THE CONCEPTS OF EQUIVALENCE, ORDER AND SCALE IN THE ANALYSIS OF PREFERENCES

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PREFACE

An act is always an act of choice: a possibility is selected, and others are rejected. Therefore an act can be associated with a set of preferences - of the selected to the rejected - and the act of choice and the set of preferences are equivalent in that they determine each other. This much, which is basic to the very notion of preference, has been suggested by Pareto[1], and enlarged on by many economic theorists, especially Samuelson[2] and Houthakker[3], in the particular matter of consumers' behaviour. A thought which seems to be underlying most of the familiar arguments is that choices "reveal" preferences. What seems to be neglected is the fact, or the truism, that to make choices is not necessarily to have preferences. Choices can be made at random. This gives an ever-present limitation to the application of the concept of preference to choices.

However, the most universal scheme for the analysis of choices is in terms of preference; and, because of this, the somewhat intractable concept penetrates everywhere in economics. The supposition of given preferences is a normal starting-point for economic argument. But little is said about how it is to be decided in any real matter that preferences have a significant existence, and, granted this, how, in what manner and extent, they are to be known. Without such processes, all economic theory depending on preferences can be nothing more than mathematics.

But the work here is just precisely that: a building up of certain formal concepts, and of analytic propositions which involve them. It is part of the ground-work which is necessary before anything practical can be truly understood and done. Everything depends on the use of concepts about relations, choice and preference, for want of which, apparently, the subject has suffered for over fifty years. The concern here is to present these abstract basic concepts systematically, with generality, but finally with reference to the consumer. Propositions are given unencumbered by demonstrations, the lines of which are for the most part shown elsewhere[4,5], so as the better to present the essential structure and substance of the subject.

Most of the material can be almost found in the literature; and little of it exactly. To give some of the closest antecedents: the main problem can, with certain supplements and adjustments of thought, be read into the writings of Pareto, Samuelson, and, most especially, the well-known paper of Houthakker, but nowhere quite plainly. The main theorem concludes with a bit less than that with which Slutzky[6] starts. We suppose coherence, which is a way of looking at Houthakker's axiom, and add responsivity, which ensures that one change, in the price-expenditure ratios, always produces another, in commodity quantities; and we conclude that the quantities give the absolute maximum of a differentiable function subject to the price-expenditure constraint. Slutzky needs a twice-differentiable function.

Although, as precise statements for which there is a rigorous proof, just about all the main theorems stated here seem, strictly speaking, to be new, there is only one which is anything of a real surprise: the theorem on "total" incoherence. (All the same, the theorem that responsivity implies the equivalence of the order and scale conditions is surprising enough.) This theorem came to light during a correspondence with Gale, concerning a controversy about the equivalence of the Samuelson Weak Axiom of Revealed Preference with the Houthakker Strong Axiom, for which the possibility was entertained by Arrow[7] and Uzawa[8]. Gale[9] settled the issue by producing a counter-example, a system which he showed to satisfy the Weak, but not the Strong Axiom. To resolve a certain perplexity in our discussions, he was then led to conjecture that, for his example, every point is preferred to every other. My theorem, which was thus suggested by his example, established the conjecture as true.

Concerning the concept of a scale, it is often taken that preferences have the defining properties of a scale, but then it is assumed, in addition, without realizing that it is already implied, that indifference is transitive[10]. Or again, Birkhoff[11] realizes this implication, but does not then go on to show that the classes in the equivalence are completely ordered. Generally, the proposition that a scale, as defined here, is an order, whose complete negation is an equivalence, and which reduces to a complete order of the classes in that equivalence, is a nice, and needed proposition, that seems everywhere to have been missed. One of our principal theorems is concerned with the distinction between a general order, and a scale, and peculiar conditions under which this distinction vanishes.

Acyclicity, which is a basic condition for preferences, has had treatment by Von Neumann and Morgenstern[12]. Houthakker's axiom is in the nature of an acyclicity condition. It is not, as seems sometimes to be supposed, akin to a transitivity condition; but, rather, it is an irreflexivity condition, applied to the, in any case transitive, relation determined between the extremities of preference chains.

Integrability has been a sore question in consumer theory, ever since Hadamard said something about it in his review of Pareto's Manuale di Economia Politica. Here, by statement of a definition of integrability, together with the propositions that, for a responsive system, it is necessary for coherence, and that its negation, together with the weak axiom, is sufficient for complete incoherence, a clarification of its significance is obtained.

In regard to the concepts of local and global coherence, which are introduced, they are generally distinct conditions, with the global conditions implying the local one, but not conversely; and it is not trivial that they should be equivalent for responsive systems.

Responsivity implies invertability and continuity, and is implied by invertability and differentiability. Investigation of differentiable systems gives a continuation of the theory for responsive systems. If they are invertable, which condition is necessary even for integrability to have definition, then they are responsive, and have all the established consequent properties. The local behaviour of the system is specified by the partial derivatives; and the central problem is to obtain the condition on the partial derivatives which is necessary and sufficient for coherence, both locally and globally, the two forms of the condition being now equivalent. This condition appears as a condition on the familiar Slutzky coefficients, the properties of which constitute the main part of the theory.

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1. Structure of Relations

For a relation R between the elements of a set C to be defined, it just has to be decidable whether or not it holds, from any one element x to another y. In the one case there is the assertion, written xRy, of a propositional function of ordered pairs of elements; and in the other case there is the denial $\sim xRy$. In other words, relation is defined just when a propositional function, or equivalently a set of ordered pairs of elements, is defined. This set, by which the relation is defined, may be called the graph of the relation. It is the set of all the ordered couples of the elements, with the first element in the relation R to the second. Accordingly there are three equivalent statements

$$xRy,\ R(x,y) \text{ and } (x,y) \in R$$

that x has the relation R to y, that the propositional function R is true for (x,y) and that (x,y) belongs to the set R.

