j, given that the antecedent event was a true event.

These probabilities are assumed to have the following characteristics:

$$\sum_{j_i=1}^{m_i} p_i(j_i) = \sum_{j_i=1}^{m_i} q_i(j_i) = 1 , \qquad 0 < p_i(j) < 1$$

$$0 < q_i(j) < 1 .$$

Because the number of possible, mutually exclusive states of an indicator i is m_i, the probabilities that indicator i is in state j must sum to one, given a true event and a violation, respectively.

It need not be true that $p_i(j) + q_i(j) = 1$ because there may be states for which an indicator will tend to assume a high probability, whether a violation or no violation has occurred. Further, the joint probabilities are $p_i(j_i)\,p_k(j_k)$ and $q_i(j_i)q_k(j_k)$, considering the states of indicators as random variables, independently distributed for different indicators. Given these assumptions, the conditional probability that a violation V has occurred can be derived if the system of indicators is in state S .

5.2 The Model

Given two events S and V , $P(S \cap V)$ is the joint probability of the event defined by the occurrence of S and V . $P(S|V)$ is the conditional probability of event S , given that event V has occurred. $P(S|\tilde{V})$ is the conditional probability of S , given event V has not occurred, i.e., $\tilde{V}$ is the complement of V . P(S) is the probability that S occurs.

Let $S = (j_1, \ldots, j_k)$ be the state of the system of indicators, and let V be a violation of a tax law, then the a posteriori probability of a violation, given a particular state of the system of indicators, is[16]

$$P(V|S) = \frac{P(V \cap S)}{P(S)} \quad . \tag{1.1}$$

This conditional probability can be stated in terms of known probabilities by using a simple geometrical representation.[17] Assuming that the total possibility space, the unit square, is divided into the subset of violations V , the subset on non-violations $\tilde{V}$, and in states of the system of indicators S and $\tilde{S}$. The 4 disjoint subspaces constitute independent sets of probabilities.

[16] W. Feller (1950, Second Edition 1966, pp. 108-114).

[17] Compare J. G. Kemeny, J. L. Snell, G. T. Thompson (1963, pp. 138-143).

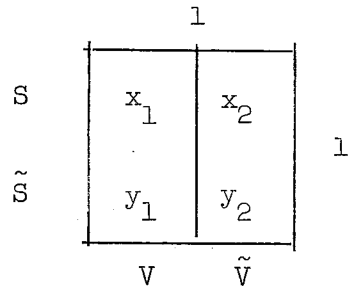

Figure 1

One can infer from Figure 1 the following probabilities:

$$P(V) = x_1 + y_1$$
$$P(\tilde{V}) = x_2 + y_2$$
$$P(S|V) = x_1/x_1 + y_1$$
$$P(S|\tilde{V}) = x_2/x_2 + y_2 ,$$

furthermore

$$x_1 = P(S|V) \cdot P(V)$$
$$x_2 = P(S|\tilde{V}) \cdot P(\tilde{V}) .$$

Hence

$$P(V|S) = \frac{P(V \cap S)}{P(S)} = \frac{x_1}{x_1+x_2} = \frac{P(S|V) \cdot P(V)}{P(S|V) \cdot P(V) + P(S|\tilde{V}) \cdot P(\tilde{V})} \qquad (1.2)$$

which is a simple form of Bayes' theorem.

Assuming now that t is the a priori probability of a violation, i.e., a violation prior to consulting the evidence of the indicators, there is

$$P(S|V) \cdot P(V) = t \prod_{i=1}^{k} p_i(j_i) , \qquad \text{for } 1 > t > 0 ,$$

$$P(S|\tilde{V}) \cdot P(\tilde{V}) = (1-t) \prod_{i=1}^{k} q_i(j_i) , \qquad (1.3)$$

where Π is the common notation for a product. Substituting (1.3) into (1.2) one obtains:

$$P(V|S) = \frac{t \prod_{i=1}^{k} p_i(j_i)}{t \prod_{i=1}^{k} p_i(j_i) + (1-t) \prod_{i=1}^{k} q_i(j_i)} . \qquad (1.4)$$

For $t \neq 0$ and each $p_i(j_i) \neq 0$, $i=1,\ldots,k$, (1.4) can be rewritten:

$$P(V|S) = \frac{1}{1 + (\frac{1}{t} - 1) \prod_{i=1}^{k} r_i(j_i)} \qquad (1.5)$$

where $r_i(j_i)$ is defined as $q_i(j_i)/p_i(j_i)$. Given a state S of the indicators and the probabilities $p_i(j_i)$, $q_i(j_i)$ and t, the (conditional) a posteriori probability that a violation has occurred can be determined.

