A GAME IN ECONOMICS

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Econometric Research Program
Research Memorandum No. 111

February 1970

The research described in this paper was supported by the Office of Naval Research N00014-67 A-0151-0007, Task No. 047-086.

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Let us suppose that all of the coal deposits in a small country, isolated from the rest of the world by a range of high mountains, are owned by two competing firms. And let us suppose that a tunnel is under construction which, when completed at time $t = T$ in the not-too-distant future, will link the country with the outside world. At that time, the internal price of coal will become equal to the world price of $\$Q_W$ per ton. But until then, because the demand for coal in the country is highly inelastic, it will be possible for the firms to charge more.

We shall also assume that, at time $t = T$, it will become possible to sell an operating coal mine for a price $\$ P_W$, to foreign investors who wish to become exporters of coal from our small but coal-rich nation. But we shall not admit the possibility of selling mines prior to the completion of the tunnel, as it seems unlikely that either firm would allow a third competitor to enter the domestic market while they can so easily prevent it.

We shall assume further, that both firms mine coal in essentially the same way, and that by burning one ton of coal as fuel to operate its mining machinery, and by spending $\$ k$ in labor and other costs, 1 can produce b-1 (net) tons of new coal, while the same expenditure nets c tons for 2 . But we suppose that their prospects differ in that 1 has only a few mines in operation at the time $t=t_o$ when the firms begin to compete, but owns large undeveloped coal fields, whereas 2 starts the game with numerous mines in operation, but no undeveloped resources. Thus we should expect 1 , at least during the early stages of the competition, to be opening new mines, and to be equipping them with the newest and most efficient equipment available. And for this reason, we shall assume $b-1 > c$.

Finally, we assume that, in order to place a single new coal mine in operation, firm 1 must invest A tons of coal and $\$_K$ in labor and other costs. Thus, on each day of the period $t_o \leq t \leq T$, which we take to be of several years duration, firm 1 must decide how much of its present stock of coal to (1) allocate to the production of coal for present market consumption, how much to (2) invest, along with the required amounts of labor and other inputs, in the development of new mines, and how much (3) to stockpile against future market demands.

That is, if M(t) is the number of mines 1 has in operation on day t , and S(t) is the supply of coal it has on hand, and if the directors of 1 elect to allocate $u_1(t)$ tons of that coal to the development of new coal mines, and consume $u_2(t)$ tons in the production of new coal on that day, then firm 1 will have

$$(1a) \qquad M(t+1) = M(t) + u_1(t)/A = M(t) + a\,u_1(t)$$

mines in operation and

$$(1b) \qquad S(t+1) = S(t) + b\,u_2(t) - (u_1(t) + u_2(t))$$

tons of coal on hand at the start of day $t+1$. And if we append to the finite difference equations (1a) and (1b) the inequality constraints

$$(2) \qquad \begin{array}{ll} u_1(t) \geq 0 , & u_2(t) \geq 0 , \\ u_1(t)+u_2(t) \leq S(t) , \quad \text{and} & u_1(t) \leq \alpha\, M(t) , \end{array}$$

we obtain a complete description of 1's technological alternatives on day t . For clearly, 1 can not allot a negative amount of coal to either of its productive activities, consume more coal in a day as fuel than it had at the start of that day, or extract more than $b\alpha$ (gross) tons of coal in a day from a single mine, if α is the maximum number of tons of coal which may be burned for fuel in a day in a single mine.

However, since the competition is to extend over a large number of days, we may approximate $S(t+1) - S(t)$ and $M(t+1) - M(t)$ by $\dot{S}(t)$ and $\dot{M}(t)$, and so replace the difference equations (1a) and (1b) by the system

$$(1) \qquad \begin{aligned} \dot{M}(t) &= a\, u_1(t) \\ \dot{S}(t) &= b\, u_2(t) - (u_1(t) + u_2(t)) \end{aligned}$$

of ordinary differential equations. But we emphasize that it is really the discrete time process (1a) and (1b) which we wish better to understand; the introduction of continuous time is merely an analytical device which, in this case, facilitates the solution of the problem.

As to firm 2, its technology is even simpler, for it does not have the option of developing new mines. Indeed, if $I(t)$ is 2's coal inventory at time t, we may write simply

$$(3) \qquad \begin{aligned} &\dot{I}(t) = c\, v(t) \quad , \\ &0 \le v(t) \le I(t) \quad , \quad \text{and} \quad v(t) \le \beta \ , \end{aligned}$$

where $v(t)$ is the quantity of coal consumed on day t as fuel to power 2's machines, and β is the maximum number of tons of coal which can be burned in a day as fuel in 2's mines, which are fixed in number.

Next we assume that the demand for coal in the country in which they operate is totally inelastic, and is equal to $m_o = m\sqrt{\pi}$ tons of coal per day, regardless of the prices the two firms choose to charge. However, if firm 1's price $p_o(t)$ for a ton of coal on day t is higher than firm 2's price $q_o(t)$, then the demand $\delta(t)$ on firm 2 for coal on day t would exceed $m\sqrt{\pi} - \delta(t) = d(t)$, the demand on firm 1. More precisely, we assume that

$$(4) \qquad \begin{aligned} d(t) &= m\,\Phi\,(q_o(t) - p_o(t)) \\ \delta(t) &= m[\sqrt{\pi} - \Phi(q_o(t) - p_o(t)] = m\,\Phi\,(p_o(t) - q_o(t)), \end{aligned}$$

where the function Φ is defined for every real number x by the relation

$$\Phi(x) = \int_{-\infty}^{x} e^{-u^2}\,du .$$

Thus $0 < \Phi(x) < \sqrt{\pi}$ and $\Phi(0) = \sqrt{\pi}/2$. Also $\Phi(-x) = \sqrt{\pi} - \Phi(x)$, $\Phi'(x) = e^{-x^2}$, and $\Phi''(x) = -2\,x\,e^{-x^2} = -2\,x\,\Phi'(x)$. We shall assume too that $m_o = m\sqrt{\pi} < c\beta$, so that firm 2 has sufficient productive capacity to fill the entire market's demand. This would be the case, for instance, if 2 had historically been the only coal producer in the nation, and 1 had come into being largely as a speculation, seeking to exploit the opportunity afforded by the new tunnel.

The particular forms (4) for the demand functions $d(t)$ and $\delta(t)$ will not, of course, be appropriate in all situations. But they would be, for instance, if the two firms were located at opposite "ends" of the country, and each forced their customers to bear all of the transportation costs. For then, as indicated schematically in figure 0 below, there could be a point E somewhere between them where the price $p_o(t)$ plus the cost of transporting a ton of coal from 1 to E exactly equals $q_o(t)$ plus the cost of transportation from 2 to E . So

Figure 0.

the customers to the left of E will buy from 1 , while those to the right will buy from 2 . The location of E will depend, in simple cases at least, only on the difference $p_o(t) - q_o(t)$, and will move continuously and monotonically from 2 to 1 if that difference is allowed gradually to increase from $-\infty$ to $+\infty$.

Now let $P = P_w - K$, $Q = Q_w - k$, $p(t) = p_o(t) - k$, and $q(t) = q_o(t) - k$. P, Q, $p(t)$, and $q(t)$ are the profits associated with the world and domestic market prices P_w , Q_w , $p_o(t)$, and $q_o(t)$. And in terms of them, the firms' profits over the period $t_o \leq t \leq T$ may be expressed as

$$(5a) \quad J_1 = P(M(t) - M(t_o)) + Q(S(T) - S(t_o)) + m \int_{t_o}^{T} p(t)\, \Phi(q(t) - p(t))\, dt$$

and

$$(5b) \quad J_2 = Q(I(T) - I(t_o)) + m \int_{t_o}^{T} q(t)\, \Phi(p(t) - q(t))\, dt \quad ,$$

since $p_o(t) = p(t) - q(t)$. The complete evolution (often called "kinematic") equations for $M(t)$, $S(t)$, and $I(t)$ by appending the demand terms to the $\dot{S}$ and $\dot{I}$ equations given earlier. This yields

$$(6a) \quad \dot{M}(t) = a\, u_1(t) \qquad \dot{I}(t) = c\, v(t) - m\, \Phi(p(t) - q(t))$$

$$\dot{S}(t) = b\, u_2(t) - (u_1(t) + u_2(t)) - m\, \Phi\, (q(t) - p(t)).$$

The problem of maximizing the functionals (5a) and (5b) subject to the differential equation constraints (6a) and the inequality constraints

$$u_1(t) \geq 0 , \quad u_2(t) \geq 0 , \quad v(t) \leq \beta ,$$

(6b)

$$u_1(t) + u_2(t) , \quad u_2(t) \leq \alpha M(t) , \quad \text{and } 0 \leq v(t) \leq I(t) ,$$

is a differential game of the sort treated in [1]. So we shall seek to find a Nash equilibrium point for it in the class of "switching strategies" there defined. But first, a few simple observations are in order.