Two important special relations are the universal and the null relations ∇ and Δ, which always and never hold, respectively, between any pair of elements. Also, as two basic relations between the elements in any set, there are the relations I and D of identification and distinction, thus,

$$xIy \equiv x = y, \quad xDy = x \neq y.$$

The <u>negation</u> of the relation R is the relation $\bar{R}$ defined by

$$x\bar{R}y \equiv \sim xRy,$$

which holds just when R does not hold. The <u>reverse</u> R' of R is defined by

$$xR'y \equiv yRx;$$

and it holds from x to y just when R holds from y to x. The reverse of the negation of relation is the same as the negation of the reverse; and so, without ambiguity, both may be denoted by $\bar{R}'$. The <u>complete negation</u> of R is defined as the relation $\tilde{\bar{R}}$ which holds just when neither R nor its reverse R' hold; so it is defined by

$$x\tilde{\bar{R}}y \equiv \sim xRy \wedge \sim yRx.$$

A pair of relation are <u>complementary</u> if each is the negation of the other. Thus, the universal and null relations, and the relations of identification and distinction, are each complementary pairs of relations.

Two relations have the relation of <u>implication</u>, of one to the other, if the one holds only if the other holds:

$$Q \Rightarrow R \equiv xQy \Rightarrow xRy.$$

If the relations are considered as graphs, rather than propositional functions, implication becomes set-inclusion $Q \subseteq R$.

Equivalence of relations is defined by mutual implication; for the graphs, it is identity.

A relation is reflexive if every element has that relation to itself; so, xRx for all x, if R is reflexive, which condition has the statement

$$x = y \Rightarrow xRy,$$

and equivalently,

$$I \Rightarrow R.$$

A relation which no element has to itself is called irreflexive. Thus, elements which stand in an irreflexive relation R must be distinct; that is,

$$xRy \Rightarrow x \neq y,$$

and equivalently,

$$R \Rightarrow D.$$

A symmetrical relation is defined to be such that if it holds one way, between a pair of elements, then it also holds the other way; thus,

$$xRy \Rightarrow yRx,$$

or equivalently,

$$R \Rightarrow R'.$$

Thus a symmetrical relation is unchanged by reversal. An antisymmetric relation is such that if it holds one way, then it cannot hold the other; thus,

$$xRy \Rightarrow \sim yRx,$$

which is

$$R \Rightarrow \bar{R}',$$

or that R implies the negation of its reverse. A relation is called <u>complete</u> if it holds one way if it does not hold the other, between any pair of distinct elements:

$$\sim xRy \Rightarrow yRx,$$

or,

$$\bar{R}^{-1} \Rightarrow R.$$

A complete, antisymmetric relation R, holding just one way between any pairs of elements, thus satisfies the condition

$$R = \bar{R}^{-1}$$

2. <u>Composition of Relations</u>

Given two relations Q, R, they may be composed together in any of three different ways, by operations to be called <u>conjunction</u>, <u>disjunction</u> and <u>adjunction</u>, to form a third relation, called their <u>sum</u>, <u>product</u> and <u>resultant</u>, respectively, and denoted by $Q \vee R$, $Q \wedge R$ and QR, thus:

$$x(Q \wedge R)y = xQy \wedge xRy, \quad x(Q \vee R)y = xQy \vee xRy,$$

$$x(QR)y = \bigvee_z xQz \wedge zRy$$

Conjunction and disjunction are commutative operations, but adjunction is not. However, all are associative; and adjunction is distributive over disjunction. Any sequence of relations $R_1, \ldots, R_m$ has a well defined resultant $R_1 \ldots R_m$; and the mth power R^m if any relation R can be defined as the resultant of a sequence of m relations, all of which are identical with R, thus,

$$R^m = \overbrace{R \ldots R}^{m}.$$

Though the adjunction of different relations is not generally commutative, the adjunction of different powers of the sum relation is commutative. For

$$R^m R^n = R^{m+n}$$

and

$$m + n = n + m$$

3. Links, Chains and Cycles

A pair of elements ordered in sequence may be called a _link_; and then a sequence of any number of elements may be called a _chain_, with all successive pairs of elements defining its links, and the initial and final elements giving its _extremities_. The _span_ of a chain is to be the link formed between its extremities. Two links, or chains, are said to be _coupled_, in a given order, if the final element of the first, in that order, coincides with the initial element of the second. Thus (x,y), (y,z) represent a coupled pair of links. A coupled pair of chains may be directly joined, in the order in which they are coupled, to form a third chain; and the spans of coupled chains are coupled links, which join to give the same span as the join of the chains. A chain determines, and is determined by, a series of links, in which each link is coupled with its successor.

A chain with coincident extremities defines a <u>cycle</u>. It follows that a cycle can be represented as a chain, with extremities coinciding at any of its elements. The links of a cycle are given by the links of any such chain representing it, these themselves forming a similar cycle.

Given a relation R, let any ordered pair of elements be said to determine an R- <u>link</u> if the first has the relation R to the second. An R- <u>chain</u> can then be defined as a chain each link of which is an R- link. Thus, given a binary relation R, that is, a relation between pairs of elements, there can, for any m, be formed an m-axy relation, between any m elements, by the condition that these elements form an R-chain. The condition that a chain $(x_1, \ldots ,x_m)$ be an R-chain, of some m elements, is

$$x_1Rx_2, \ldots\ldots , x_{m-1}Rx_m$$

Thus, from a propositional function R(x,y) of pairs, there is formed a propositional function

$$R(x_o, \ldots , x_m) \equiv x_oRx, \wedge \ldots \wedge X_{m-1}Rx_m$$

of sequences. From the given binary relation, there has been formed a multiple relation, which is that possessed by elements which form a chain in the given relation.

An R-cycle is defined as a cycle whose links are R-links. Accordingly, ... , x_m, x_1, ... , x_m, ... is an R-cycle of m elements if

$$x_m R x_1, \ldots, x_{m-1} R x_m, \text{ that is } R(x_m, x_1, \ldots, x_m)$$

4. Acyclicity

A relation R is defined to be acyclic if no R-cycles exist. Less restrictively, it could be said to be non-k-cyclic if no cycles of k distinct elements exist. The conditions of non-2-cyclicity and antisymmetry are the same. Acyclicity is non-k-cyclicity for every k. For a complete, antisymmetric relation, non-3-cyclicity implies acyclicity.

