OPTIMAL CONTROL OF LINEAR ECONOMETRIC SYSTEMS WITH FINITE TIME HORIZON*

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1. THE PROBLEM

Given a linear econometric model in its reduced form

$$(1.1) \quad y_t = Ay_{t-1} + Cx_t + b + u_t ,$$

where y_t is a vector of p dependent variables, x_t is a vector of q variables subject to control, and where $Eu_t = 0$, $Eu_t u_t' = V$ and $Eu_t u_s' = 0$ for $t \neq s$, and given a quadratic welfare function for T periods

$$(1.2) \quad W = E \sum_{t=1}^{T} (-y_t' K_t y_t + 2 k_t' y_t) ,$$

K_t being symmetric and positive semi-definite, an important, and familiar, problem is to find $x_1, \ldots, x_T$ to maximize expected welfare W , the expectation being conditional on the given initial condition y_o . However, this problem can be formulated in different ways, two of which will now be stated. The solutions obtained by these formulations will be different and will serve different purposes.

The formulation familiar to economists is the one due to Simon [5], and adopted to linear econometric systems by Theil [6]. It requires expressing the econometric model in final form, i.e.,

$$(1.3) \quad \begin{aligned} y_1 &= Cx_1 + (Ay_o + b + u_1) = R_{11}x_1 + s_1 \\ y_2 &= ACx_1 + Cx_2 + (As_1 + b + u_2) = R_{21}x_1 + R_{22}x_2 + s_2 \\ y_3 &= A^2Cx_1 + ACx_2 + Cx_3 + (As_2 + b + u_3) \\ &= R_{31}x_1 + R_{32}x_2 + R_{33}x_3 + s_3 , \\ &\ \ \cdot \quad \cdot \quad \cdot \end{aligned}$$

where the notations R_{ij} and s_j are Theil's [6]. It treats each x_t $(t=1,\ldots,T)$ as a function of past $s_1,\ldots,s_{t-1}$ and the parameters in the conditional distribution of the future $s_t,\ldots,s_T$. Its main result is concerned with the optimal x_1 for the first period. This optimal x_1 is expressed as a linear function of the expectations $Es_1,\ldots,Es_T$ of all future random vectors s_t in (1.3). If the economic model were deterministic, with u_t in (1.1) being zero, i.e., if s_t in (1.3), which are functions of $u_1,\ldots,u_t$, were non-stochastic, the optimal x_1 for the maximization of (1.2) would be the same linear function of $s_1,\ldots,s_T$ as the above linear function of $Es_1,\ldots,Es_T$. This result is the well known first-period certainty equivalence.

A second formulation of the maximization problem, the one chosen in this paper, is to retain the reduced form (1.1) and to express each optimal x_t as a function of y_{t-1}. It seems natural to ask how the control variables x_t, such as government expenditures and money supply, should respond to recent observations y_{t-1} on economic variables such as GNP, employment, and the price level. It seems less natural to express the optimal x_t as a function of the (not directly observed) variables $s_1,\ldots,s_{t-1}$ and $Es_t,\ldots,Es_T$.[1]

In the solution, we obtain simultaneously the optimal feedback relations between x_t and y_{t-1} for all periods from 1 to T, and not just the optimal x_1 for the first period. True, in the beginning of period 1, the decision maker does not have to act on x_2 (if his decision can be implimented without delay), and he can do better by waiting till the beginning of period 2 when more information shall

have become available. Yet, if the econometric model (1.1) remains valid, all useful additional information for period 2 will be contained in y_1 (since y_2 depends only on y_1, x_2, and the random vector u_t with a known distribution); our solution tells how he should respond optimally to additional information in his future decisions. Thus the problem is solved for all periods.

If an economic decision maker were willing to take an econometric model (and an accompanying welfare function) so seriously as to accept the maximizing x_t for each period without questions, then the difference between the above two approaches would reduce simply to a difference in methods of computation. In the first approach, a T-period problem would be solved only to yield the solution for the first period. In the second approach, the T-period problem would be solved to yield answers for future periods - until, of course, the econometric model or the welfare function is revised. Insofar as the decision maker has to be convinced of the validity of the optimal solution, the second form is easier to communicate to him since he is likely to think in terms of reacting to recent observations contained in y_{t-1}. Furthermore, even if existing econometric models are not accurate enough for fine tuning, they may be used, in conjunction with the second approach, to indicate possibly desirable patterns of policy responses to recent economic events, and thus also to evaluate the observed patterns of responses by actual decision makers. It is for these applications that our formulation is intended.