For instance, the quantities $P\ M(t_o)$, $Q\ S(t_o)$, and $Q\ I(t_o)$ are constants known at the outset of the game, and can have no effect on the players strategic decisions. Thus we may ignore them, and write more simply

$$J_1 = P\ M(T) + Q\ S(T) + m \int_{t_o}^{T} p(t)\Phi\ (q(t) - p(t))\ dt$$

(5)

$$J_2 = Q\ I(t) + m \int_{t_o}^{T} q(t)\ \Phi\ (p(t) - q(t))\ dt$$

Next, observe that it can never be optimal for 1, against any strategy for player 2 , to allow his supply of coal $S(t)$ to be exhausted before the end of the game. For 1's production process requires coal to make coal, so that if $S(t_1) = 0$ for any $t_1 < T$, then $S(t)$

remains zero and $M(t)$ has some constant value for $t > t_1$ as well. But if 1 had started, at time $t_1 - h$ to increase its price slightly, it would have run out of coal more slowly. Hence it would have had more coal to use in its production processes, during the period $t_1 - h < t \leq t_1$, and so have produced at least as much coal and as many mines as previously, before going out of business. And it would have sold the coal at a higher price. Thus $S(t_o) > 0$ implies $S(t) > 0$ for all $t_o < t < T$, if 1 is playing rationally, and a similar argument establishes that, if $I(t_o) > 0$, then $I(t) > 0$ for $t_o < t < T$.

And lastly, it seems apparent that if $P = P_W - K > AQ$, 1 would never choose its decision variables $u_1(t)$ and $u_2(t)$ in such a way that $u_1(t) + u_2(t) < S(t)$. For if, by doing so, 1 had finished the game with $S(T) > 0$ tons of coal in its yards, and $M(T)$ mines in operation, it would have forgone the profit $P \cdot a \Delta S$ to be earned by converting ΔS tons of coal into coal mines, in favor of the profit $Q \cdot \Delta S$ earned by selling the coal at the world price of $\$Q_W$ per ton. Because, had 1 decided, at some instant when $u_1(t) + u_2(t) < S(t)$, to increase $u_1(t)$ slightly, it would have ended the game with ΔS fewer tons of coal, but $a \cdot \Delta S$ more mines in operation. And since $P > QA$ implies $P/A = P \cdot a > Q$, such behaviour could never be optimal against any strategy which

2 might adopt. The argument in case $S(T) = 0$ is similar, but involves a simultaneous increase in both $p(t)$ and $u_1(t)$, so that the perturbed strategy does not lead 1 to run out of coal before the end of the game.

The above heuristics do not constitute a proof that $u_1(t) = S(t) - u_2(t)$, as we have not actually exhibited strategies by which 1 could guarantee himself the gains we have asserted are possible. But we shall accept them as ample justification for the introduction of a single new decision (control) variable $u(t) = u_2(t)$ for 1, and assuming $u_1(t) = S(t) - u(t)$ hereafter. When we do so, the kinematic equations (6a) become

$$(6) \qquad \begin{aligned} \dot{M}(t) &= a(S(t) - u(t)) \\ \dot{S}(t) &= bu(t) - S(t) - m\,\Phi\,(q(t) - p(t)) \\ \dot{I}(t) &= c\,v(t) - m\,\Phi\,(p(t) - q(t)) \quad , \end{aligned}$$

and the inequality constraints (6b) become

$$(7) \qquad \begin{aligned} & u(t) \geq 0 && v(t) \leq 0 \\ & S(t) - u(t) \leq 0 && I(t) - v(t) \leq 0 \\ & \alpha M(t) - u(t) \leq 0 && \beta - v(t) \leq 0 \quad . \end{aligned}$$

Hereafter, we will discuss only the simplified differential game defined by the payoffs (5), the equations (6), and the inequalities (7). And we will seek to solve it in the class of strategies $(u(t,x),\ p(t,x))$ and $(v(t,x), q(t,x))$ which depend on time and the state variable $x = (M,S,I)$ alone.

To indicate that the assumption $P > QA$, by which we justified setting $u_1(t) = S(t) - u_2(t)$, is not too unrealistic, we suggest that the values $A = \$4 \times 10^5$ tons of coal per mine, $K = \$10^6$ per mine, $P_w = \$2.5 \times 10^6$ per mine, $Q_w = \$11$ per ton of coal, and $k = \$8$ per ton of coal are not implausible. And then $\$A\ Q = A(Q_w - k) = 4 \times 10^5 x(11-8) = 1.2 \times 10^6 < 1.5 \times 10^6 = (2.5-1) x\ 10^6 = \$P = \$P_w - \$K$. We also note, in connection with these figures, that if b is at all large (say $b > 5$), then $b\ Q > 15$ is considerably greater than $a\ P = 15/12 = 5/4$. So we will not hesitate to assume $b\ Q > a\ P$, and to ignore the case $b\ Q \leq a\ P$ when it arises later on.

The fact that neither firm can produce any coal without some coal to start with is, of course, not realistic. It is a consequence of our assumption that coal is mined only with the aid of machines. The defect could easily be remedied by the incorporation of two small decision variables $\epsilon_1(t)$ and $\epsilon_2(t)$ on the right sides of the $\dot{S}$ and $\dot{I}$ equations (6), to allow the firms to dig coal by hand if necessary. But as it is clear that the firms never will need to dig coal by hand, if only we allow them positive initial stocks $S(t_o)$ and $I(t_o)$, it seems reasonable to suppose that they will not.

Most emphatically, neither will ever knowingly sell coal to the other to help him escape from the predicament of having no coal, should he be so unwise to fall into it. For during any period $A < t < B$ in which one of them has no coal, the other may exploit the inelasticity of demand to sell as much as $m_o = m\sqrt{\pi}$ tons of coal at any price it may wish to charge. Total inelasticity is not, of course, a realistic assumption but has proved to be a useful approximation in many cases, and we doubt that it does serious damage here.

Our confidence rests on the fact that a Nash equilibrium point is a highly competitive solution concept. Indeed, if our coal mining firms were disposed to act in concert, they could jointly name an arbitrarily high price of $\$_\pi$ per ton (thus becoming, in effect, a monopoly) and agree to split the profits evenly. But in fact, as we shall see shortly, the competitive desire of the players to play "rationally" (that is, in such a way that each player's strategy is the best possible against that of his opponent) will prevent them from exploiting the inelasticity of the market. This, in itself, seems to us an interesting assertion about the nature of "pure competition", and its effect on free market prices. The theory of many player differential games is developed in [1], and in particular, the definition of a switching strategy will be found there, and a brief description of the proceedure by which we shall solve the present game follows the statement of theorem 1.

To begin the solution process, we write down the two Hamiltonian functions

$$
\begin{aligned}
H_1 &= mp\Phi(q-p) + a\lambda_1(S-u) + \lambda_2(bu - S - m\Phi(q-p)) + \lambda_3(cv-m\Phi(p-q)) \\
&= m(p-\lambda_2+\lambda_3)\,\Phi(q-p) + (b\lambda_2-a\lambda_1)u + c\lambda_3 v + (a\lambda_1-\lambda_2)S - \sqrt{\pi}\, m\lambda_3
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
H_2 &= mq\Phi(p-q) + a\mu_1(S-u) + \mu_2(bu-S-m\Phi(q-p)) + \mu_3(cv-m\Phi(q-p)) \\
&= m(q+\mu_2-\mu_3)\,\Phi(p-q) + (b\mu_2-a\mu_1)u + c\mu_3 v + (a\mu_1-\mu_2)S - \sqrt{\pi}\, m\mu_2 \,.
\end{aligned}
$$

Next we find the values of (u,p) and (v,q) which maximize H_1 and H_2 respectively. Clearly the appropriate values $u*$ and $v*$ are

$$
\begin{aligned}
u^* &= 0 && \text{if } b\lambda_2 - a\lambda_1 < 0 \\
u^* &= S && \text{if } b\lambda_2 - a\lambda_1 > 0 \text{ and } S < \alpha M \\
u^* &= \alpha M && \text{if } b\lambda_2 - a\lambda_1 > 0 \text{ and } S > \alpha M
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
v^* &= 0 && \text{if } \mu_3 < 0 \\
v^* &= \beta && \text{if } \mu_3 > 0 \quad \text{and } I \geq \beta \\
v^* &= I && \text{if } \mu_3 > 0 \quad \text{and } I \leq \beta \,.
\end{aligned}
\tag{10}
$$

To calculate p^* and q^*, we introduce the variables $r = \lambda_2 - \lambda_3$ and $s = \mu_3 - \mu_2$, and observe that for fixed r and s, the expressions

(11) $$Q_1 = (p^* - r)\,\Phi(q^* - p^*) \quad \text{and} \quad Q_2 = (q^* - s)\,\Phi(p^* - q^*)$$

must be in equilibrium. That is, if p^* is replaced by any other real number p, the value of Q_1 must be reduced. And if q^* is replaced by any other q, Q_2 must be reduced. In short, we must find an equilibrium point for the game in which the payoffs are the functions (11) and the strategy sets are the p and q axes respectively. It will be shown that the game has a unique equilibrium point $(p^*, q^*) = (p^*(r,s), q^*(r,s))$ for every pair of non-negative real numbers r and s. Moreover p^* and q^* will always be positive. Necessary conditions that (p^*, q^*) be an equilibrium point for the game (11) are

(12) $$-\frac{\partial Q_1}{\partial p} = (p^* - r)\,\Phi'(q^* - p^*) - \Phi(q^* - p^*) = 0$$
$$-\frac{\partial Q_2}{\partial q} = (q^* - s)\,\Phi'(p^* - q^*) - \Phi(p^* - q^*) = 0 \ .$$

We shall first seek a function $p(q;r)$ which satisfies the first of equations (12) and a function $q(p;s)$ which satisfies the second. And then we shall show that the unique point of intersection of the graphs of the two functions is indeed an equilibrium point of the game.