Any chain would become a cycle, were it extended by the link from the final element to the initial. Thus, the absence of cycles requires the break between the extremities of every chain. Put in the form of such a requirement, the acyclicity condition is

$$x_o R x_1 \wedge \cdots \wedge x_{m-1} R x_m \Rightarrow \sim x_m R x_o \text{ , } R(x_o, \ldots, x_m) \Rightarrow \bar{R}(x_m x_o).$$

Equivalently, acyclicity means that the extremities of a chain must be distinct, that is

$$x_o R x_1 \wedge \cdots \wedge x_{m-1} R x_m \Rightarrow x_o \neq x_m \text{ , } R(x_o, \ldots, x_m) \Rightarrow x_o \neq x_m$$

5. Chain Extensions

Pairs of elements in a given relation to each other are also said to be linked in that relation. Now pairs of elements which are the extremities of a chain in a relation will be said to be chained, in that relation, from the initial element of the chain to the final; and then they are defined to be in a chain relation, which is called the chain extension of that relation. Thus, if $\vec{R}$ denotes the chain extension of any relation R, then, for its definition, $\vec{R}$-links are given as the spans of R-chains; and $\vec{R}$ is thus defined by its links. Since R-links are particular R-chains, it follows that R-links are $\vec{R}$-links, or that R implies $\vec{R}$:

$$R \Rightarrow \vec{R}.$$

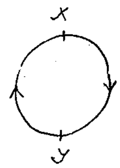
Now the span of a given $\vec{R}$-chain is an $\vec{R}$-link. For, any links of an $\vec{R}$-chain, these being $\vec{R}$-links, are the spans of R-chains which are coupled in a sequence, and join together in that sequence to give an R-chain, whose span, defining an $\vec{R}$-link, is the same as for the given R-chain. Thus the chain extension of any relation has the property that its links are included among the spans of its chains. Any relation with this property is the same as its own chain extension. It follows that no new relation is obtained by repeating the operation of taking the chain extension, or that multiple chain extension operations are equivalent to the simple one.

Acyclicity, or the absence of cycles, requires that every chain have its extremities distinct. But the extremities of R-chains are precisely those elements forming $\vec{R}$-links. Thus, for acyclicity, there is the condition

$$x\vec{R}y \Rightarrow x \neq y,$$

or

$$\vec{R} \Rightarrow D;$$



that is, the non-reflexivity of $\vec{R}$. Thus it appears that the acyclicity of relation R is equivalent to the non-reflexivity of the chain extension $\vec{R}$. Consider an R-cycle on two elements x, y. It is assembled from two chains between x and y, one from x to y and the other from y to x, obtaining x,y related both ways in the chain extension relation $\vec{R}$, that is $x\vec{R}y$ and $y\vec{R}x$. Without cycles, such symmetries are impossible. Hence, the conditions of the acyclicity of R and the antisymmetry of $\vec{R}$ are equivalent.

If elements are defined to be <u>chained in order</u> m in a relation if they are the extremities of a chain of that relation of m+1 elements, then, since

$$xR^m y \Leftrightarrow \bigvee_{z_o, \dots, z_m} R(z_o, \dots, z_m) \wedge z_o = x \wedge z_m = y,$$

it appears that the relation thus defined, by elements being chained in order m, is the same as the mth power R^m of the relation. Now elements may be defined to be chained if they are chained in any order; so that

$$\vec{R} \equiv \bigvee_{m=1}^{\infty} R^m ,$$

since

$$xR^m y \wedge yR^n z \Rightarrow xR^{m+n} z ,$$

which is

$$R^m R^n = R^{m+n} ,$$

it follows that

$$x\vec{R}y \wedge y\vec{R}z \Rightarrow x\vec{R}z ,$$

for any relation R .

6. Transitivity

A relation R with the property

$$x_o Rx_1 \wedge x_1 Rx_2 \Rightarrow x_o Rx_2$$

is called transitive. This property, by induction, is equivalent to

$$x_o Rx_1 \wedge \dots\dots \wedge x_{m-1} Rx_m \Rightarrow x_o Rx_m ,$$

and this asserts that the span of an R-chain is an R-link; equivalently, $\vec{R} \Rightarrow R$. Since $R \Rightarrow \vec{R}$ anyway, the transitivity condition has the expression

$$R = \vec{R}.$$

The transitivity condition is that if a relation holds between successive elements of a chain then it carries over between the extremities. The transitivity of the chain extension of any relation has been shown; for any relation R, and its chain extension $\vec{R}$, it appeared that $R \Rightarrow \vec{R}$, where $\vec{R}$ is transitive. It will now appear that any transitive relation T implied by R is implied by $\vec{R}$, so that $\vec{R}$ appears as the minimal transitive relation implied by R, by which property it is called the transitive closure of R. Thus, if $R \Rightarrow T$ where T is transitive, then an R-chain is a T-chain; and the span of an R-chain, that is an $\vec{R}$-link, is the span of a T-chain; which must be a T-link, by the transitivity of T. Therefore an $\vec{R}$-link is a T-link; and this shows that $\vec{R} \Rightarrow T$.

For a transitive relation, the conditions of irreflexivity and antisymmetry are equivalent. For a complete relation, acyclicity implies transitivity; and for an antisymmetric relation, transitivity implies acyclicity.

A relation is said to have the property of negative transitivity if its negation is transitive. Negative transitivity together with antisymmetry implies transitivity.

The transitivity condition can be written

$$R^2 \Rightarrow R,$$

implying

$$R^m \Rightarrow R \quad (m=1,2, \ldots),$$

and therefore equivalent to the condition $\vec{R} \Rightarrow R$, where

$$\vec{R} = \bigvee_{m=1}^{\infty} R^m .$$

7. Equivalence

A reflexive, symmetric and transitive relation defines an equivalence. For the simplest possible examples of equivalence, there is the relation of identification I, in which each element is equivalent only to itself; then also the universal relation ∇, in which all elements are equivalent to each other.

A partition resolves a set into a union of disjoint subsets, called the components in the partition.

The relation which elements have by the condition that they belong to the same component in a partition is an equivalence, determined by that partition. Reversely, to every equivalence, there is determined a partition, from which it can be thus derived, whose components are to define equivalence classes. Every element belongs to just one equivalence class, of which it is representative, and which is composed of all elements with which it has equivalence.

Thus, equivalence Q and partition Π have a mutually determining association, in which

$$xQy \Leftrightarrow \Pi_x = \Pi_y ,$$

when Π_x denotes the component of Π to which x belongs. Thus equivalence between elements is reduced to an identity between classes; and, reversely, a partition into classes defines an equivalence, asserted between elements in the same class, and denied between elements in different classes, any element belongs to one and only one class.