5.3 An Example

Let there be three indicators and four possible, mutually exclusive states of each indicator. Suppose the respective probabilities depending upon whether the event was true or not, are as follows:

	State 1	State 2	State 3	State 4
Indicator 1	$p_1(1) = 0.88$	$p_1(2) = 0.08$	$p_1(3) = 0.03$	$p_1(4) = 0.01$
	$q_1(1) = 0.02$	$q_1(2) = 0.03$	$q_1(3) = 0.05$	$q_1(4) = 0.90$
Indicator 2	$p_2(1) = 0.40$	$p_2(2) = 0.10$	$p_2(3) = 0.20$	$p_2(4) = 0.30$
	$q_2(1) = 0.40$	$q_2(2) = 0.15$	$q_2(3) = 0.20$	$q_2(4) = 0.25$
Indicator 3	$p_3(1) = 0.10$	$p_3(2) = 0.40$	$p_3(3) = 0.50$	$p_3(4) = 0.00$
	$q_3(1) = 0.20$	$q_3(2) = 0.20$	$q_3(3) = 0.60$	$q_3(4) = 0.00$

Assume further that the a priori probability of a violation $t = 0.5$ and that the system of indicators is in states $S_1(1,1,1)$, $S_2(1,2,1)$, $S_3(2,2,2)$, $S_4 = (4.2,3)$, respectively. Then, applying (1.5) one obtains:

$$P(V|S_1) = \frac{1}{1 + \frac{0.02}{0.88} \cdot \frac{0.40}{0.40} \cdot \frac{0.20}{0.10}} = 0.956 \ .$$

In the same manner, one obtains for

$$P(V|S_2) = \frac{1}{1 + \frac{0.02}{0.88} \cdot \frac{0.15}{0.10} \cdot \frac{0.20}{0.10}} = 0.935 \ ,$$

$$P(V|S_3) = \frac{1}{1 + \frac{0.03}{0.08} \cdot \frac{0.15}{0.10} \cdot \frac{0.20}{0.40}} = 0.780 \ ,$$

$$P(V|S_4) = \frac{1}{1 + \frac{0.90}{0.01} \cdot \frac{0.15}{0.10} \cdot \frac{0.60}{0.50}} = 0.006 \ .$$

To show how these results are influenced by the a priori probability t, let $t = 0.40$. Then

$$P(V|S_1) = \frac{1}{1 + (\frac{1}{0.40} - 1) \frac{0.02}{0.88} \cdot \frac{0.40}{0.40} \cdot \frac{0.20}{0.10}} = 0.935$$

$$P(V|S_4) = \frac{1}{1 + (\frac{1}{0.40} - 1) \frac{0.90}{0.01} \cdot \frac{0.15}{0.10} \cdot \frac{0.60}{0.50}} = 0.004 .$$

Several points are worth noting. The distribution of the random variables' states shows different degrees of reliability for the different indicators. The most reliable seems to be indicator 1 . Less reliable are indicators 2 and 3 . Indicator 2 shows little difference in probabilities between states 1 and 4 , whether a violation has occurred or not, and is therefore much less reliable than indicator 1 . Thus in instances like $S_2(1,2,1)$ where the readings conflict with each other, indicator 1 tends to dominate the other two indicators.

The derived a posteriori probabilities depend on the a priori probability t that a violation has occurred. In some instances, given system $S_1(1,1,1)$ and $t = 0.40$, the percentage change of the conditional probability is small; in the case of $S_4(4,2,3)$ and $t = 0.40$, the percentage change of the conditional probability is 33.3 percent - which is considerable. Various other examples may easily be provided. However, the relative conditional probability that a violation has occurred does not depend on the a priori probability t that a violation has occurred.[18] Denoting $\prod_{i=1}^{k} r_i(j_i)$ by R_i, then a violation is more likely to have occurred if

$$\frac{1}{1 + (\frac{1}{t} - 1) R_1} \geqq \frac{1}{1 + (\frac{1}{t} - 1) R_2} \qquad (1.6)$$

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[18] M. Davis and H. W. Kuhn (1966, p. 226).

This follows from (1.5), but this relative conditional probability is independent of a given t, which means that a violation was equally likely for the two suspicious events. Because $\frac{1}{t} > 0$, (1.6) is true if and only if

$$R_1 \leqq R_2 .$$

However, this condition is given independently of t.

Therefore, the decision rule of the inspector requires: to check all tax returns for suspicious events, to order the events according to their relative suspiciousness, and select the relatively most suspicious tax returns for further auditing. If large resources are devoted to inspection, the inspector may carefully inspect every suspicious event. Such behavior is strongly endorsed by equity considerations. By cost-effectiveness considerations, however, the inspector normally will confine inspections to the relatively most suspicious events and especially to the higher income brackets.

5.4 The Degenerate cases.

One of the simplifying assumptions in section 5.1 was

$$0 < p_i(j) < 1$$

and

$$0 < q_i(j) < 1 \qquad \text{for} \qquad \begin{matrix} i=1,\ldots,k \\ j=1,\ldots,m_i \end{matrix} .$$

Interesting cases are where $p_i(j) = 0$ and $q_i(j) > 0$, or where $p_i(j) > 0$ and $q_i(j) = 0$. The first state of indicator i clearly implies a true event, and hence from (1.4) follows $P(V|S_1) = 0$. The second state of indicator i shows a violation has occurred and hence $P(V|S_2) = 1$. The second possibility is of particular interest for the inspector. This is one of the cases where either mistakenly (imperfect information, problem of interpretation or arithmetic error) or deliberately tax was evaded. Of minor interest are the cases where $p_i(j) = 1$ and $q_i(j) < 1$ and vice versa. High probabilities $p_i(j)$ or $q_i(j)$ merely exert a greater influence on $P(V|S)$. If $p_i(j) = 1$ and $q_i(j) < 1$, then a violation is more likely to have occurred; if $p_i(j) < 1$ and $q_i(j) = 1$ the event is relatively less suspicious.