We begin with a discussion of the level curves of the function

(13) $$Q_x(t,x,a) = (x - a)\,\Phi'(t - x) - \Phi(t - x) \ ,$$

where a is a non-negative constant. In particular, we wish to determine the set $\mathcal{S}$ of points (t, x) at which Q_x is zero. $\mathcal{S}$ is closed. Also $\mathcal{S}$ lies strictly above the line $x = 0$. For if some point (t, x) of $\mathcal{S}$ lay below it, the positive number $\Phi(t-x)$ should be equal to $(x-a)\,\Phi'(t-x)$, which is negative or zero. Moreover, $\mathcal{S}$ is not contained in any right half plane. For if t is any fixed real number, the function $Q(t, x, a) = (x-a)\,\Phi(t-x)$ is negative for $x < a$, positive for all $x > a$, and tends to zero as x becomes large. Hence there is a point $x(t)$ at which Q attains an absolute maximum. And there the slope of the graph of Q, which is just $Q_x(t, x, a)$, must vanish. Thus $\mathcal{S}$ cannot be contained in any right half plane, because it must meet every vertical line in the (x, t) plane. And finally, we observe that $\mathcal{S}$ must meet every line of the form $x = t + \alpha$ at exactly one point. For on such a line, we have $Q_x(t, t+\alpha, a) = [t + (\alpha - a)]\Phi'(\alpha) - \Phi(-\alpha)$, which is a linear function of t, and hence must take on every real value c exactly once as t varies over the line. In particular, the set $\mathcal{S}$ meets the line $x = t$ at the point $(a + \frac{1}{2}\sqrt{\pi},\ a + \frac{1}{2}\sqrt{\pi})$.

Next, let us suppose that a function $x(t) = x(t; a, c)$ is defined implicitly in a neighborhood of the point $x = t = c + a + \frac{1}{2}\sqrt{\pi}$ on the line $x = t$ by the equation

$$(x(t) - a)\,\Phi'(t - x(t)) - \Phi(t - x(t)) = c \quad . \tag{14}$$

Differentiating with respect to t, we obtain

$$-2(x(t) - a)(t - x(t))\,\Phi'(t - x(t)) + \dot{x}(t)\,\Phi'(t - x(t)) - (1 - \dot{x}(t))\,\Phi'(t - x(t)) = 0 \ , \tag{15}$$

which is equivalent to

$$\frac{dx}{dt} = \frac{1 + 2(x-a)(t-x)}{2 + 2(x-a)(t-x)} \quad . \tag{15'}$$

And substituting $t = t - a$ and $x = x - a$, this becomes

(16) $$\dot{x} = \frac{dx}{dt} = \frac{\frac{1}{2} + x(t-x)}{1 + x(t-x)} ,$$

which is identical with the equation obtained by setting $a = 0$ in (15') . That is, the level curve $x(t;a,c)$ may always be obtained from the level curve $x(t;0,c)$ by translating the latter upwards $a\sqrt{2}$ units along the line $x = t$.

Now let us sketch the solution curves of (16). To do so, we observe that $\dot{x}$ vanishes only along the two branches of the hyperbola $x(x-t) = \frac{1}{2}$, and becomes infinite only on $x(x-t) = 1$. The slope of the solution curves is positive everywhere save in the two narrow crescent-shaped regions which lie between the respective branches of the hyperbolae. These are the shaded regions shown

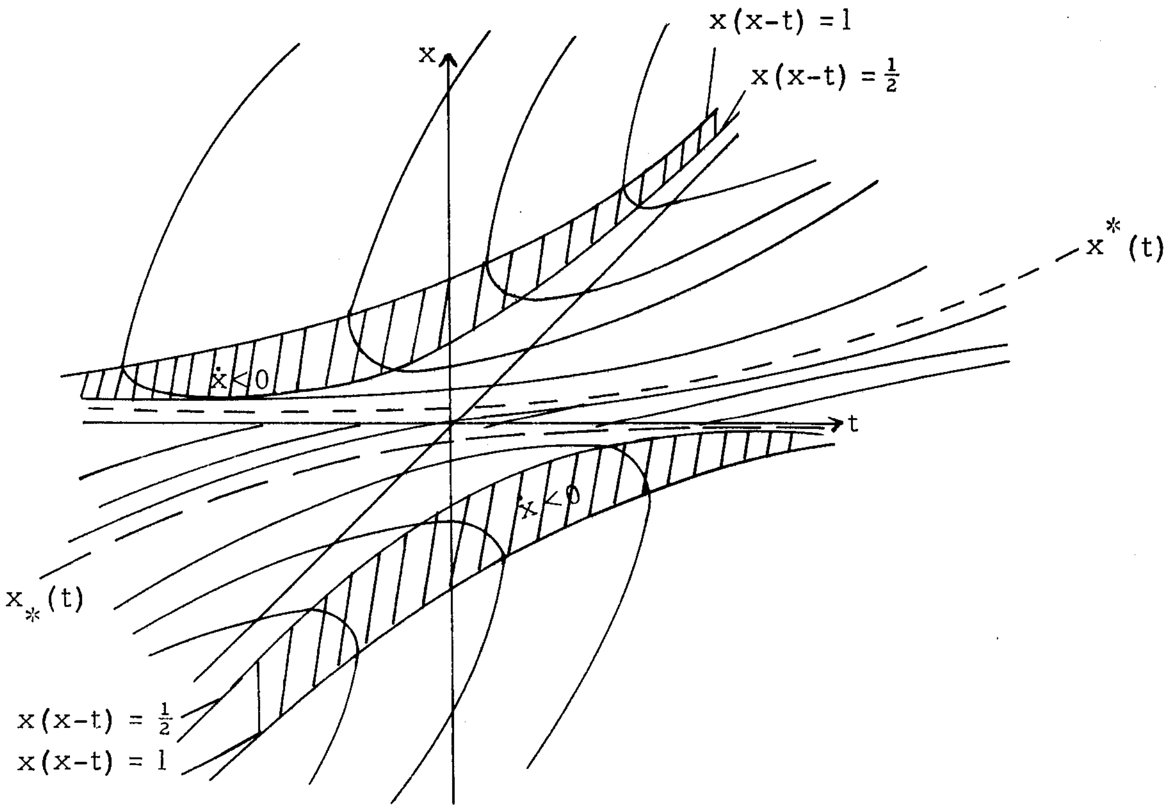

Figure 1

The level curves of $x\Phi'(t-x) - \Phi(t-x)$

in Figure 1. In particular, we observe that the lines $x = 0$ and $x = t$ are asymptotes for both the hyperbolae, and that $\frac{1}{2} \leq \dot{x} \leq 1$ in the two sectors of opening $\pi/4$ formed thereby. The above information enables us to sketch the trajectories shown in the figure. They partition the (x,t) - plane into three disjoint open sets. The uppermost of these is filled with level curves which attain their minima along the upper branch of $x(x-t) = \frac{1}{2}$. Clearly such curves can never cross the t axis. Similarly, the lowest region is filled with curves which attain their maxima along the lower branch of $x(x-t) = \frac{1}{2}$. These all lie in the lower half plane. The third region, which contains the entire t-axis, is filled with solutions of positive slope, which are bounded above and below by the upper and lower branches of $x(x-t) = \frac{1}{2}$. Hence these are defined on the entire interval $-\infty < t < \infty$. The boundaries which separate the three regions are the two solutions $x^*(t)$ and $x_*(t)$ shown in the figure.