8. Order

An irreflexive, transitive relation defines an order. Since, for a transitive relation, the conditions of antisymmetry and irreflexivity are equivalent, an order is antisymmetric. An order is called a complete order if it is complete as a relation; that is, if any pair of elements are related in it, one way or the other. Otherwise it is called a partial order.

One order is said to be a refinement of another if it is implied by the other, that is, if its links include at least all those of the other; in other words, if every pair of elements related in one order have the same relation in the other, but it is allowed that there may be elements which are related in the refined order, but not in the original.

Every order can be refined to a complete order, provided the axiom of choice is assumed, which is that there exists a rule by which an element is selected out of any subset. However, the way is not generally unique. Thus, given a partial order, in which only certain pairs of elements are related, there can be found a total order, one relating every pair, which agrees with the original partial order, in regard to those pairs which it relates; and this is provided that there exists a rule by which a well-defined element can be chosen from any subset.

Given any relation, in general not an order, it may or may not be possible to enlarge it so as to obtain an order. The condition that this be possible is acyclicity. For, in this case, the chain extension, already transitive in its construction, is irreflexive, and gives an order implied by the relation; and this is, moreover, the coarsest such order, since every order which is implied by the relation is a refinement of this order.

9. Scale

An antisymmetric, negatively transitive relation defines a scale. The complete negation of a scale defines the relation of indifference in that scale. Thus, given a scale S, and its relation of indifference $\tilde{S}$, there is, for any pair of elements, resolution into the three mutually exclusive, complementary possibilities

$$xSy, \quad x\tilde{S}y, \quad xS'y \quad ,$$

From these possibilities, there is obtained either one or the other of the pair of opposite preferences, and, otherwise, indifference.

Since, with antisymmetry given, negative transitivity implies transitivity, and since antisymmetry implies irreflexivity, a scale appears as irreflexive and transitive, and therefore an order. However, an order is not necessarily negatively transitive, and therefore not necessarily a scale. Thus a scale appears as just a special kind of order. Nevertheless, the concepts of complete order and complete scale coincide; for, given completeness and antisymmetry, transitivity implies negative transitivity.

If S is a scale, then the relation $\tilde{S}$ of indifference in S must be an equivalence; and the classes of this equivalence $\tilde{S}$ are to define the indifference classes of the scale S. Every element belongs to one and only one of the indifference class, since these classes form the components in a partition.

If the components of a partition of a set are put in a complete order, then the elements of the set are put in a partial order, with the relation between elements decided by the relation between the components in the partition to which they belong. Such a partial order, thus determined by a completely ordered partition, turns out to be a scale. Thus a completely ordered partition determines a scale. Reversely, given a scale, there is obtained a partition, the components of which are the indifference classes of the scale, and then there is obtained a unique complete order of the components in the partition, from which the scale can again be derived. Therefore, a scale, and a completely ordered partition, applied to a set, are logically identical concepts, a scale relation between elements being identified with a complete order relation between the components to which they belong in a partition.

There is now the equivalence between the concepts of a scale S applied to the elements of a set, and a complete order $\mathcal{S}$ applied to the components of a partition Σ of that set, with

$$xSy \Leftrightarrow \sigma_x \mathcal{S} \sigma_y ,$$

where σ_x is the component in the partition Σ with representative x. Any scale S is in fact also an order, and its indifference relation $\tilde{S}$ is an equivalence, the classes of which determine the partition Σ whose components are to have a complete order $\mathcal{S}$.

A scale is thus a kind of partial order, which has that most concise representation given by a completely ordered partition.

A <u>gauge</u> of a scale applied to a set is defined as a numerical function of the elements which is the greater or the less for one element or another only when the one or the other is preferred in the scale. Thus, for a gauge ϕ, measuring a scale S,

$$\phi(x) < \phi(y) \Rightarrow xSy,$$

and a gauge Φ is said to <u>completely measure</u> S if it completely represents S by its magnitudes:

$$\phi(x) < \phi(y) \Leftrightarrow xSy.$$

II. PREFERENCE AND CHOICE SYSTEMS

1. Preference and Choice

A set R, and an element x of that set, represents a choice, denoted by [x; R], with R called the range and x called the object of choice.

In any choice [x; R], it being understood in this notation that $x \in R$, the object of choice x is considered selected, from the range R; and other elements in the range, that is, elements y with $y \in R$ and $y \neq x$, forming the set R - x, are considered rejected.

In any choice, there is determined a relation of preference, of the selected element to each of the rejected ones; thus,

$$\{x, R - x\} = \{(x,y);\ y \in R - x\}$$

will denote the set of preferences associated with a choice [x; R], each preference being given by an ordered couple (x,y) with $y \in R - x$.

Any set of ordered couples is the graph of a relation; and the set $\{x, R - x\}$ of preferences associated with a choice [x;R] is the graph of a relation which defines the preference relation of the choice. But the set $\{x, R - x\}$ may itself be considered to denote the relation. A choice [x; R] thus determines and is determined by a relation $\{x, R - x\}$ in which the selected element, or the object of the choice, is represented as superior, or preferred, to each of the rejected elements, these being the other elements in the range of the choice.

The preference relation associated with a single choice is thus a relation with the characteristic form that the first term of any pair is a fixed element, the selected one, while the second term is variable, but always different from it, and ranging over the rejected elements.

Now consider any set of choices, say

$$K = \{[x_i;\ R_i]\}_{i \in I}$$

composed of choice $[x_i;\ R_i]$ indexed in a set I; and consider the union

$$Q = \bigcup_{i \in I} \{x_i,\ R_i - x_i\}$$

of the sets of preferences associated with each of the choices separately. Alternatively, consider the sum $Q = \bigvee_{i \in I} Q_i$ obtained by taking the disjunction of the individual preference relations of the choices. The preferences in Q may be called the <u>base preferences</u> of the set of choices K.