The most clear-cut case occurs when $p_i(j) = 0$ and $q_i(j) = 1$, or $p_i(j) = 1$ and $q_i(j) = 0$. Under these circumstances true events and violations, respectively, are proven to be certain. Wages, salaries, and dividend income are important examples. Because of the system of tax deduction at the source and the reporting of this income to centralized federal revenue agencies, there is almost no possibility of reporting incorrect income figures in income tax returns. Receiving income through various expense accounts is another matter, of course. It causes no difficulties to check the centrally stored income data with the figures reported in federal income tax returns by computerized methods.

To be complete, the possibility of $0 \leqq t \leqq 1$ has to be mentioned. If $t = 1$, the violation is sure and no index of suspiciousness has to be evaluated. $t = 0$ proves that there exists no suspiciousness whatsoever, and the event is not audited.

5.5 Summary

The basic idea of the preceding model is that mass auditing of the individual income tax should rely on scientific criteria of effectiveness. Present auditing techniques used by the Internal Revenue Service already fulfill this requirement to some extent.

The proposed course of action is to select suspicious events and hence income tax returns, to evaluate the relative suspiciousness of each event, and to order them according to their relative suspiciousness. Cost and equity considerations will then single out those events and returns which are subjected to further investigation. It is hoped that the evaluation of the relative suspiciousness should help to make auditing more efficient.

6. Auditing of Corporation Income Based on Recursive Inspection Games

6.1 Definitions and Assumptions

The conditional probability model of section 5 dealt with the auditing of a mass tax. The model presented in this section depicts auditing of corporations where one or several inspectors have to do field work, checking books and receipts. This situation can best be described as a face-to-face strategic problem: i.e., a two-person game for at least those cases where there exists a violator who deliberately tries to be non-compliant in his income tax payments.

The main body of assumptions in sections 3 and 4 is applicable. It will be assumed that the director of finance, who will be called the violator, has two possible strategies in this game: either to violate or not to violate. This choice of action repreats itself during a number of ℓ stages. Hence there exists a recursive relation between the ℓ stages. These different stages may, but need not, be conceived as a sequence of time intervals. In the case of one particular audit of a given corporation, it may be assumed that the balance sheet and the income statement contain ℓ events which are either true or violations.

The inspector is assumed to have m inspections available where he checks an event carefully. If the number of inspections match exactly the number of events, there does not exist a strategical problem. The inspector is very likely to find every violation under these circumstances. In the case of $m < \ell$, a true choice problem emerges for both the inspector and the violator. The violator faces the problem of optimally allocating his n violations over ℓ stages, i.e., with the highest assured probability of not being detected. Likewise, the inspector must adopt an optimal decision rule which assures him a maximum probability of detecting every violation.

It is almost completely unlikely that the inspector audits every single event when field auditing. The main reason is a cost-effectiveness consideration. Generally, the inspector will audit the main items, or events, which imply high probabilities of at least partial non-compliance. Hence, even in individual field-auditing, the inspector normally inspects a sample of events and not all events. Thus it is reasonable to assume that $\ell > m \geq n$ and that both the violator and the inspector have two strategies, respectively: to violate or not to violate, and to inspect or not to inspect.

As in section 4, it may be assumed that an event puts one or several indicators in various states. The probabilities that the indicators are in these states depend on whether the event was true or a violation. A balance sheet and an income statement exhibit a large number of possible events and hence possible violations. The probabilities that indicators are in various states according to whether the event was true or a violation will probably be different for every event. This fact makes the problem very difficult to manage.

It is only for simplicity that these different indicators and their possible states will be combined to a single index of suspiciousness, which is assumed constant over all stages of the game. This index may be computed by taking some average joint probability of indicators and states, given a violation or given a non-violation, respectively. However, this seems to be a very strong assumption which has to be eliminated if a more realistic model is developed. But given these simplifying assumptions a minimax probability of detecting at least one violation can be derived. This minimax probability implies, as mentioned above, the highest assured probability of catching the violator. Simultaneously it implies the (qualitative) decision rule for the inspector: to save an inspection for the "last" event.

6.2 The Model

Suppose there are ℓ events. The game occurs in ℓ stages. At each stage the inspector faces one of three signals: a true event, a violation, or a doubtful event. The latter is expressed in the index of suspiciousness. Assume that a true event produces a suspicious signal with probability $1-Q$, and the signal true event with probability Q. The violation produces a suspicious signal with probability $1-P$, and the signal violation with probability P.

At each stage the violator may have violated or not. If he decided to act lawfully then a true event occurs at that stage. At each stage, the inspector has the option of inspecting or not. If he inspects a doubtful signal generated by a violation, he finds the violation with probability r. The inspector is allowed at most m inspections; the violator breaks the law at least n times.