Next, observe that every solution of (16) crosses the line $x = t$ exactly once. Hence each of the sets $\mathcal{S}_c = \{(x, t) : Q_x(t,x,0) = c\}$ consists of just one of the solutions of (16). In particular $\mathcal{S}$ is so constituted. But since $\mathcal{S}$ must lie in the upper half plane, it must lie on or above $x^*(t)$. And since $\mathcal{S}$ is not contained in any right half plane, it can not be any of the curves crossing the hyperbolae $x(x-t) = \frac{1}{2}$ and $x(x-t) = 1$. Therefore $\mathcal{S}$ can only be the graph of $x^*(t)$ itself.

Reverting to our previous notation, we may express this fact by writing $x^*(t) = x(t;0,0)$. The latter function is now globally defined. And we may calculate $x(t;a,0)$ by translating $x(t;0,0)$ along the line $x=t$ as described previously. Hence we may obtain $p(q;r)$ from $x(t;a,0)$ by substituting $x=p$, $t=q$, and $a=r$. And $q(p;s)$ is got by setting $x=q$, $t=p$, and $a=s$. The resulting state of affairs is indicated in Figure 2. There the curves Γ_p and Γ_q are just the graphs of $p(q;r)$ and $q(p;s)$ respectively. Clearly the curves do cross. We call the coordinates of the point A at which they do so $p^* = p^*(r,s)$

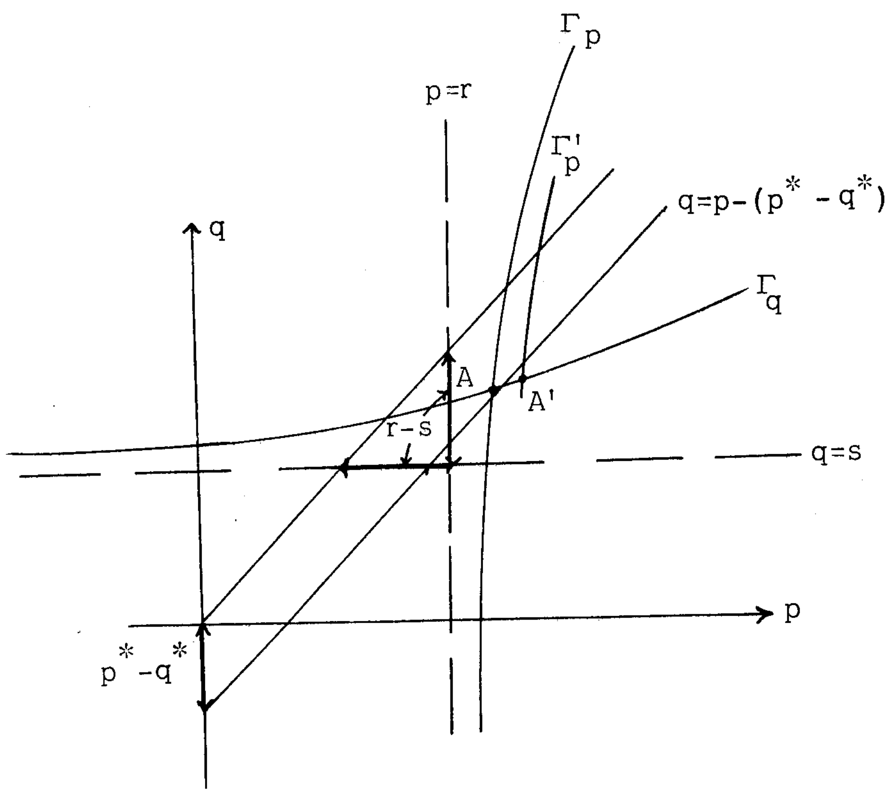

Figure 2

The location of the point $A(p^*, q^*)$ from r and s

and $q^* = q^*(r,s)$. To see that there is only one such point, construct the line $q = p - (p^* - q^*)$ thru $A(p^*, q^*)$. If Γ_p and Γ_q should cross at another point, one of them would have to cross this line a second time. But both Γ_p

and Γ_q cross every line of unit slope exactly once. Thus $A(p^*, q^*)$ is the unique point of intersection for Γ_p and Γ_q, and hence the only point at which equations (12) both hold. Moreover it is a true equilibrium point because Q_1 and Q_2 attain their absolute maxima (in p and q respectively) there.

Knowing (p^*, u^*) and (q^*, v^*), we may now proceed to write and solve the Hamilton-Jacobi equations of the game. But before we do so, we deduce a growth property of the functions p^* and q^* of r and s which we shall need later. We deduce that if $r-s$ increases monotonically and without bound, the difference $p^*(r,s) - q^*(r,s)$ must do so too. For if we add any number Δ to r and to s, the effect on the curves Γ_p and Γ_q is to slide them upwards and to the right $\Delta\sqrt{2}$ units along the line $p = q$. And so the point $A(p^*, q^*)$ where they cross slides along $q = p - (p^* - q^*)$, leaving $p^* - q^*$ unchanged. Therefore $p^* - q^*$ depends only on the difference $r - s$. And if we increase r unilaterally, the effect is to move Γ_p along $p = q$ to the new position Γ'_p , which meets Γ_q at a point $A' = A(p^{**}, q^{**})$ below $q = p - (p^* - q^*)$. Then $p^{**} - q^{**} > p^* - q^*$, so that the dependence of $p^*(r,s) - q^*(r,s)$ on $r - s$ is monotone and increasing. Also, the increase is without bound. For given any number α, the curve Γ_q meets the line $q = p - \alpha^2$ at just one point A_α . So if we choose r so large that the line $p = r$ lies to the right of A_α, the point (p^*, q^*) must also lie to the right of A_α, because the entire curve Γ_p lies to the right of $p = r$. Hence the line $q = p - (p^* - q^*)$ lies below $q = p - \alpha^2$, and $p^* - q^* > \alpha^2$. This fact will be useful in discovering the properties of the optimal strategies.

Let us now consider the value functions $V(t, x)$ and $W(t, x)$ for players 1 and 2 respectively. These must[†] obey the Hamilton-Jacobi system of partial differential equations, as well as the initial conditions

$$
\begin{aligned}
V(T,M,S,I) &= PM + QS \\
W(T,M,S,I) &= QI \ .
\end{aligned}
\tag{17}
$$

The Hamilton-Jacobi equations are obtained by inserting the strategy pairs (p^*, u^*) and (q^*, v^*) into the Hamiltonian functions (8), and equating the results to $-V_t$ and $-W_t$ respectively. Recalling[†] that the vectors $(\lambda_1, \lambda_2, \lambda_3)$ and (μ_1, μ_2, μ_3) are in reality the gradients $V_x = (V_M, V_S, V_I)$ and $W_x = (W_M, W_S, W_I)$ we have for the present case

$$
\begin{aligned}
&V_t + m(p^*(r,s) - r)\Phi(q^*(r,s) - p^*(r,s)) + (bV_S - aV_M)u^* \\
&\qquad + cV_I v^* + (aV_M - V_S)S - m\sqrt{\pi}\ V_I = 0 \\
&W_t + m(q^*(r,s) - s)\Phi(p^*(r,s) - q^*(r,s)) + (bW_S - aW_M)u^* \\
&\qquad + cW_I v^* + (aW_M - W_S)S - m\sqrt{\pi}\ W_S = 0 \ .
\end{aligned}
\tag{18}
$$

Here we have retained the notations $r = \lambda_2 - \lambda_3 = V_S - V_I$ and $s = \lambda_3 - \lambda_2 = W_I - W_S$ for simplicity. Into the equations (18) we now substitute the trial solutions

$$
\begin{aligned}
V &= \lambda_1(\tau)M + \lambda_2(\tau)S + \varphi(\tau) \\
W &= \qquad \mu_3(\tau)I + \psi(\tau) \ ,
\end{aligned}
\tag{19}
$$

where τ is the backwards time variable defined by $\tau = T - t$, and $\lambda_1(\tau), \lambda_2(\tau), \mu_3(\tau)$, $\varphi(\tau)$ and $\psi(\tau)$ are functions to be determined. We now have

† See [1], Theorem 1.