From the fundamental set Q of preferences, directly associated with the choices of the set K takes separately, there may now be formed the larger set P of <u>derived preferences</u>, which span the chains of preferences of Q. By this process, from any Q-chain $(z_o,\ z_i,\ldots,z_m)$ there is derived the spanning P-link, $(z_o,\ z_m)$, between the extremities z_o, z_m. In other words, the operation by which P is formed from Q is such that, with any set of preferences of Q which are coupled together in pairs in a sequence so as to form a chain, there is included the preference between the end-elements.

The fundamental preferences of the choices K are defined for the individual choices taken separately; but the derived preferences P have reference to the entire collection of the choices, and belongs in the structure of the set K that they form jointly.

By the form of the construction from Q, the relation P is necessarily transitive, any P-chain being spanned by a P-link. For, P-links are defined as the spans of Q-chains; and therefore the span of a P-chain is also the span of an Q-chain, namely, that which is obtained by joining all the Q-chains which determine the links in the P-chain, and it is therefore a P-link. In fact, P is the chain extension of Q, the same thing as the transitive closure of Q, the minimal extension of Q which is transitive.

2. Efficacy

A preference system is a relation which is instrumental for the determination of choice; and not all relations can be effective for such a purpose. To state those which are, it is only necessary to examine the manner of this instrumentality.

Let a preference system P be said to be effective on a range of choice R if, firstly, there exists one, and only one element $x \in R$ such that

$$\{x, R - x\} \subset P.$$

That is, the preferences associated with the choice [x, R] are a part of P; and secondly

$$\{R - x, x\} \cap P = 0;$$

that is, none of the opposite preferences of the choice, forming the set $\{x, R - x\}' = \{R - x, x\}$ occur in P.

Thus a preference system P which is effective on a range R decides a choice [x; R] with that range, in which the object x is distinguished as superior to every other element in R, by preferences all contained in P, while none of the other elements in R are at the same time distinguished as superior to x by contrary preferences also in P.

It will be evident that a necessary condition that a preference system be effective on every subset of elements of given set is that it give a complete order of the elements of that set.

Let a preference system be called <u>completely effective</u> on a set if it is effective on every subset of that set. Then the proposition is that a preference system is completely effective only if it gives a complete order. More generally, a preference system may be effective for just an arbitrary subset of the set of all subsets.

It is plain that for a relation to be completely effective as a preference system, it must be complete, and antisymmetric; that is, give one or the other of the two possible preferences between any pair of elements, but not both. Otherwise, it would not be effective for deciding the choice between any pair of elements. Also it must be transitive. For, in the case of a complete, antisymmetric relation, transitivity is equivalent to acyclicity, and this condition is obviously necessary if the relation is to be completely effective. Should there be a cycle of preferences involving three elements, no choice is possible between the three, since each is inferior to one of the two others, and so none is superior to all. Moreover, the absence of three-cycles, in the case of a complete relation, implies transitivity. Transitivity, in the case of an antisymmetric relation, implies acyclicity. Thus a completely effective relation must be a complete order. Now a complete order will be completely effective if and only if every subset has an element which is maximal in the order, which is to say that the set is well-rdered by the relation. A preference system may give a complete order, without it being completely effective: still every element may be inferior to some other element, but the infinite sequences of elements generated by taking, with every element, another one superior to it will never contain repetitions, and never close into cycles, and there will be no termination in a maximal element.

A complete order is at the same time a complete scale. So the condition for a relation to be completely effective can be relaxed in two ways: by requiring it just to be a scale, and then by requiring it just to be an order. Assuming the axiom of choice, that there is a rule selecting a well-defined element from any subset, any relation satisfying the weaker conditions can be represented as part of a relation satisfying the strong condition. With this in view, by a preference system will be meant a relation which is generally an order, and possibly a scale, but need not give a well-ordering of the set, nor even a complete ordering.

3. Feasibility, Compatibility and Coherence

The question arises as to the conditions which must be satsified by a set of preferences, for it to be permissible for them to be considered as belonging together, and in such a way as arising out of the same system. Choices are observed; and a choice is equivalent to a set of preferences, the observation of which goes with observation of the choice. Different choices are generally dissociated, in that they are made generally at different times, between which there are various changes of circumstance. However, it is possible that the same system of preferences may, more or less rigidly, persist through these changes, in time and contingent circumstances. The rigidity of the operative preference system is the ground for a coherence of these manifested preferences, appearing as operative in the different choices.

Thus, it may be required to know whether or not there exists an effective system to which could belong all the base preference Q, formed from a set K of choices, with each choice taken separately. There is formed from these preferences the set P of derived preferences, constructed for the choices all taken together.

If any preferences belong to one system, they will be said to cohere in that system. Also the system will be said to fulfill the general condition of coherence if there exists at least one such system, in which they can be made to cohere.

Firstly, in an effective preference system, there cannot be admitted an absurd preference, for any element in regard to itself, which requires at the same time the selection and rejection of that element. By this condition applied to each, a set of preferences is to be called feasible, there being no members which are absurd. Then further, there cannot be allowed any opposite pairs of preferences, which are incompatible with each other, in that they obtain conflict of choice between a pair of elements. Preferences which contain no contrary pairs will be called compatible.

Now an effective preference system is given by a transitive relation. Therefore, if any given preferences, being coherent, belong to such a system, so also must the derived preferences, which span the chains formed by the given preferences. And these derived preferences, belonging to the same system, must be compatible. Because a derived system of preferences is, inherently, in the form of the derivation, transitive, compatibility is equivalent to feasibility, and then equivalent to the condition for the system to give an order. But the derived system is then an order, containing the given preferences, and obtaining their coherence. Hence, a necessary and sufficient condition that preferences be coherent is that the derived preferences give an order. This condition is that the derived preferences be feasible, or, equivalently, compatible; and this order, containing the preferences, and obtaining their coherences, is contained in every order in which they are contained. Any preference between distinct objects has been admitted as feasible, but a preference between one object and itself is absurd, and a set of preferences is called absurd or feasible according as it does or does not contain absurd preferences. So a preference relation is thus feasible if it is irreflexive, and otherwise absurd.

The opposite preference between any two objects have been taken to be incompatible; and preferences are to from a compatible set so long as no pairs of them are incompatible. A compatible preference relation is thus one that is antisymmetric.

Now a transitive set of preferences, this being one which contains, with every chain of preferences, the preference which spans the chain, is feasible if and only if it is consistent. Equivalently, a transitive relation is irreflexive if and only if it is antisymmetric.