Our formulation of the problem is certainly not new - feedback control based on observations on the state variables is a familiar concept in the literature of control theory, e.g., [7], [8], and [9]. Neither do we claim that the solution to optimal feedback control with a linear stochastic model and a quadratic welfare function is new - it is in fact well-known in control theory - although, perhaps, we have stated the problem and expressed the solution in a simpler and more convenient form than are often found in the literature. However, we do claim, first, that the solution to this problem by the elementary technique of Lagrange multipliers given in Sections 2 and 3 is new. Second, although dynamic programming is a technique commonly used for control problems, the exposition in Section 4 is simpler and more transparent than what the author can find in the literature. Third, we have also applied the method of Lagrange multipliers to solve the optimal control problem when there are delays in obtaining information or in carrying out decisions, thus providing a method for evaluating the cost of delays in terms of the change in expected welfare.

Partly to achieve simplicity, we have considered only linear feedback control equations, i.e.,

$$x_t = G_t y_{t-1} + g_t \tag{1.4}$$

and posed the problem of maximizing W with respect to the matrices G_t and the vectors g_t $(t=1,\ldots,T)$. We believe that the solution to linear feedback control is itself important, for the purposes of comminicating the result to decision makers and of evaluating the

observed patterns of policy responses to recent economic data. In other words, even if the optimal feedback equation is non-linear, the optimal linear feedback equation is of interest. It so happens that the optimal feedback equation is linear for a linear model and a quadratic welfare function, as is well-known in the control literature, e.g. [9], but we do not feel apologetic for not dealing with this mathematical point in the present paper.

In a previous paper [3], the steady-state solution to the problem was obtained by the elementary method of Lagrange multipliers. When a steady state is reached, we assume expected welfare to be

$$W = E(-y_t'Ky_t + 2k'y_t) \tag{1.5}$$

and the feedback control equation to be

$$x_t = Gy_{t-1} + g \tag{1.6}$$

for all t. A steady-state solution could be obtained only by assuming, first, that the parameters K_t and k_t in the welfare function (1.2) are unchanging through time and, second, that the system with suitable control, i.e., the system (1.1) with (1.6) substituted, is covariance-stationary. By treating a finite time horizon, this paper relaxes both of these assumptions, and will serve as a continuation of the previous paper [3].

We will find it convenient to decompose y_t into its mean Ey_t and deviation from mean y_t^*,

$$y_t = y_t^* + Ey_t = y_t^* + \mu_t . \tag{1.7}$$

Since the welfare function will accordingly be decomposed into the two parts involving y_t^* and μ_t respectively, there is no cost in replacing (1.2) by the following slightly more general welfare function

$$(1.8) \qquad W = - \sum_{t=1}^{T} Ey_t^{*\prime} K_{1t} y_t^* + \sum_{t=1}^{T} (- \mu_t' K_{2t} \mu_t + 2k_t' \mu_t) ,$$

which will reduce to (1.2) when the symmetric matrices K_{1t} and K_{2t} are equal. The econometric model (1.1), when the control equation (1.4) is applied, becomes

$$(1.9) \qquad y_t = (A + CG_t) y_{t-1} + Cg_t + b + u_t .$$

The problem then is to maximize (1.8) with respect to G_t and g_t, given the system (1.9). Section 2 will present a solution to the partial problem of maximizing the first term on the right of (1.8), while Section 3 will give a solution to the complete problem. Section 4 provides a solution by dynamic programming, for the purpose of relating the present work with the more standard works on the control of stochastic as well as non-stochastic systems. Section 5 contains a few concluding remarks.