$V_M = \lambda_1(\tau)$, $V_S = \lambda_2(\tau)$, $W_I = \mu_3(\tau)$, $r = \lambda_2(\tau)$, $s = \lambda_3(\tau)$, and $V_I = W_M = W_S = 0$, so that the equations (18) become

$$m(p^*(\tau) - \lambda_2(\tau))\,\Phi(q^*(\tau) - p^*(\tau)) + S(a\lambda_1(\tau) - \lambda_2(\tau)) + (b\lambda_2(\tau) - a\lambda_1(\tau))u^*$$

(20a)

$$= \lambda_1'(\tau)M + \lambda_2'(\tau)S + \varphi'(\tau)$$

(20b) $$m(q^*(\tau) - \mu_3(\tau))\Phi(p^*(\tau) - q^*(\tau)) + cv^*\mu_3(\tau) = \mu_3'(\tau)I + \psi'(\tau) \quad .$$

Here we have written $p^*(\tau)$ and $q^*(\tau)$ instead of $p^*(\lambda_2(\tau), \lambda_3(\tau))$ and $q^*(\lambda_2(\tau), \mu_3(\tau))$ for convenience, and introduced the operator $' = d/d\tau$. The equations (20) actually represent six separate pairs of equations which result from the six possible control pairs (u^*, v^*), namely

(21) (i) $u^* = 0,\ v^* = \beta$ (ii) $u^* = S,\ v^* = \beta$ (iii) $u^* = \alpha M,\ v^* = \beta$

(i') $u^* = 0,\ v^* = I$ (ii') $u^* = S,\ v^* = I$ (iii') $u^* = \alpha M,\ v^* = I$.

The functions $\lambda_1(\tau)$, $\lambda_2(\tau)$, and $\mu_3(\tau)$ may now be determined for each pair (u^*, v^*), by equating the coefficients of M, S, and I on the right and on the left. And once this has been done, it will be possible to obtain $\varphi(\tau)$ and $\psi(\tau)$ by quadrature. For instance if $I > \beta$, then $v^* = \beta$ so that I appears only on the right in (20b), and not at all in (20a). Then $\mu_3(\tau)$ must satisfy the equation

(22) $$\mu_3'(\tau) = 0 \quad .$$

Or if $I < \beta$, then $v^* = I$, so that $\mu_3(\tau)$ obeys

(23) $$\mu_3'(\tau) = c\mu_3(\tau) \quad .$$

Similarly, if $b\lambda_2 - a\lambda_1 < 0$, then $u^* = 0$, so that the equations for $\lambda_1(\tau)$ and $\lambda_2(\tau)$ are

(24)
$$\lambda_1'(\tau) = 0$$
$$\lambda_2'(\tau) = a\lambda_1(\tau) - \lambda_2(\tau) \quad .$$

But if $b\lambda_2 - a\lambda_1 > 0$, then $\lambda_1(\tau)$ and $\lambda_2(\tau)$ obey

(25)
$$\lambda_1'(\tau) = 0$$
$$\lambda_2'(\tau) = (b-1)\lambda_2(\tau)$$

if $S < \alpha M$, $\mu^* = S$, or

(26)
$$\lambda_1'(\tau) = -a\alpha\lambda_1(\tau) + b\alpha\lambda_2(\tau)$$
$$\lambda_2'(\tau) = a\lambda_1 - \lambda_2$$

if $S > \alpha M$ and $\mu^* = \alpha M$. Then $\varphi(\tau)$ is given by

(27)
$$\varphi(\tau) = m\int_0^\tau (p^*(t) - \lambda_2(t))\Phi(q^*(t) - p^*(t))\,dt \quad ,$$

where the functions p^* and q^* are as above. Of course we can not expect to write down $\varphi(\tau)$ explicitly, as we have only qualitative information about p^* and q^* . But the required integration can in theory be carried out over any interval $0 \leq \tau \leq B$ on which $\lambda_2(\tau)$ and $\mu_3(\tau)$ are known. $\psi(\tau)$ is obtained in a similar manner. And since the equilibrium strategies do not involve the functions φ and ψ, it shall suffice for our purposes to be assured of their existence. For then the process which we have described does in fact lead to the solutions of the Hamilton-Jacobi equations (18). We now pass to the actual construction of these solutions.

First, let us suppose that on some backwards time interval $A \leq \tau \leq B$, we have $b\lambda_2(\tau) < a\lambda_1(\tau)$. Then the equations (24) hold, so that $\lambda_1(\tau)$ is constant and $\lambda_2(\tau)$ is a decreasing function of τ†. Therefore the above inequality

† In this problem, the adjoint variables $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3 are shadow prices, and so must eventually turn out to be non-negative.

actually holds on the entire half-line $A \leq \tau < \infty$, and if $u^*(A) = 0$, then $u^*(\tau)$ must vanish also for every $\tau > A$. If we call the region of (τ, M, S, I)-space wherein $u^* \equiv 0$ the "region of rapid growth", or RG (because 1 is devoting himself exclusively to the expansion of his productive facilities there), we may rephrase the above fact by saying that once the state $S(\tau)$, $M(\tau)$, $I(\tau)$ of the game enters RG, it never again will leave it. In particular, if $bQ < aP$, the entire game is played in RG. This corresponds to the fact that if the world price P_W of coal mines is sufficiently high relative to that of coal, 1 will go out of the coal business entirely in order to become a builder of mines. To rule out this uninteresting possibility, we shall assume hereafter that $bQ > aP$.

Our procedure now will be to solve the equations (18) locally at first (that is, in a neighborhood of the initial manifold $\tau = 0$), and then to extend our solutions as far as we are able. Now at the instant $\tau = 0$, we have $\Phi(p-q) = \sqrt{\pi} - \Phi(q-p) = \sqrt{\pi} - \Phi(q^*(0) - p^*(0)) = \sqrt{\pi} - \Phi^* < \sqrt{\pi}$. And since we assumed that $m\sqrt{\pi} < c\beta$, there is a line $I = I^*$ in the (I, τ) plane above which the solution $I(\tau)$ of (6) decreases with τ near the I-axis, and below which $I(\tau)$ is initially increasing. Meanwhile, the (S, M)-plane will be divided into two regions by the line $S = \alpha M$. In the region where $S \leq \alpha M$, which we shall call the "region of no growth" (NG for short), because $M(\tau)$ is constant there, the equations for $S(\tau)$ and $M(\tau)$ are just

$$(28) \qquad M'(\tau) = 0 \qquad S'(\tau) = m\Phi(q^*(\tau) - p^*(\tau)) - (b-1)S(\tau) ,$$

while in $S \geq \alpha M$ (which we shall call the "region of slow growth", or SG), we have

$$(29) \qquad M'(\tau) = a(\alpha M(\tau) - S(\tau)) \quad S'(\tau) = m\Phi(p^*(\tau) - q^*(\tau)) + S(\tau) - b\alpha M(\tau) .$$

Thus the behavior of the trajectories $S(\tau)$, $M(\tau)$ for small values of τ is as indicated in Figure 3.

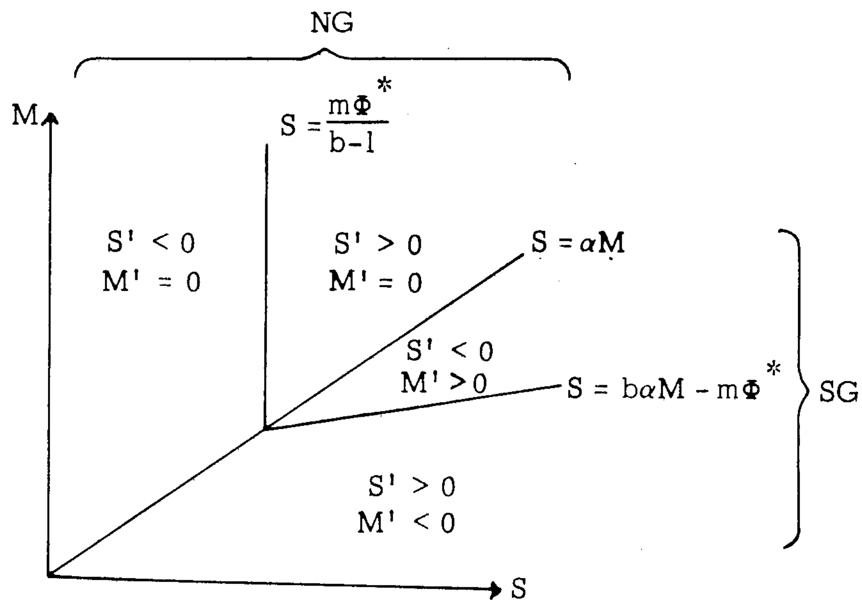

Figure 3

The flow in the (S, M)-plane near $\tau = 0$

Next, let us study the curves $S(\tau)$, $M(\tau)$ which lie in NG for larger values of τ. Here $\lambda_1(\tau)$ and $\lambda_2(\tau)$ satisfy the equations (25), so that $\lambda_1(\tau) = P$ and $\lambda_2(\tau) = Qe^{(b-1)\tau}$. Therefore $b\lambda_2(\tau) - a\lambda_1(\tau)$ is both positive and increasing on any interval $0 \leq \tau \leq B$ on which $S(\tau)$, $M(\tau)$ remains in NG, so that the switch from $u^* > 0$ to $u^* = 0$ (which takes place when $b\lambda_2 - a\lambda_1$ changes sign) never occurs in NG. Also, by the growth property established above for $p^* - q^*$ as a function of $r - s$, $p^*(\tau) - q^*(\tau)$ must increase monotonically and without bound. For here $r - s = \lambda_2(\tau) - \mu_3(\tau) = Q(e^{(b-1)\tau} - e^{c\tau})$ increases monotonically and without bound because of our assumption (namely that $b - 1 > c$) on the relative efficiencies of the two firms' production processes.