Accordingly, give any preferences, forming at set Q, they are coherent if the derived preferences, which span the chains formed by the given preferences, and forming a set $P = \vec{Q}$, are compatible. Since P is, by construction, necessary transitive, an equivalent of the coherence condition for Q is the feasibility of P. With Q coherent, P is an order containing Q, and obtaining the coherence. A further, more direct form of the coherence condition for a set of preferences is given by acyclicity.

4. Goods

Consider a set C, whose elements are to be called states, and a scale S, applied to C, which is to be a value scale; that is, the scale is to decide the better and the worse between states, or which one is to be taken preferable to which other, in any choice. So any two states are decided, one better or worse than the other, by their relation in S, or if neither of these possibilities, then as equivalent in value, by relation in $\tilde{S} = \bar{S} \wedge \bar{S}^{-1}$!.

A state is supposed specified by a variety of quantities, measuring different components, into which the state is effectively resolved. The assemblage of these quantities, say $x_1, \ldots x_n$, which can be given as the elements of a vector x, defines the composition of the state; and different possible states are to be distinguished from each other just by the distinctness of their compositions. In this way, distinct states are identified with distinct points in a space of vectors of order n.

Let x, y be two states, with components x_i, y_j. They will be defined to have the relation

$$x \subset y,$$

of inclusion of x in y, if each component of x is at most the corresponding component of y, and not all such are equal; that is,

$$x_i \leqslant y_i \quad (i = 1,\ldots,n),\ x \neq y.$$

A component in a state has the attribute of a good if more of it is always better.

Thus, if a state can be resolved into an assemblage of goods, and if x, y are any two state-compositions, the elements of which are now all measures of goods, then

$$x \supset y \Rightarrow xSy,$$

for any operative value scale S; that is, a state with composition of goods which contains that of another must be better than that other. A value scale applied to composition vectors which satisfies this condition, of being a refinement of the relation of inclusion, is defined to be increasing, or is said to satisfy the law of increase which is just that more of a good is better.

The question about whether or not certain measured quantities are in fact measures of goods, either always, or perhaps just conditional on some particular circumstance or connexion, is usually a complex question. But, in the case of possession of material commodities, more of a possession is usually recognized as better; because there is always liberty, belonging to the character of material possession, for its voluntary disposal; so that actual possession is proof of the preference for it.

5. Compensation

Consider a value scale S applied to states, each supposed represented by a composition of goods. Then any change of state is a composite variation of goods, expressible as a combination of losses and gains. A combination δx of losses and gains, applied to a state x , is said to be compensating, in a scale S, if any further gain γ leads to a better state than x, and any further loss λ leads to a worse state, that is

$$(x + \delta x + \gamma) \quad Sx, \qquad xS(x + \delta x + \lambda)$$

for any non-null γ, λ whose elements are non-negative, non-positive respectively; but either of the possibilities

$$(x + \delta x)\ Sx, \qquad xS\ (s + \delta x)$$

can be allowed in this condition.

In an important class of scales, the compensation condition is equivalent to indifference. That is, states x, y are derived from each other by compensating variations if and only if they are indifferent, $x \tilde{\bar{S}} y$. Thus, if the sets x S, S x of elements inferior and superior to a given element are open, in which case S will be called an open scale, then S has this property.

Consider a state with goods in composition x, and a gain δx_i in the ith good, compensated by a loss $- \delta x_j$ in the jth. If the limit of the ratio $\frac{\delta x_j}{\delta x_i}$ of the compensating amounts exists, as $\delta x_j \to 0$, it will be called the compensation ratio of the jth good on the ith.

Consider a scale in which the finite ratio $\frac{\delta x_j}{\delta x_i}$, as just x_i, or x_j increases, decreases or increases, respectively. This condition means that the greater the measure in which a given good is held, then, other goods being equal, the less any marginal change of it has the power of compensating for given marginal changes of other goods.

In other words, the scarcer a good, the more precious it is; and reversely, the more abundant a good, the less its increase is desired against increase in other goods. This condition will be called the law of diminishing compensation. It is a form of the general law of diminishing returns, in which the "essential value" gained by a given increase in a good, measured in terms of what increases in other goods would be sacrificed to gain it, diminishes as the amount of that good held increases. Generally, the only way in which the value of a change can be measured is in terms of the sacrifices which would be made to obtain it.

A scale S of composition will be called convex if

$$y, zSx \Rightarrow [y, z]\, Sx$$

that is, if two compositions y, z are preferred to a third x, then so is any composition on the setment [y, z] joining them. Equivalently, the domain S x of composition superior to x is convex. Convexity is the analytical statement for the law of diminishing compensation; and for a convex scale, the limiting compensation ratios exist almost everywhere, and the finite ratio condition for diminishing compensation is equivalent to the implied limiting condition, on the limiting ratios.

The increasing, open and convex scales of compositions represent a most important class of scales, arising in the theory of preferences for combinations of measured goods. They satisfy the laws of increase, and of diminishing compensation; and have the property that compensating changes are those which result in indifference.

III. EXPENDITURE SYSTEMS AND COHERENCE

1. Purchase as Allocation and as Choice

Consider a purchase, by which certain amounts of commodities are obtained, at certain prices. The amounts can be supposed given by a vector x, and the prices by a vector p, and then the purchase is represented by the vector pair (p,x). The expenditure involved in the purchase is $e = p'x$.

Associated with the purchase, there is the set $\{y;\ p'y \leqslant e\}$ of compositions y requiring at most the same expenditure e, at the prices p. This set is identical with

$$R_u = \{y;\ u'y \leqslant 1\} ,$$

where

$$u = p/e$$

for which

$$u'x = 1$$

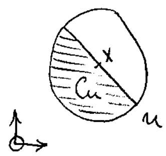
The vector u gives the prices expressed as fractions of the expenditure e, or with this expenditure taken as unit of money; and these define the <u>relative prices</u> of the purchase.

A purchase which is represented by a vector pair

(u,x) such as satisfies the condition $u'x = 1$, of having the total expenditure as unity, will be instead denoted by $[u;x]$, and will be termed an expenditure allocation. It determines the distribution in which the total expenditure is allocated to the different commodities.