For the readers who are not familiar with the previous paper [3], it should be pointed out that the problem here treated is more general than it might first appear. The econometric model may contain lagged variables of more periods, such as

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + C_0 x_t + C_1 x_{t-1} + b + u_t ;$$

the control equation may actually be

$$x_t = G_{1t}y_{t-1} + G_{2t}y_{t-2} + G_{3t}x_{t-1} + g_t ;$$

the welfare function may involve both y_t and x_t ; and the residuals u_t may be serially correlated. By redefining a new vector y_t to include the previous y_t , y_{t-1}, and x_t in the above example, we are able to convert the system into the form (1.1) and to deal only with y_t in the welfare function. In this conversion, we note that the number p of output variables y_t must be larger than the number q of control variables x_t which are imbedded in the former. Serial correlations in u_t can also be eliminated by suitable manipulations if u_t satisfies an autoregressive scheme with known parameters. The reader may wish to refer to the previous paper [3] for further motivation and for discussion of economic applications.

2. SOLUTION TO A PARTIAL PROBLEM

As in the previous paper [3], we first consider maximizing the welfare function

$$W_1 = - \sum_{t=1}^{T} Ey_t^{*\prime} K_{1t} y_t^* = - \sum_{t=1}^{T} \operatorname{tr} K_{1t} E y_t^* y_t^{*\prime} . \tag{2.1}$$

To obtain the model governing the deviation y_t^* from mean, assuming the control equation (1.4) is used, we take expectation of (1.9) to yield

(2.2) $$\mu_t = (A + CG_t)\mu_{t-1} + Cg_t + b$$

and subtract (2.2) from (1.9):

(2.3) $$y_t^* = (A + CG_t)y_{t-1}^* + u_t = R_t y_{t-1}^* + u_t$$

where R_t is defined as

(2.4) $$R_t = (A + CG_t) \quad .$$

Since the welfare function (2.1) involves the covariance matrices $Ey_t^* y_t^{*\prime} = \Gamma_t$, say, and each Γ_t is a function of $G_1, \ldots, G_t$ by (2.3), one may try to write W_1 as a function of $G_1, \ldots, G_T$, and proceed to maximize. This direct approach turns out to be extremely difficult. The simple approach adopted, as in the previous paper [3], is to transform the problem to one of constrained maximization by treating, somewhat artificially, an additional set of variables $\Gamma_1, \ldots, \Gamma_T$, and to consider the welfare function as a function of both sets of variables $W_1 = W_1(\Gamma_1, \ldots, \Gamma_T; G_1, \ldots, G_T)$ while these variables are subject to constraints.

We will derive a set of constraint equations by first post-multiplying (2.3) by y_t' and taking expectation

(2.5) $$Ey_t^* y_t^{*\prime} = R_t \, Ey_{t-1}^* y_t^{*\prime} + V \; ;$$

then by premultiplying the transpose of (2.3) by y_{t-1}^* and taking

expectation

$$(2.6) \qquad Ey^*_{t-1}y^{*\prime}_t = (Ey^*_{t-1}y^{*\prime}_{t-1})R'_t \quad .$$

Substitution of (2.6) into (2.5) gives the constraints,

$$(2.7) \qquad Ey^*_t y^{*\prime}_t = V + R_t(Ey^*_{t-1}y^{*\prime}_{t-1})R'_t \quad ,$$

or

$$\Gamma_t = V + (A + CG_t)\,\Gamma_{t-1}\,(A + CG_t)' \qquad (t=1,\ldots,T) \quad .$$

Note that $\Gamma_0 = 0$, since $y_0 = \mu_0$ and $y^*_0 = 0$, and that $\Gamma_1 = V$.