Thus $\delta(\tau) = m\Phi(q^*(\tau) - p^*(\tau))$ must decrease monotonically to zero. And differentiating again the second of equations (28), we have $S''(\tau) = \delta'(\tau) - (b-1)S'(\tau)$ and $S''(\tau) = \delta'(\tau) < 0$ for any value of τ for which $S'(\tau) = 0$. Hence $S(\tau)$ may have one local maximum, but no local minima in NG. In order to interpret these facts, let us suppose that the play begins at $\tau = 0$ at a point (M_0, S_0) in NG. Then the entire subsequent motion in NG takes place in the plane $M = M_0$, so that the problem of sketching the trajectories in NG reduces to plotting $S(\tau)$

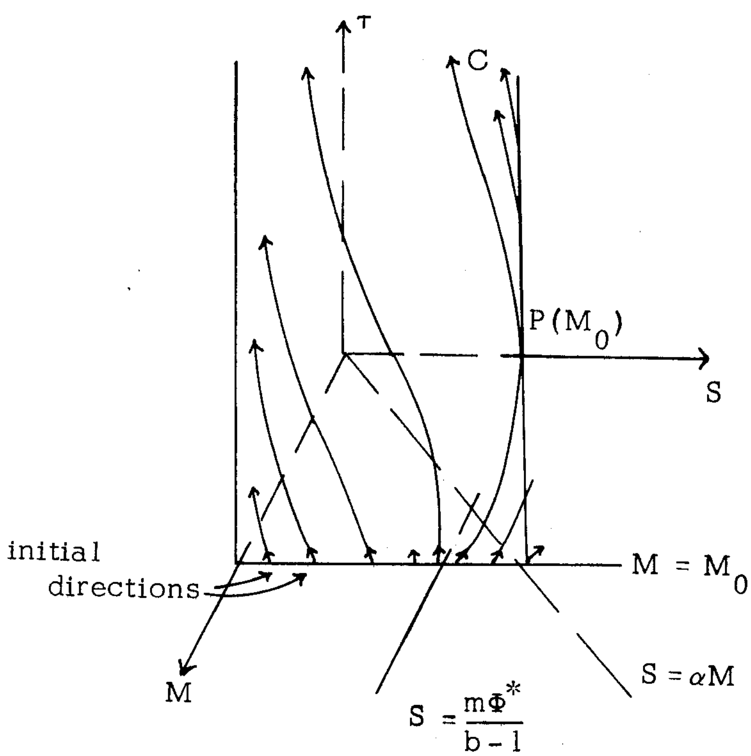

Figure 4

The optimal trajectories in NG

against τ in the following figure (Figure 4). Here the fact that the trajectories beginning to the left of $S = m\Phi^*/(b-1)$ decrease monotonically is due to the facts that they decrease initially and that they can have no minima. Those beginning close to $(M_0, \alpha M_0)$ must escape from $S \leq \alpha M$ because of the smooth dependence of the solutions of (28) on initial conditions. In particular (see Figure 3), if $M_0 \geq m\Phi^*/\alpha(b-1)$, there must be a curve C which strikes the plane $S = \alpha M$ tangentially, and then returns to NG . Let us denote by $P(M_0)$ the point at which this contact takes place. Then if we let M_0 increase from $m\Phi^*/\alpha(b-1)$ to infinity, the locus of $P(M_0)$ will be a curve Γ in the plane $S = \alpha M$, as shown in Figure 5. Every point in $S = \alpha M$ which lies below Γ is

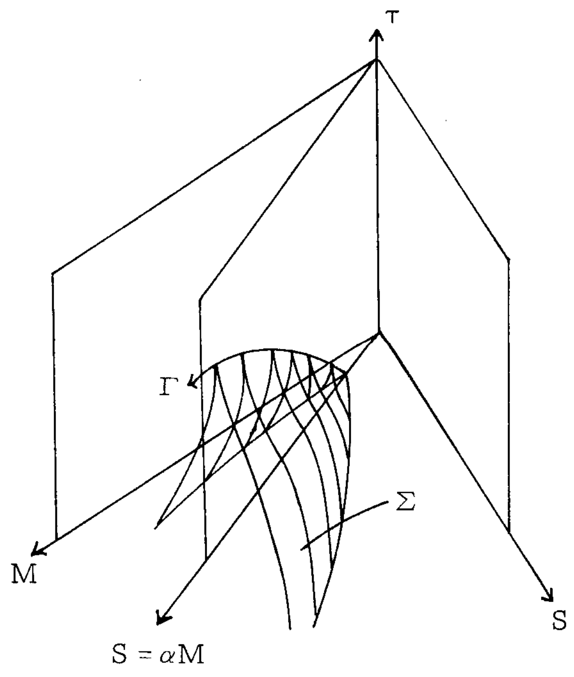

Figure 5

The surface Σ in SG

the end point of a trajectory emerging from NG, while the points above Γ are the initial points of trajectories moving back into NG . Let P^* be such a point, and let $S(t)$, $M(t)$ be a forward time trajectory out of NG thru P^* . Because the right hand sides of the equations (28) and (29) are continuous across $S = \alpha M$, the curve $S(t)$, $M(t)$ must be continuable into $S \geq \alpha M$. Now consider the solutions of (29) which pass thru points P^* on Γ . They form a surface Σ in $S \geq \alpha M$, which divides the latter into two regions. The region under Σ is filled with solutions of (29) which begin on $S = \alpha M$ and descend to the plane $\tau = 0$ (or rather $t = T$, since we are speaking now of forward time). What this means is that if firm 1 begins the game at a point P^* in $S = \alpha M$ which lies under Γ, he may choose to proceed either into $S < \alpha M$ or $S > \alpha M$, his payoff being the same in either case. In order to better describe the behavior of the curves lying under Σ, and indeed in all of $S > \alpha M$, it is necessary to discuss the behavior of the solutions of (29) with some care.

For this purpose, it is convenient to make use of the following theorem on differential inequalities.

<u>Theorem</u>: Let $f = (f^1, \ldots, f^n)$ be a function defined and continuously differentiable on an open set $R\times[0,T]$ of E_{n+1} which takes values in E_n, and suppose that each f^i is monotone increasing in each of its first n variables. For any continuously differentiable function $u(t)$ from the interval $[0,T]$ into R , define $P(u) = (\dot{u}(t) - f(u(t),t))$. Then the inequalities $u(0) \leq v(0)$ and $P(u) \leq P(v)$ imply that $u(t) \leq v(t)$ for all $0 \leq t \leq T$. This is a slight specialization of a theorem to be found on page 85 of Walter [6]. A more general version is to be

found on page 25 of Szarski [5]. In order to apply it to the system (29), we introduce (temporarily) the new variables $x = M$ and $y = -S$. Then (29) takes the form

$$\text{(30)} \qquad \begin{aligned} x'(\tau) &= a\alpha x(\tau) + ay(\tau) \\ y'(\tau) &= b\alpha x(\tau) + y(\tau) - \delta(\tau) \end{aligned}$$

and the right hand sides are indeed increasing in both x and y. So let $(x(\tau), y(\tau))$ be a solution of (30) starting at (x_0, y_0), and let $(x_*(\tau), y_*(\tau))$ and $(x^*(\tau), y^*(\tau))$ be solutions of the related systems

$$\text{(31)} \qquad \begin{aligned} x_*'(\tau) &= a\alpha x_*(\tau) + ay_*(\tau) \\ y_*'(\tau) &= b\alpha x_*(\tau) + y_*(\tau) - m\sqrt{\pi} \end{aligned} \qquad \text{and} \qquad \begin{aligned} x^{*\prime}(\tau) &= a\alpha x^*(\tau) + ay^*(\tau) \\ y^{*\prime}(\tau) &= b\alpha x^*(\tau) + y^*(\tau) \end{aligned},$$

which also pass thru (x_0, y_0) when $\tau = 0$. Now write

$$\text{(32)} \qquad P\begin{pmatrix} \rho(\tau) \\ \sigma(\tau) \end{pmatrix} = \begin{pmatrix} \rho'(\tau) - a\alpha\rho(\tau) - a\sigma(\tau) \\ \sigma'(\tau) - b\alpha\rho(\tau) - \sigma(\tau) \end{pmatrix}$$

for any two functions $\rho(\tau)$, $\sigma(\tau)$ defined on $0 \leq \tau < \infty$. Then

$$\text{(33)} \qquad P\begin{pmatrix} x_*(\tau) \\ y_*(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ -m\sqrt{\pi} \end{pmatrix} \leq \begin{pmatrix} 0 \\ -\delta(\tau) \end{pmatrix} = P\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} = P\begin{pmatrix} x^*(\tau) \\ y^*(\tau) \end{pmatrix},$$

so that $x_*(\tau) \leq x(\tau) \leq x^*(\tau)$ and $y_*(\tau) \leq y(\tau) \leq y^*(\tau)$. Thus if we write the systems

$$\text{(34)} \qquad \text{(a)} \quad \begin{aligned} M' &= a\alpha M - aS \\ S' &= -b\alpha M + S \end{aligned} \qquad \text{(b)} \quad \begin{aligned} M' &= a\alpha M - aS \\ S' &= -b\alpha M + S + m\sqrt{\pi} \end{aligned},$$

and let $(M^*(\tau), S^*(\tau))$ and $(M_*(\tau), S_*(\tau))$ be solutions of the homogeneous and inhomogeneous systems (34) respectively, which pass thru (M_0, S_0) when