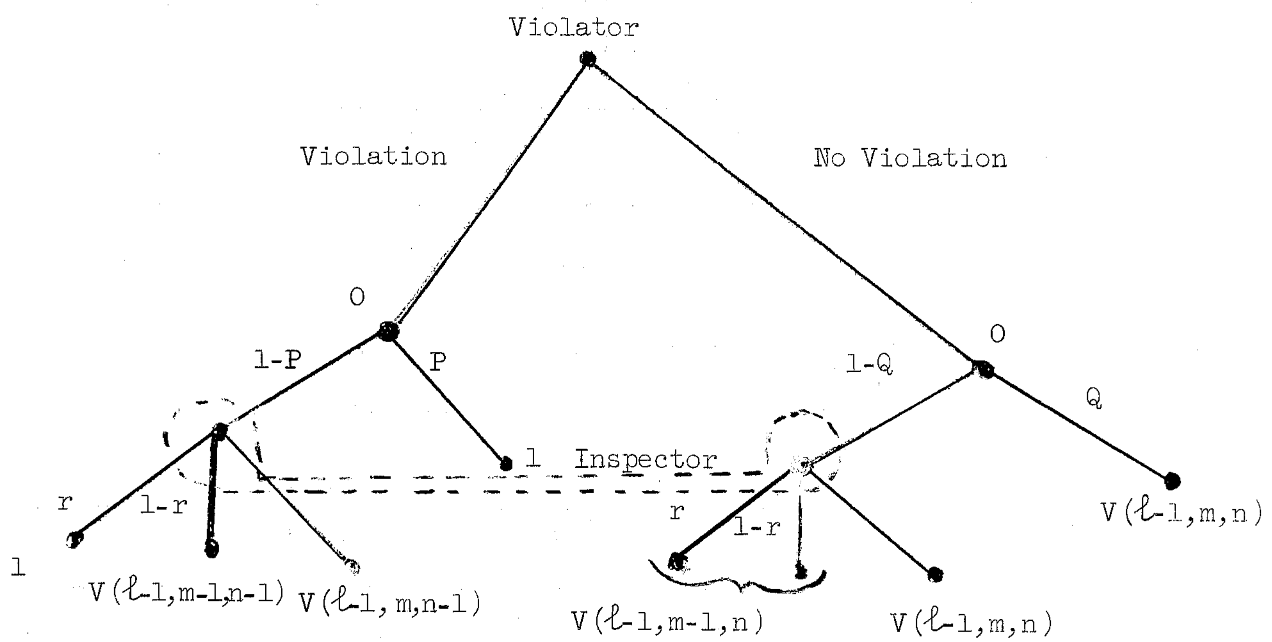
The two person constant-sum recursive game depends on the parameters P, Q, r, ℓ, m, and n. It will be denoted as $\Gamma(\ell,m,n)$. The payoff in this game is assumed to be 1 if a violation is detected, and 0 if no violation is detected. <u>The value of game $\Gamma(\ell,m,n) = V(\ell,m,n)$ is then the minimax probability that at least one violation is discovered.</u> Simultaneously this is the highest assured probability of detecting one violation that the inspector can achieve. The informational pattern of this game clearly shows the recursive nature. By using a game tree, the first stage of $\Gamma(\ell,m,n)$ is drawn. The tree graph is just another way of expressing the rules of this recursive game.

The first move is made by the violator. If he made a violation, chance (0) makes a move: it either signals a suspicious event with probability $(1-p)$, or a violation with probability P. The next move is made by the inspector: either to inspect or not to inspect the suspicious event. If he inspects, he finds the violation with probability r or he faces the game $\Gamma(\ell-1, m-1, n-1)$, which has the value $V(\ell-1, m-1, n-1)$. In case he does not inspect, he plays $\Gamma(\ell-1,m,n-1)$ with value $V(\ell-1, m, n-1)$.

If chance produces the signal violation, he then knows definitely that the violator broke the law; hence the payoff to the inspector will be 1.

Similar considerations apply to the move of no violation by the violator. It should be mentioned that the suspicious event signal constitutes an information set for the inspector, i.e., he does not know at which of the two vertices he is. If $r = 1$, then after having made an inspection he will know a posteriori which move he made. This will give him the necessary amount of information to know which game $\Gamma(\ell,m,n)$ he will play next.

Figure 2: Tree of first stage of $\Gamma(\ell,m,n)$



The normal form of this recursive game is equal to the following 2 x 2 matrix game,[19] where both players have 2 strategies at each stage of the game.

		Violator: violate	Violator: do not violate
Inspector	inspect the suspicious event	$(1-P)(1-r)\,V(\ell-1,m-1,n-1) + (1-P)\,r + P$	$(1-Q)\,V(\ell-1,m-1,n) + Q\,V(\ell-1,m,n)$
	do not inspect the suspicious event	$(1-P)V(\ell-1,m,n-1) + P$	$V(\ell-1,m,n)$

One difficulty already mentioned above is implied by this matrix: namely that the inspector knows the violator has made a violation even without having detected this violation.[20] This is a strong assumption but may be fulfilled in reality, if intelligence or denunciation is present. However, this difficulty can be disposed of either by assuming $r = 1$, or by reducing the number of possible

[19] M. Dresher (1962), and H. W. Kuhn (1966).

[20] H. W. Kuhn (1966, p. 174).

violations to $n = 1$. For this special situation where the violator has but one violation to perform over the period of the game, the above matrix reduces to:

		Violator: violate	Violator: do not violate
Inspector	inspect the suspicious event	$(1-P)\,r + P$	$(1-Q)\,V(\ell-1,m-1,1) + Q\,V(\ell-1,m,1)$
	do not inspect the suspicious event	P	$V(\ell-1,m,1)$

For this game solutions can be generated by proper assumptions for P, Q and r .