Thus, with a purchase (p,x) having expenditure $e = p'x$, there is determined an expenditure allocation $[u;x]$, where $u = p/e$. Now, further, with this allocation, there is associated the choice $[x;C_u]$, of the composition x, out of the set of all compositions costing as most as much; and this choice is equivalent to the set $x,C_u - x$ of the preferences (x,y) $(y \in C_u - x)$, of x over every other composition y in C_u.

2. Expenditure Systems



In an expenditure allocation $[u;x]$, the vector can be considered as specifying the condition $u'x = 1$ applied to the composition vector x, called the balance condition, since it is the condition that the cost of the composition x exactly exhausts the expenditure. The vector u is to be called a balance vector; and any composition y is said to be within on or over a balance u according as $u'y \lesseqgtr 1$. An expenditure allocation is thus given by a balance, together with a composition on that balance; and the allocation is regarded as a choice by regarding this composition as chosen out of all those compositions within the balance.

Balances and composition are given by vectors u and x with positive element, which can be considered belonging to certain balance and composition regions B and C:

$$u \in B, \quad x \in C.$$

An expenditure system is defined as a rule E by which, with every balance u, there is associated a composition x = Eu, which is on that balance, so that u'x = 1, and thus obtaining an expenditure allocation [u;x] . It is, accordingly, a mapping

$$E:B \rightarrow C \ (u \rightarrow x; \ u'x = 1)$$

of the balance region into the composition region, the image of each balance u being a composition x on that balance, or such that u'x = 1. The expenditure system E can also be considered and a set of allocations $\{[u;x] \ ; \ x = Eu, \ u \in B\}$. Also it can be considered as a choice system, giving determination of an object of choice x from a range of choice C_u and, as such, it is represented as a set of choices

$$\{[x;C_u] \ ; \ x = Eu, \ u \in B\}$$

The union

$$Q = \bigcup_{u \in B, \ x = Eu} \{x, C_u - x\}$$

of the sets $\{x, C_u - x\}$ of preferences which are each equivalent to a choice $[x;C_u]$ of the system defines the base preferences of the system.

3. Responsivity

An expenditure system is said to be responsive in a given balance region, with coefficient given by certain positive constant, if the distance moved by composition, in response to a movement of balance, is at least a fixed multiple of the distance moved by the balance.

Thus, if E is an expenditure system obtaining composition x,y on any given balances u, v ∈ B such that

$$0 < \lambda_B \leqslant \frac{|x - y|}{|u - v|}$$

then E is responsive in B, with coefficient λ_B. Accordingly, the distance $|x - y|$ between compositions x,y determined by E in balances u,v ∈B is at least the positive multiple λ_B of the distance $|u - v|$ between u,v.

Let C = EB be the image of the region by the mapping E. Then the responsivity of E in B implies that E is an invertible mapping between B and C, and that the inverse is continuous.

4. Integrability

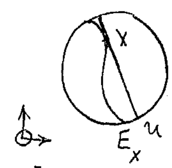
Let E be an invertible expenditure system; so, beside giving x an a function of u, it also obtains u as a function of x. Then there can be formed the differential form u'dx, associated with the system with coefficients u a function of the coordinates x. The differential form is said to be integrable if it is proportional by a factor λ, called an integratory factor, to the total differential $d\phi$ of a differentiable function ϕ, called an integral, thus,

$$\lambda u'dx = d\phi$$

The integrability of the expenditure system is defined by the integrability of its associated differential form.

Any integrals of such a differential form are functionally dependent on each other. Therefore, if there are any at all, there is almost one functionally independent integral ϕ, defining a unique level surface

$$E_x = \{y;\ \phi(y) = \phi(x)\},$$



that is, a surface on which ϕ takes a constant value, through any point x. These level surfaces are independent of which integral ϕ is used to construct them, by the functional dependence between all integrals; whence they are directly characteristic of the differential form. In fact, they may themselves be characterized as integral surfaces of the differential equation u'dx = o, with E_x as a unique such integral surface through any point x, and not requiring reference to the integral ϕ.

5. Order and Scale Coherence

The coherence of an expenditure system E is defined by the coherence of its base preference $Q = \bigcup_{u \in B} \{x, C_u - x\}$, which is given by the condition that the derived preferences $P = \vec{Q}$ form an order. With P automatically transitive, the order condition reduces to the requirement that P be irreflexive, or equivalently, that it be antisymmetric. Since

$$Q \supseteq P$$

the antisymmetry of P requires the antisymmetry of Q; but it is not equivalent to it. Moreover, while Q is necessarily not transitive*, it is, however, necessarily irreflexive; while P, necessarily transitive, is generally not irreflexive, and may possibly be reflexive.

Next to the general coherence condition, which is that P be an order, there may be considered the condition, stronger than mere coherence, that P be a scale. Any relation which is a scale is necessarily an order, but not conversely, and this is true for P. Though, of

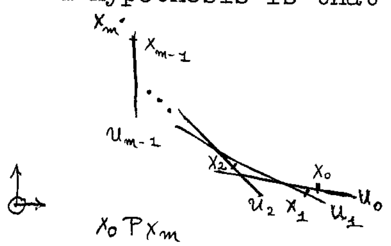
*At least with E responsive in a set C with interior points.

course, any order can be defined to a scale, so coherence in an order implies coherence in a scale, the requirement that P be that scale is more than just coherence.

In the order condition on P, there is transitivity, which holds anyway, and irreflexivity, which is equivalent to antisymmetry, by the transitivity, and which may or may not hold. The scale condition is negative transitivity together with antisymmetry. Thus, given the order condition, there is negative transitivity wanting, if the scale condition is to be established. But the given antisymmetry is not then generally enough to deduce it from transitivity.

A consequence of the preference relation P being a scale is that the indifference relation $\tilde{P} = \bar{P} \wedge \bar{P}'$ is an equivalence; and this is not at all inevitable if P is just an order. With P an order, and therefore irreflexive, $\tilde{P}$ is reflexive; also it is symmetric by construction, and therefore it is an equivalence if and only if it is transitive. But the transitivity of $\tilde{P}$ cannot be deduced from the transitivity of P; though, however, it can be deduced from the negative transitivity.

Thus, there is not a partitioning of C into indifference classes, to be derived just on the hypothesis that P is an order. The needed hypothesis is that P be a scale.