The Lagrangian expression, with H_t denoting symmetric matrices of Lagrange multipliers, is

$$(2.8) \qquad L_1 = -\sum_{t=1}^{T} \operatorname{tr} K_{1t}\Gamma_t + \sum_{t=1}^{T} \operatorname{tr} H_t[\Gamma_t - (A + CG_t)\Gamma_{t-1}(A + CG_t)' - V] \quad .$$

We differentiate L_1 with respect to G_t and Γ_t $(t=2,\ldots,T)$, using the differentiation rule

$$(2.9) \qquad \frac{\partial}{\partial G} \operatorname{tr} BG = \frac{\partial}{\partial G} \operatorname{tr} GB = B' \quad ,$$

yielding

$$(2.10) \qquad \frac{\partial L_1}{\partial G_t} = -2C'H_tA\Gamma_{t-1} - 2C'H_tCG_t\Gamma_{t-1} = 0 \qquad (t=2,\ldots,T);$$

$$(2.11a) \qquad \frac{\partial L_1}{\partial \Gamma_t} = -K_{1t} + H_t - (A + CG_{t+1})'\,H_{t+1}(A + CG_{t+1}) = 0 \qquad (t=2,\ldots,T-1)$$

$$(2.11b) \qquad \frac{\partial L_1}{\partial \Gamma_T} = -K_{1T} + H_T = 0 \quad .$$

Equations (2.10), (2.11) and (2.7) will determine the unknowns G_t , H_t , and Γ_t .

To solve for H_t and G_t, we simply use (2.11) and (2.10) and work backwards. Given $H_T = K_{1T}$ from (2.11b), we set, from (2.10),[2]

$$(2.12) \qquad G_T = -(C'H_TC)^{-1} C'H_TA .$$

Then from (2.11a), we obtain

$$(2.13) \qquad H_{T-1} = K_{1,T-1} + (A + CG_T)' H_T(A + CG_T) .$$

The process continues until H_2 and G_2 are obtained. As we have noted in connection with equation (2.7), $\Gamma_0 = 0$ and $\Gamma_1 = V$. Hence G_1 has no effect on Γ_1, and we may as well set $G_1 = 0$. This does not mean that nothing is done to control the economy for the first period; any control can be exercised through the choice of g_1 in the control equation (1.4).

In the previous paper [3], we dealt with the steady-state solution, with the control matrix G invariant through time after equilibrium is reached. The optimal G was obtained by solving the two equations

$$G = -(C'HC)^{-1}C'HA$$

and

$$H = K_1 + (A + CG)' H(A + CG) .$$

These equations will give the steady-state solution to the above matrix difference equations in G_t and H_t, if such a steady state exists for sufficiently large T and for $K_{1t} = K_1$ for all t. We also note that, in the present formulation, the matrices A and C can be functions of t; the subscript t can be added to A and C in the appropriate places in equations (2.8), (2.10) and (2.11).

As it has been pointed out in the previous paper [3], if there are delays in obtaining information and/or in carrying out decisions, the control matrix G_t can be partitioned into

$$G_t = (G_{1t}\ G_{2t}) = (0\ G_{2t}) , \tag{2.14}$$

$G_{1t} = 0$ being coefficients of the components of y_{t-1} in the control equation (1.4) for which no information is available at time t ; the covariance matrix Γ_t can be accordingly partitioned

$$\Gamma_t = (\Gamma_{1,t}\ \Gamma_{2,t}) = \begin{pmatrix} \Gamma_{11,t} & \Gamma_{21,t} \\ \Gamma_{12,t} & \Gamma_{22,t} \end{pmatrix} ; \tag{2.15}$$

and equation (2.10) will be replaced by

$$\frac{\partial L_1}{\partial G_{2t}} = -2C'H_tA\Gamma_{2,t-1} - 2C'H_tCG_{2t}\Gamma_{22,t-1} = 0 , \tag{2.16}$$

but equations (2.11) and (2.7) will remain the same.

To solve equations (2.16), (2.11) and (2.7) for G_{2t} , H_t and Γ_t $(t=2,\ldots,T)$, the following iterative method is suggested. Start with an initial guess $G_{2,t}^{(1)}$ $(t=2,\ldots,T)$ in the first iteration. For example, these matrices could be the appropriate columns selected from the solution for G_t $(t=2,\ldots,T)$ obtained by equations (2.10) and (2.11) when there are no information delays. Using these

$G_{2,t}^{(1)}$, calculate $H_t^{(1)}$ $(t=T,T-1,\ldots,2)$ by equation (2.11), and $\Gamma_t^{(1)}$ $(t=2,3,\ldots,T)$ by equation (2.7). In the second iteration, compute $G_{2t}^{(2)}$ by equation (2.16), i.e.,

$$G_{2t}^{(2)} = -(C'H_t^{(1)}C)^{-1}\, C'H_t^{(1)}A\Gamma_{2,t-1}^{(1)}\Gamma_{22,t-1}^{(1)^{-1}} \qquad (t=2,\ldots,T) .$$

These matrices will serve as inputs for calculating $H_t^{(2)}$ by equation (2.11) and $\Gamma_t^{(2)}$ by equation (2.7), and so forth.