Assume $P = Q = \frac{1}{2}$ and $r = 1$, then the recursive game matrix becomes

		Violator: violate	Violator: do not violate
Inspector	inspect the suspicious event	1	$\frac{1}{2}\,V(\ell-1,m-1,1) + \frac{1}{2}\,V(\ell-1,m,1)$
	do not inspect the suspicious event	$\frac{1}{2}$	$V(\ell-1,m,1)$

Let further be $V(\ell,1,1) = V_\ell$, i.e., it will be assumed that there is 1 possible violation and 1 possible inspection in ℓ stages. This is an additional simplifying assumption which may be dropped in a more detailed analysis. The modified payoff matrix is

		Violator: violate	Violator: do not violate
Inspector	inspect the suspicious event	1	$\frac{1}{2}\,V_{\ell-1}$
	do not inspect the suspicious event	$\frac{1}{2}$	$V_{\ell-1}$

because $V(\ell-1,m-1,1) = 0$. The value of this game is[21]

$$V_\ell = \frac{3\,V_{\ell-1}}{2 + 2\,V_{\ell-1}} .$$

[21] J. G. Kemeny, J. L. Snell, G. T. Thompson (1963, pp. 283-284). To obtain the value one just has to apply the algorithm for determining the value of a game without a saddle-point.

To find a solution to this non-linear difference equation is not a simple matter.[22] The multitude of possible solutions can be restricted by setting $v_\ell = \frac{x_\ell}{y_\ell}$. It follows

$$\frac{x_\ell}{y_\ell} = \frac{3\,x_{\ell-1}}{2\,x_{\ell-1} + y_{\ell-1}} \quad .$$

By setting

$$x_\ell = 3\,x_{\ell-1}$$
$$y_\ell = 2\,x_{\ell-1} + 2\,y_{\ell-1}$$

one obtains a system of 2 linear first-order difference equations, the solution of which can be determined in the following manner.

The general solution to this system of linear first-order difference equations is

$$\binom{x}{y}_\ell = A\binom{x}{y}_{\ell-1} \quad .$$

But $A = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}$. The characteristic values of A are 2 and 3; the associated eigenvectors are $\binom{0}{1}$ and $\binom{1}{2}$. Hence,

$$A\binom{0}{1} = 2\binom{0}{1} \quad \text{and} \quad A\binom{1}{2} = 3\binom{1}{2} \quad .$$

But it can be proven[23] that any set of eigenvectors for the square matrix A, no two of which correspond to the same eigenvalues, is linearly independent. Hence, if $\binom{x}{y}_\ell$ is a solution for stage ℓ, then $\binom{x}{y}_\ell$ can be written as a linear combination of the two distinct eigenvectors of A:

$$\binom{x}{y}_\ell = \alpha_1\binom{0}{1} + \alpha_2\binom{1}{2} \quad .$$

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[22] H. W. Kuhn (1966, pp. 175-176).

[23] G. Hadley (1961, pp. 249-250).

However, $V_1(1,1,1) = 1$ is a boundary condition. If there is only 1 event which is a violation, inspected by the inspector, then this violation is sure to be detected. Hence the payoff to the inspector of $\Gamma(1,1,1)$ is 1. Therefore the starting vector can be written as

$$\binom{1}{1}_1 = \alpha_1 \binom{0}{1} + \alpha_2 \binom{1}{2} .$$

By solving this system of two linear equations one obtains $\alpha_1 = -1$ and $\alpha_2 = 1$.

Therefore,
$$\binom{1}{1}_1 = -1 \binom{0}{1} + 1 \binom{1}{2} .$$

The solution for stage 2 or $\Gamma(2, 1, 1)$ is:

$$\binom{x}{y}_2 = A \binom{x}{y}_1 = -2 \binom{0}{1} + 3 \binom{1}{2} .$$

Then
$$\binom{x}{y}_\ell = A \binom{x}{y}_{\ell-1} = -2^{\ell-1} \binom{0}{1} + 3^{\ell-1} \binom{1}{2} , \text{ and}$$

hence,

$$V_\ell = \frac{3^{\ell-1}}{2(3^{\ell-1})-2^{\ell-1}} = \frac{3^{\ell-1}}{2(3^{\ell-1}-2^{\ell-2})} = \frac{1}{2(1-\frac{1}{2}(\frac{2}{3})^{\ell-1})} ,$$

and $\lim_{\ell \to \infty} V_\ell = \frac{1}{2}$.

What do these results imply? The solutions to the game $\Gamma(\ell,1,1)$ are: $V(1,1,1) = 1$, $V(2,1,1) = \frac{3}{4}$, $V(3,1,1) = \frac{9}{14}$ etc. Therefore, given 1 violation, 1 inspection, and 1 event, the assured probability of detecting this violation is 1, or the violation is sure to be detected. Given 2 events, 1 violation and 1 inspection, the minimum assured probability of detecting the violation is $\frac{3}{4}$, not $\frac{1}{2}$ as might be expected by some common sense consideration. Thus, even this simple model serves the purpose of correcting and giving more preciseness to our intuitive expectations. The limit of $V(\ell,1,1)$ converges toward $\frac{1}{2}$. This implies that if ℓ is very large, additional true events neither influence the value of the game for the inspector, or for the violator.

From this fact one may draw a conclusion concerning the qualitative nature of the optimal strategies of both the inspector and the violator. The optimal strategies of this game could be determined quantitatively,[24] but this would not lead to a deeper insight within the present framework. Both the inspector and the violator can assure a payoff of $\frac{1}{2}$ if they wait longer and longer to use the inspection or violation, respectively, provided the number of stages is large. This can be interpreted as implying that the inspector does not use a predetermined course of action in auditing (such as fixing in advance the events which he will always inspect), but that he uses a random behavior of selecting the one particular event to inspect.