However, if the system is given as responsive, the generally effective distinction between the order and scale conditions loses its effect; and they become the same condition. Thus, though generally

$$\text{scale } P \Rightarrow \text{order } P$$

but not conversely, nevertheless,

$$\text{responsive } E\ .\Rightarrow.\ \text{order } P \Leftrightarrow \text{scale } P$$

By this proposition, with E responsive, it is only necessary to suppose P irreflexive for it to be a scale, and to be antisymmetric and negatively transitive; that is, to obtain $\tilde{\bar{P}}$ transitive, and therefore an equivalence, and then to obtain the complete order of the classes in this equivalence which they represent the scale.

6. Gauges of Preference

It has appeared that, for a responsive expenditure system E, if the preference relation P is irreflexive, which is the condition for coherence, then it is a scale, with indifference relation $\tilde{\bar{P}}$ which is an equivalence. There arises the question as to methods by which this scale can be actually known. The direct construction, contained in the definition, is plainly impossible, for it involves the formation of every chain, of limitless length, descending from every point. However, there is the advantage that P now is not just the order, that general cascading system of chains, formed in this manner, which is obtained by coherence without proximativity. It has that more presentable structure, given by a scale, and expressible as a complete order of the components in a partition; that is, a complete preference order of indifference classes. The problem can now be put more plainly, as that of the identification of the indifference classes, and of their complete order. One form of complete solution is to give construction for a gauge, which measures the scale, that is a function which is the greater or less for the better or worse in preference, and which takes equal values where there is no preference at all. It thus identifies the classes, by equality of values, and then order, by order of values.

If a system is responsive and coherent, it can be shown then to be necessarily integrable. Thus,

$$\text{responsive } E \wedge \text{irreflexive } P \Rightarrow \text{integrable } u'dx$$

Now, to the conclusion of integrability, known from the hypotheses of responsivity and irreflexivity, can be added the conclusion ϕ <u>is a gauge of</u> P <u>if it is an integral of</u> $u'dx$.

That is, any integral of the differential form of the system is a gauge of the preference relation of the system. The complete conclusion is that the differential form is integrable, and any integral gives a gauge for the scale, the conclusion that the order relation is in fact a scale appearing as a part of the conclusion. Hence,

$$\phi(x) > \phi(y) \Leftrightarrow xPy,$$

where the condition for ϕ to be an integral is that it be differentiable, having vector ϕ_x of partial derivatives $\partial\phi/\partial x_i$ with respect to the elements x_i of x, and

$$u\lambda = \phi_x$$

where $\lambda = x'\phi_x$ since $u'x = 1$. This is the condition for Φ to be stationary under the constraint $u'x = 1$ applied to x. From the condition

$$y \neq x \wedge u'y \leqslant 1 \Rightarrow xPy,$$

it appears that this stationary value must be an absolute maximum. Under the hypotheses of proximativity and coherence, the expenditure system is presented with the property that there exists a differentiable function Φ, and the composition x determined on any balance u is such as to give an absolute maximum of ϕ under the constraint $u'x = 1$. Thus, for a proximative system, the conditions that it have irreflexive preference relation, and that it be derivable from a differentiable function, by obtaining the x which gives maximum of that function under every condition $u'x = 1$, are altogether equivalent.

7. Local and Global Coherence

An expenditure system E defined on any balance region B is also defined for any subregion of B; and its coherence can be considered just in reference to any such subregion. Now the coherence of the expenditure system at any point may be defined by the condition that there exists an open neighborhood of that point in which the system is coherent.

The local coherence of a system in a region may be defined by the coherence of the system at each of its points; and global coherence, by coherence in the region as a whole.

Global coherence in an open region certainly implies local coherence; for the region itself is an open neighborhood of each of its points. But the converse, that local coherence implies global coherence, is not immediately evident, and very likely is not true, except under some further hypotheses. Proximativity provides one such hypothesis, and obtains the equivalence of local and global coherence in an open region. Thus, subject to proximativity, the distinction between local and global coherences disappears, and the two conditions become equivalent.

8. Total Incoherence

The condition of total incoherence for an expenditure system E may be defined as that in which every composition is preferred to every other; in other words, in which the preference relation P is universal: $P = \nabla$. Or, for another statement, there exists a preference cycle through any given pair of compositions. In this case, from any composition to any other, there are at least two chains, one ascending and the other descending, which therefore join to form a cycle, and which separately give the two opposite preferences between the compositions.

Total incoherence, which is $P = \nabla$, requires the reflexivity of P, that is $I \Rightarrow P$; but not conversely.

Let two compositions be said to be <u>encycled</u> together if they are elements of a preference cycle. Thus, if $\overset{\circ}{P}$ is the relation defined by

$$x\overset{\circ}{P}y \equiv xPy \wedge yPx$$

then two compositions have this relation if and only if they are encycled. Now, the thus defined relation of encyclement $\overset{\circ}{P}$ is symmetric, in the immediate form of its definition; and, by the transitivity of P, it is transitive. Another view of the transitivity is that two cycles with a common element describe a third cycle, which is that obtained by joining them, crossing itself at that element.

The symmetry and transitivity of the relation of encyclement is thus inherent in its definition. Therefore it only has to be reflexive to be an equivalence, and this condition is equivalent to the reflexivity of P.

Accordingly, if P is reflexive, then, and only then, there is defined a partition of the compositions, within any component of which all elements are connected by being encycled together, and between components of which there are no such connexions. In the case of complete incoherence, the partition is trivial, there being only one component, or one encyclement class, since all pairs of compositions are encycled together.

Consider an expenditure system E on an open balance region B, and let its preference relation P be everywhere reflexive. Then the encyclement relation $\overset{o}{P}$ is an equivalence. The condition that every composition be an interior point of its encyclement class is only possible if there is just one encyclement class, that is, if there is total incoherence. But this condition can be deduced from the antisymmetry of the base preferences Q, together with both responsivity and non-integrability in the region. Since integrability is implied by coherence, which is the non-reflexivity of P, taken together with responsivity, non-integrability implies the reflexivity which obtains encyclicity as an equivalence.

In this way there is arrived at the following proposition: If an expenditure system E is responsive in an open region, where there is non-integrability, but where the base preferences Q are antisymmetric, then it is totally incoherent, that is, every composition is preferred to every other.