3. SOLUTION TO THE COMPLETE PROBLEM

Return now to the maximization of the original welfare function

$$(3.1) \qquad W = -\sum_{t=1}^{T} \operatorname{tr} K_{1t}\Gamma_t - \sum_{t=1}^{T} \mu'_t K_{2t}\mu_t + \sum_{t=1}^{T} 2k'_t\mu_t .$$

Since we have the constraints (2.2) and (2.7) respectively for μ_t and Γ_t, one possible approach is to form the Lagrange expression, using additional vectors λ_t of multipliers,

$$(3.2) \qquad L = \frac{1}{2}W + \frac{1}{2}\sum_{t=1}^{T} \operatorname{tr} H_t\,[\Gamma_t - (A+CG_t)\Gamma_{t-1}(A+CG_t)' - V]$$

$$- \sum_{t=1}^{T} \lambda'_t\,[\mu_t - (A+CG_t)\mu_{t-1} - Cg_t - b] .$$

Straightforward differentiation yields

$$(3.3) \qquad \frac{\partial L}{\partial G_t} = \frac{1}{2}\frac{\partial L_1}{\partial G_t} + C'\lambda_t\mu'_{t-1} = 0 \qquad (t=2,\ldots,T)$$

$$(3.4) \qquad \frac{\partial L}{\partial \Gamma_t} = \frac{1}{2}\frac{\partial L_1}{\partial \Gamma_t} = 0 \qquad (t=2,\ldots,T)$$

$$(3.5) \qquad \frac{\partial L}{\partial g_t} = C'\lambda_t = 0 \qquad (t=1,\ldots,T)$$

$$(3.6a) \qquad \frac{\partial L}{\partial \mu_t} = K_{2t}\mu_t + k_t - \lambda_t + (A + CG_{t+1})'\lambda_{t+1} = 0 \qquad (t=1,\ldots,T-1)$$

$$(3.6b) \qquad \frac{\partial L}{\partial \mu_T} = K_{2T}\mu_T + k_T - \lambda_T = 0 \ .$$

Using equation (3.5), we find that (3.3) and (3.4) are equivalent to (2.10) and (2.11) for the solution of optimal G_t and H_t in section 2. Thus policies to control the covariances are still obtained by the previous method and are unaffected by the need for controlling the mean μ_t. This point can be observed directly from equation (2.2). Suppose that G_t maximize W_1, and that G_t^*, together with g_t^* say, maximize the remaining part of W, i.e.,

$$(3.7) \qquad W_2 = -\sum_t \mu_t' K_{2t}\mu_t + \sum_t 2k'\mu_t \ .$$

But G_t can always be used to generate the same μ_t as G_t^* can, since, by (2.2),

$$(3.8) \qquad \mu_t = (A + CG_t^*)\mu_{t-1} + Cg_t^* + b$$

$$= (A + CG_t)\mu_{t-1} + C[(G_t^* - G_t)\mu_{t-1} + g_t^*] + b \ .$$

All that needs to be done is to replace g_t^* by $g_t = (G_t^* - G_t)\mu_{t-1} + g_t^*$. This also shows that the maximum of W_2 can be achieved for any arbitrary set of G_t.