This idea becomes clearer if the case $\Gamma(\ell,m,1)$ is considered, where m may be a constant. Now the inspector has m inspections available to find the unique violation. It can be shown[25] that for a fixed m. $\lim_{\ell \to \infty} V(\ell,m,1) = \frac{1}{2}$. If the inspector does not use a predetermined auditing procedure, the violator should try to place his violation outside the set of events which is most likely to be inspected by the inspector. In this case there exists a chance of not being detected, provided no outside information is available to the inspector. Hence the above value of the game implies that the inspector uses a random strategy, selecting randomly those suspicious events which are to be inspected. Then only can he guarantee himself the above determined values.

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[24] C. f. M. Maschler (1967, pp. 15-25. H. Everett (1957, pp. 24-32).
[25] H. Kuhn (1966, p. 175).

6.3 Extension I.

A further assumption will now be introduced. There may be true events or violations which are never to be inspected[26] because of legal restrictions or cost considerations. Such a legal restriction may be absolute banking secrecy, such as exists in Switzerland. No matter how evident a violation may be, if it has to be proven by violating banking secrecy the inspector is not allowed to inspect the violation.

Another reason for not inspecting a violation may be the relatively low evidence of a violation and the relatively high costs of investigating whether a violation actually exists.

If such a class of states for the system of indicators can be determined, the rules of the game and the value of the game change.

Suppose there are two classes of signals, class A and class B . The inspector may inspect class A but is not allowed to inspect class B . For simplicity it is assumed that true events and violations generate signals which are always in class B . Suspicious events always generate signals which are in class A .

Assume that a true event produces a suspicious signal of class A with probability $1-Q$, and the true event signal in class B with probability Q . A violation produces a suspicious signal in class A with probability $1-P$, and the violation signal which is contained in class B with probability P . Again it will be assumed that the inspector detects a violation with probability r . The rules of the game $\Gamma(\ell,m,n)$ are specified in the following graph.

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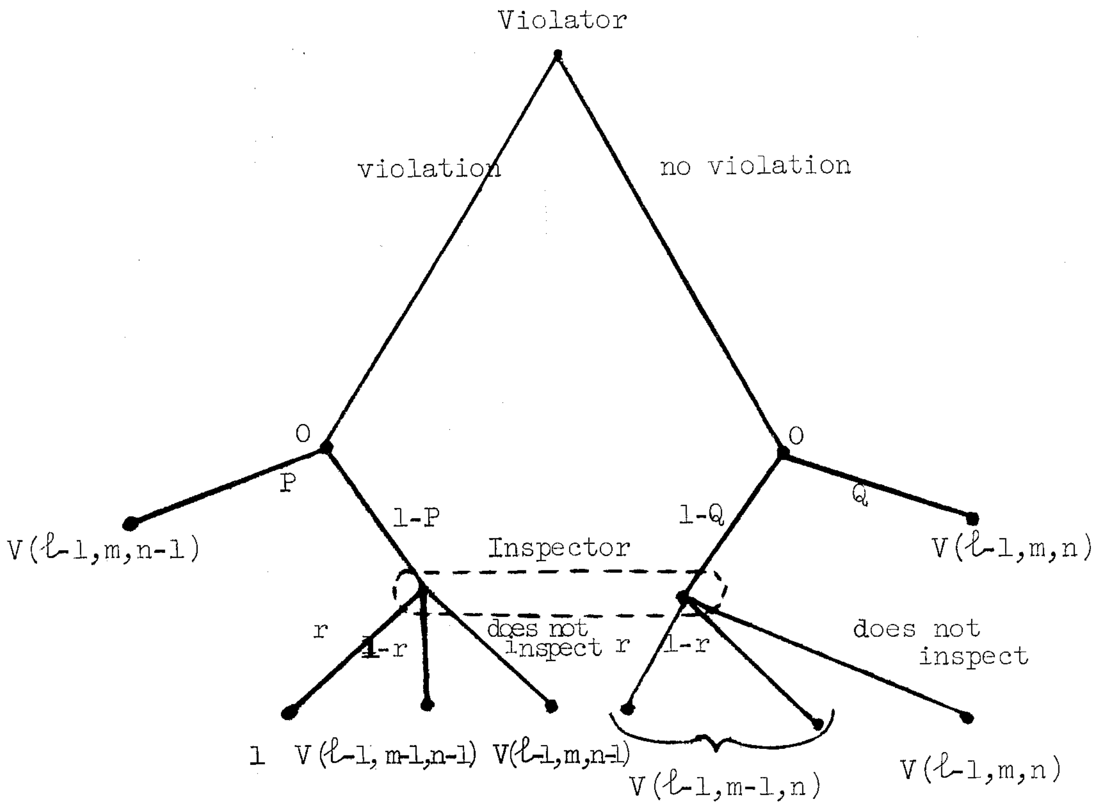
[26] M. Maschler (1967, p. 4).