$\tau = 0$, while $(M(\tau), S(\tau))$ is the solution of (29) which satisfies the same initial condition, we have

$$M_*(\tau) \leq M(\tau) \leq M^*(\tau)$$
(35) $$S_*(\tau) \geq S(\tau) \geq S^*(\tau) \ ,$$

for all $0 \leq \tau < \infty$. Thus the point $M(\tau), S(\tau)$ is always contained in the rectangle in the (M,S)-plane whose upper left-hand corner is $(M_*(\tau), S_*(\tau))$ and whose lower right hand corner is $(M^*(\tau), S^*(\tau))$. Moreover, if $M^*(\tau), S^*(\tau)$ is any solution of the homogeneous system (34a), then $(M^*(\tau) + \frac{m\sqrt{\pi}}{(b-1)\alpha}$, $S^*(\tau) + \frac{m\sqrt{\pi}}{(b-1)}$) is a solution of the inhomogeneous system (34b).

Now since the determinant of the inhomogeneous system does not vanish, it has a single stagnation point at the origin. And since the characteristic equation is

(36) $$\lambda^2 - (1+a\alpha)\lambda + (1-b)a\alpha = 0 \ ,$$

there are always two real characteristic roots of opposite sign. Moreover there are two straight line solutions of the form $S^*(\tau) = k_1 M^*(\tau)$ and $S^*(\tau) = k_2 M^*(\tau)$, where k_1 is a negative number and k_2 is greater than α . Hence the origin becomes a saddle point (see Coddington and Levinson [3] page 373), and the solution curves have the form indicated in Figure 6. And by the remark of the previous paragraph, the curves for the inhomogeneous system are obtained by translating the origin $m\sqrt{\pi(1+\alpha^2)}/\alpha(b-1)$ units to the right along the line $S = \alpha M$. The resulting state of affairs is indicated in Figure 7 . If play begins at time $\tau = 0$ at a point P_1^* above the line $S = k_2 M$, both the curve Γ^* (which is a trajectory of the homogeneous system) and the curve Γ_* (inhomogeneous) cross

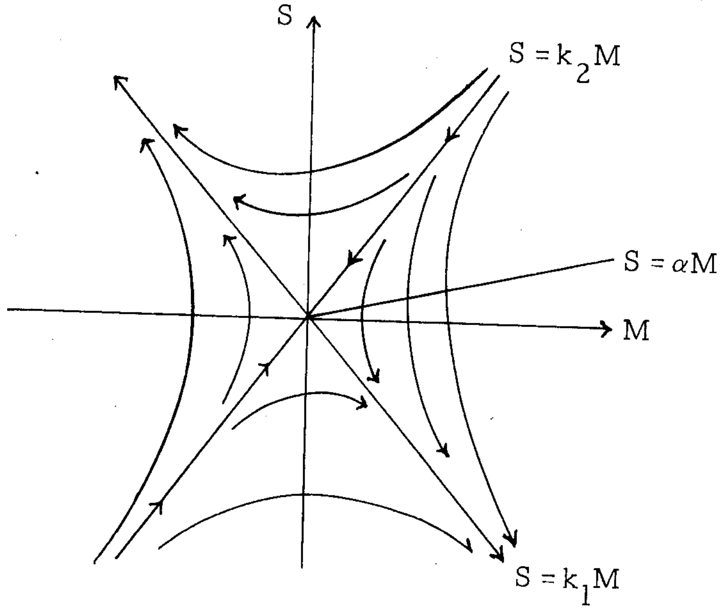

Figure 6

The solutions of the homogeneous system (34)

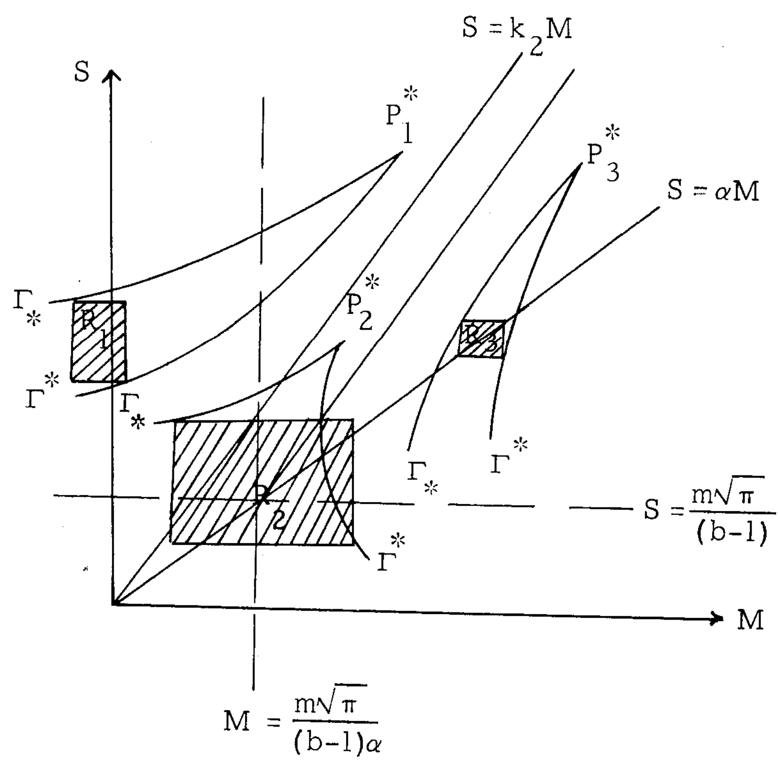

Figure 7

The estimation of the point $(M(\tau), S(\tau))$

the S axis after a finite time. And since the rectangle R_1, which contains the point $(M(\tau), S(\tau))$, must cross it with them, we may conclude that the solution of (29) beginning at P_1^* escapes from the first quadrant no later than the solution Γ^* of the homogeneous system. And similarly, the solutions of (29) beginning at points P_3^* beneath the line $S + m\sqrt{\pi}/(b-1) = k_2(M + m\sqrt{\pi}/(b-1)\alpha)$ must cross the line $S = \alpha M$ no later than the solution Γ_* of the inhomogeneous system (34). About the solutions of (29) beginning at points P_2^* between the lines $S = k_2 M$ and $S + m\sqrt{\pi}/(b-1) = k_2(M + m\sqrt{\pi}/(b-1)\alpha)$ however, we have no such accurate information. For here the curves Γ^* and Γ_* diverge rapidly, so that the rectangle R_2 of Figure 7 grows quickly and without bound.

Nevertheless, we are now in a position to complete the sketching of the equilibrium trajectories begun in Figure 5. For above the surface Σ shown there, there must be a second such surface Σ', looking very much like Σ, which is the boundary of that portion of SG which is filled with trajectories beginning on $S = \alpha M$ and descending (in forward time) to the plane $\tau = 0$. Above and to the right of Σ', the region $S > \alpha M$ is completely filled with trajectories beginning on $M = 0$ and descending into the plane $\tau = 0$. And close to that plane, these are indeed the equilibrium trajectories. But as τ increases, there is the possibility that $(M(\tau), S(\tau))$ may cross the boundary separating RG from SG, so that u^* switches from the value αM to zero. Beyond such a boundary, of course, the solutions of (29) are no longer optimal paths.