Figure 3: Tree of the first stage of extension I of $\Gamma(\ell,m,n)$



The rules of the game $\Gamma(\ell,m,n)$ are now different from the rules of game $\Gamma(\ell,m,n)$ of the preceding section. If the signal: "violation" occurs with probability P, the inspector is not allowed to inspect and detect the violation but faces the game $\Gamma(\ell-1,m,n-1)$. The payoff-matrix for this new game is:

		Violator: violate	Violator: do not violate
Inspector	inspect the suspicious event	$(1-P)(1-r)\,V(\ell-1,m-1,n-1) + P\,V(\ell-1,m,n-1) + (1-P)r$	$(1-Q)\,V(\ell-1,m-1,n) + Q\,V(\ell-1,m,n)$
	do not inspect the suspicious event	$V(\ell-1,m,n-1)$	$V(\ell-1,m,n)$

For $n=1$, $p=Q = \frac{1}{2}$, $r=1$ and $V(\ell,1,1) = V_\ell$ the payoff matrix reduces to

		Violator: violate	Violator: do not violate
Inspector	inspect the suspicious event	$\frac{1}{2}$	$\frac{1}{2} V_{\ell-1}$
	do not inspect the suspicious event	0	$V_{\ell-1}$

Clearly, the matrix game has a dominant main diagonal. Hence, the value of this game is:

$$V_\ell = \frac{V_{\ell-1}}{1 + V_{\ell-1}} .$$

Solving this non-linear recursive relationship it is possible to show that

$$V_\ell = \frac{1}{\ell} .$$

Hence $V(1,1,1) = 1$, $V(2,1,1) = \frac{1}{2}$, $V(3,1,1) = \frac{1}{3}$ and $\lim_{\ell \to \infty} V(\ell,1,1) = 0$.

These results differ considerably from the corresponding values in the preceding section. The values are lower and converging toward zero. Under these circumstances there exists a high pressure on the inspector to determine the number of stages. In other words, within the given setting, the need for a tax reform toward elimination of loopholes becomes very urgent.

6.4 Extension II.

In this section a class C of signals is introduced. The class C of signals includes those suspicious events for which one or several probabilities

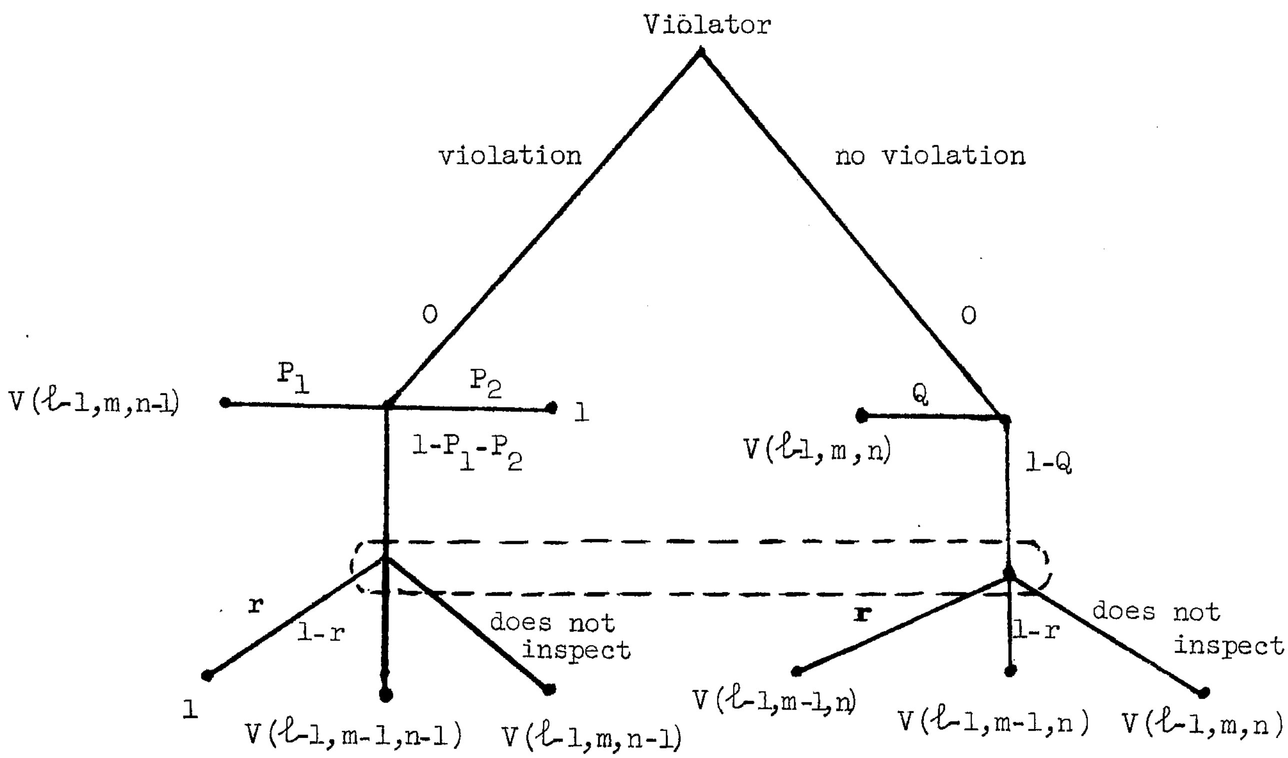
$(1-q)_i\ (j_i) = 0$ for $i = 1,\ldots,k$ and $j = 1,\ldots,m_i$ and $1 > (1-p)_i(j_i) > 0$. Under these assumptions[27] and given a true event, the system of indicators can never be in state $\{j_i\}$. Hence, the occurrence of $\{j_i\}$ with a probability $(1-p)_i\ (j_i) > 0$ implies that a violation has occurred.

This might apply to those events where either deliberately or mistakenly deductions are claimed for which a legal basis does not exist. Illegal tax evasion or imperfect information regarding the tax laws might be a rationale for the underlying behavior of the taxpayer.

Suppose there exist signals of classes A, B and C. A true event produces a suspicious signal with probability $1-Q_1 - Q_2$, the signal true event with probability Q_1, and a signal in class C with probability $Q_2 = 0$. Hence $Q_1 = Q$. A violation generates a suspicious signal with probability $(1-P_1 - P_2)$, a violation with probability P_1, and a signal in class C with probability P_2. Let it further be assumed that the inspector may inspect on a signal in class A, is not allowed to inspect on a signal in class B, and must inspect if a signal in class C occurs. The rules of this extension of game $\Gamma(\ell,m,n)$ are stated in the following game tree.

[27] M. Maschler (1967, p. 25-34).

Figure 4: Tree of the first stage of extension II of $\Gamma(\ell,m,n)$.



This change in the rules of the game $\Gamma(\ell,m,n)$ materializes in the possibility that a violation in class C signal occurs with probability P_2. This produces a new payoff-matrix for the inspector.

		Violator: violate	Violator: do not violate
Inspector	inspect suspicious event	$(1-P_1-P_2)(1-r)\,V(\ell-1,m-1,n-1)+$ $+(1-P_1-P_2)\,r + P_1\,V(\ell-1,m,n-1)$ $+\ P_2$	$(1-Q)\,V(\ell-1,m-1,n)$ $+\ Q\,V(\ell-1,m,n)$
	do not inspect suspicious event	$(1-P_1-P_2)\,V(\ell-1,m,n-1)$ $+\ P_1\,V(\ell-1,m,n-1)\ +\ P_2$	$V(\ell-1,m,n)$

For $n = 1$ this matrix reduces to

		Violator: violate	Violator: do not violate
Inspector	inspect suspicious event	$(1-P_1-P_2)r + P_2$	$(1-Q)\,V(\ell-1,m-1,n)$ $+\ Q\,V(\ell-1,m,n)$
	do not inspect suspicious event	P_2	$V(\ell-1,m,n)$

Again setting $V(\ell,1,1) = V_\ell$ and $P_1 = P_2 = \frac{1}{3}$, $Q = \frac{1}{2}$, $r = 1$, the above matrix becomes

$\frac{2}{3}$	$\frac{1}{2}V_{\ell-1}$
$\frac{1}{3}$	$V_{\ell-1}$

The value of this game is $V_\ell = \dfrac{V_{\ell-1}}{\frac{2}{3} + V_{\ell-1}}$. Solving this recursive relationship it can be shown that $V_\ell = \dfrac{V_{\ell-1}}{3(1-(\frac{2}{3})^\ell)}$.

Hence $V(1,1,1) = 1$, $V(2,1,1) = \frac{3}{5}$, $V(3,1,1) = \frac{9}{19}$, and $\lim_{\ell \to \infty} V(\ell,1,1) = \frac{1}{3}$.

The possible occurrence of events in class C increases the values of the games $\Gamma(\ell,1,1)$. One would expect this because the inspector now has an additional class of signals which increases his chances of detecting a violation. Given only 1 inspection and 1 violation, there exists a limit which is greater than zero. Hence the inspector has a positive maximin probability to detect the violation.

6. Summary

The idea of the preceding inspection game was to deal with one aspect of the free-rider problem and business corporations. The relations between the free-rider company and the Federal Revenue agents were assumed to be basically strategic. This allowed for both the inspector and the violator to adopt optimal strategies of auditing and violating.

The results of these models are certainly too simple to be readily applied to corporation auditing, but they are a starting point. Based on the simplifying assumptions of the models, both the inspector and the violator should adopt mixed strategies and wait longer and longer to use their scarce inspections and violations. The inspector should take a random selection of events to be audited rather than adopt a fixed course of action.

Finally it was shown that there exists a pressing need for simplifying the tax laws if a high degree of enforeability is to be attained. Of course this approach to the problem of free-riders and auditing is merely one of several possible alternatives. What has to be stressed is the strategic nature of this problem in the theory of public goods.

7. Conclusion

Finding the free-rider seems to involve a choice between efficiency and equity in taxation. The equity principle would require finding all free-riders no matter how small or large their tax liabilities. Efficiency requires that costs of enforcing the tax laws be weighed against losses in equity which result from auditing only certain groups with high tax potential. However, it seems that this dilemma is not unsolvable but originates chiefly from the complicated tax laws. Simpler tax laws would require less resources to enforce them.

The model presented in section 6 was a two-person constant-sum game with constant probabilities of suspicious signals being a violation or a true event. These simplifying assumptions have to be dropped in more realistic models. Each event may thus produce the signals: suspicious event, violation, or true event with different probabilities. If the balance sheet and the income statement contain ℓ events, the inspector may play any one of ℓ subgames at each stage of the recursive game $\Gamma(\ell,m,n)$. These assumptions make the computation of minimax probabilities of detection more complicated but do not make it impossible. To be satisfying, the whole question of tax evasion should also be treated in a general equilibrium setting where coalitions between violators are possible.

Further study should also be done investigating tax avoidance in a general equilibrium setting. These models should be extended by introducing imperfect information on tax laws and uncertainty about the social decision process into the system. A satisfactory treatment of these problems would not only contribute to the theory of public finance but also to the theory of general equilibrium, which until recently ignored government activities completely.

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