To preclude this phenomenon, we recall that in SG, the variables $\lambda_1(\tau)$ and $\lambda_2(\tau)$ obey the equations (26). These are linear and homogeneous, and have a single stagnation point at the origin. Moreover, the latter is again a saddle point, since the characteristic equation for this system too has one positive and one negative root. Thus there are two straight line solutions $\lambda_2(\tau) = \ell_1\lambda_1(\tau)$ and $\lambda_2(\tau) = \ell_2\lambda_1(\tau)$, with ℓ_1 being negative and ℓ_2 being in the range $a/b < \ell_2 < a$. Moreover, along the line $\lambda_2 = \ell_1\lambda_1$, we have $\lambda_1'(\tau) < 0$, while on $\lambda_2 = \ell_2\lambda_1$, $\lambda_1'(\tau) > 0$. Hence all the solutions of (26) which begin in the first quadrant of the (λ_1, λ_2)-plane tend asymptotically to the solution $\lambda_2 = \ell_2\lambda_1$, as indicated in Figure 8. This shows in particular that solutions

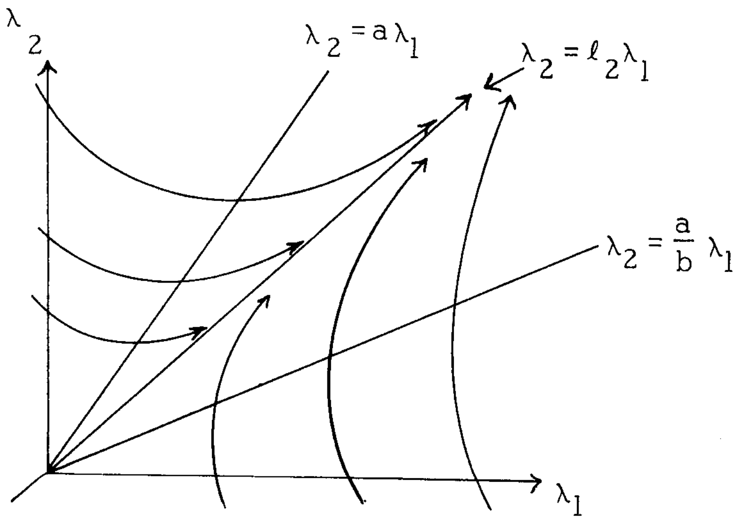

Figure 8

The positive solutions of (26)

which start from points (P, Q) above the line $\lambda_2 = (a/b)\lambda_1$ can never cross the line $\lambda_2 = \ell_2\lambda_1$, so that the indicator function $bp_2(\tau) - ap_1(\tau)$ never changes sign. Hence the control $u^* = 0$ is never optimal, either in $S \leq \alpha M$ or in $S \geq \alpha M$, and the curves sketched in the previous sections (which fill the positive orthant of (S, M, τ)-space) do indeed appear to be the optimal trajectories for firm 1. The solution process will thus be complete if we can describe the optimal trajectories $I(\tau)$ for player 2. For we will then know (qualitatively at least) the entire subsequent behavior of both players, starting from an arbitrary initial point (t_0, M_0, S_0, I_0) under optimal play.

In the (I, τ)-plane, the governing differential equation is

$$I'(\tau) = m\sqrt{\pi} - cv^* - \delta(\tau) \quad , \tag{37}$$

where $v^* = I$ if $I < \beta$ and $v^* = \beta$ if $I > \beta$. And since we assumed that $m\sqrt{\pi} < c\beta$, I' is positive along $I = 0$ and negative on $I = \beta$, so that the curves are those sketched in Figure 9. Note the two distinguished curves $I_*(\tau)$ and $I^*(\tau)$, which start

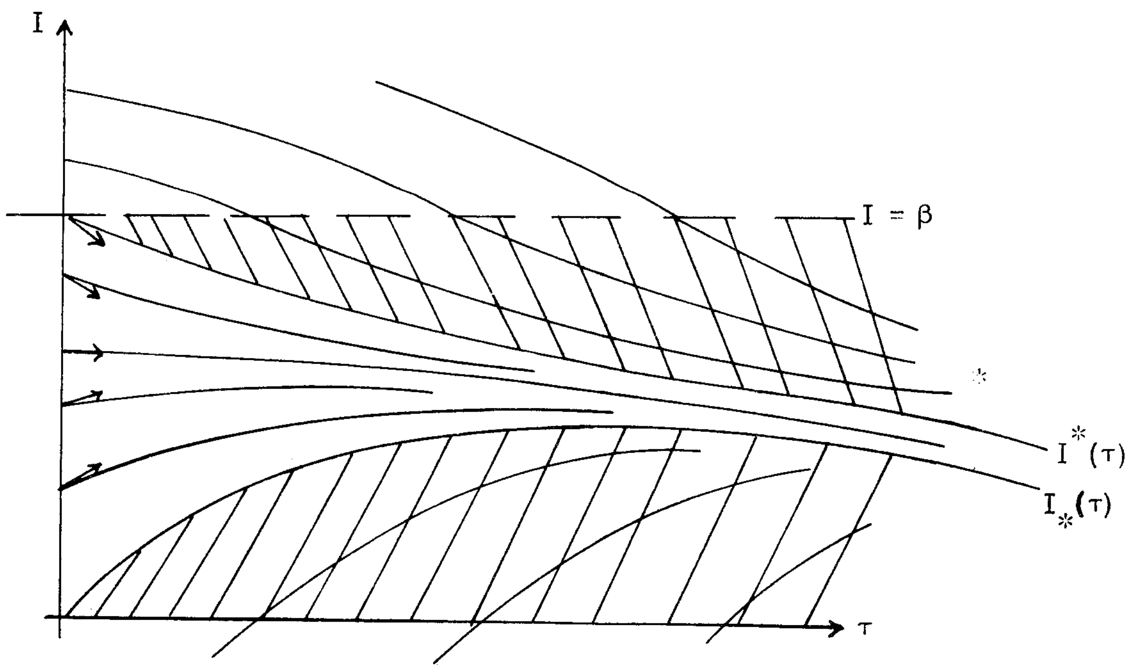

Figure 9

The solutions of (37)

from $I = 0$ and $I = \beta$ respectively. $I_*(\tau)$ is important because, if the game starts from a point (τ_o, I_o) such that $I_o < I_*(\tau_o)$, and if firm 2 adopts the same price strategy it would use on or above $I_*(\tau)$, namely $q = q^*(\tau)$, then 2 would run out of coal before the end of the game. And this, as we showed earlier, could never be optimal. Thus the optimal strategy for 2 below $I_*(\tau)$ must be to charge prices $q(\tau) > q^*(\tau)$, and so to delay the exhaustion of its coal inventory until $\tau = 0$. But the calculation of the optimum there would be difficult, as the second of the equations (18) has no solution of the form $W = \mu_3(\tau) I + \psi(\tau)$ in the region under $I_*(\tau)$. So we shall rest content with the qualitative information gained so far.

$I^*(\tau)$ is important too because, in the region RI between the line $I = \beta$ and $I^*(\tau)$, the value function $W(I,\tau)$, is again not of the form $W = \mu_3(\tau) I + \psi(\tau)$. Rather the equation for it must there be solved anew, starting from the initial data $W(\beta,\tau) = Q\,\beta + \psi(\tau)$ given on the line $I = \beta$ instead of the condition (19) given on $\tau = 0$. But we shall not do this either, as it seems apparent that the curves in RI can only be as we have already sketched them in Figure 9.

Similar remarks apply in the portion RS of NG which is filled with backwards trajectories which eventually cross the plane $S = \alpha M$ into SG. For here too, the solutions of the Hamilton-Jacobi equations (18) must satisfy boundary conditions other than (17) on the plane $S = \alpha M$. But in RS too, it seems apparent that the qualitative aspects of the trajectories can only be as previously described. So rather than face the prospect of solving the Hamilton-Jacobi equations again, we elect, at this point, to rest our case. It would surprise us if further analysis should yield other types of qualitative behaviour.

As a final remark, let us point out the applicability of our model to the theory of protective tariffs. For tariffs are important to the economic policies of many nations. And clearly, the range of mountains we postulated at the outset provides perfect protection for our firms in the period $t_o \leq t \leq T$, but no protection thereafter. So by assigning values to the physical constants a, b, c, α, and β, one could actually calculate the prices which might be expected to evolve, if the government of our small country were, in an effort to encourage the development of a coal-export industry, to promise the firms "full tariff protection" for a given period. And one could calculate a tariff rate

$R(t)$ such that $(1 + R(t))\, Q > p^*(t) \geq q^*(t) > Q$, which should be sufficient to guarantee the desired "full protection". It seems incontestible that, if such calculations could be made for more complicated and realistic models, there are a great many people who would wish to do so.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Econometric Research Program Princeton University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP None

3. REPORT TITLE

A GAME IN ECONOMICS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Summary Report: no specific reporting period.

5. AUTHOR(S) (Last name, first name, initial)

James H. Case

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 1970	36	6

8a. CONTRACT OR GRANT NO. (N00014-67 A-0151-0007 b. PROJECT NO. Task No. 047-086 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Research Memorandum No. 111 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document in unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Logistics and Mathematical Branch Office of Naval Research Washington, D.C. 20360

13. ABSTRACT

The problem of profit maximization for two firms manufacturing the same commodity is cast as a differential game. The qualitative aspects of the equilibrium point solution of the game are then discussed with the aid of some results on ordinary differential equations.