To obtain the optimal g_t, we could solve equations (3.5), (3.6) and (2.2) for g_t, μ_t and λ_t $(t=1,\ldots,T)$, assuming that the optimal G_t have already been found by the method of section 2. These are $T\times(q+2p)$ linear equations in the unknowns $g_1,\ldots,g_T$, $\mu_1,\ldots,\mu_T$, and $\lambda_1,\ldots,\lambda_T$. This problem is an example of a non-stochastic control problem in discrete time where the objective function W_2 is quadratic and the dynamic model (2.2) is a system of linear difference equations in the state variables μ_t and the control variables g_t. This is precisely the control problem when our model (1.1) becomes non-stochastic, with $\mu_t = 0$. We have just pointed out that maximum of welfare in this case can be achieved for any set of G_t, including $G_t = 0$. This means that, in the world of certainty, optimal x_t can be set independently of recent information y_{t-1}, and no feedback relation will be required.

Methods for solving this elementary non-stochastic control problem have been treated extensively in the literature, e.g. [2, Chapter 6], and will not be repeated here.[3] However, since this non-stochastic control problem is imbedded in the larger stochastic control problem, one can utilize the results on G_t and H_t to obtain an easier method of solution than merely solving a large set of simultaneous linear equations (3.5), (3.6) and (2.2). In particular, if we let $K_{1t} = K_{2t} = K_t$ in the welfare function, and apply the relationship (2.11) between H_t, G_t, and K_t, we can rewrite (3.6) as

$$(3.9) \qquad \lambda_t = k_t - K_t\mu_t + R'_{t+1}\lambda_{t+1}$$

$$= k_t - H_t\mu_t + R'_{t+1}H_{t+1}R_{t+1}\mu_t + R'_{t+1}\lambda_{t+1} \ .$$

By the use of (2.2) to replace $R_{t+1}\mu_t$, and (2.10) to nullify $C'HR_t$, (3.9) becomes

$$(3.10) \qquad (\lambda_t + H_t\mu_t) = k_t - R'_{t+1}[H_{t+1}b - (\lambda_{t+1} - H_{t+1}\mu_{t+1})]$$

or

$$h_t = k_t - R'_{t+1}(H_{t+1}b - h_{t+1}) \ ,$$

for the newly defined h_t. If we premultiply equation (2.2) by $C'H_t$ and solve for g_t, using also equations (3.10), (3.5) and (2.10), we will obtain

$$(3.11) \qquad g_t = -(C'H_tC)^{-1}C'(H_tb - h_t) \ .$$

Equations (3.10) and (3.11) can very easily be solved. The former is a linear difference equation in h_t and can be solved backward in time, from $h_T = k_T$. Given h_t, the optimal g_t will then be obtained by (3.11).

4. SOLUTION BY DYNAMIC PROGRAMMING

Recently, there has been a growing interest among economists in optimal control theory for non-stochastic systems, as exemplified by the excellent expository article of Dorfman [4]. It may be useful to provide an elementary exposition of some aspects of optimal control theory for stochastic systems as it is treated in the literature,

e.g., [7], [8], and [9], and to relate it to control theory for non-stochastic systems.

In this paper, we have treated a stochastic control problem in section 2, and a non-stochastic control problem in section 3. In the former, the dynamic model (2.3) is stochastic, and the welfare function (2.1) is in the form of a mathematical expectation. In the latter, the dynamic model (2.2), with μ_t treated as output variables and g_t as control variables, is non-stochastic or deterministic. These two problems should serve to illustrate the nature of stochastic and non-stochastic control problems. The method that we have used to solve both problems is the method of Lagrange multipliers.

In this section, we will solve the problem of maximizing the welfare function (1.2) by dynamic programming [1], which is a method commonly used to solve stochastic as well as non-stochastic control problems [4], [7], [8], and [9]. Readers familiar with Dorfman's paper [4] will see that the same principle of optimality is being applied here to a problem of stochastic control.

Let us first restate the problem. Given a dynamic stochastic model in discrete time

$$(4.1) \qquad y_t = Ay_{t-1} + Cx_t + b + u_t ,$$

and a welfare function of the form

$$(4.2) \qquad W(s, y_s; x_{s+1}, \ldots, x_T) = E \sum_{t=s}^{T} (-y'K_t y_t + 2k_t' y_t) ,$$

find $x_{s+1}, \ldots, x_T$ to maximize W, with each x_t to be determined by the linear feedback control equation

$$(4.3) \qquad x_t = G_t y_{t-1} + g_t .$$

Note that welfare depends on the initial period s, the initial value y_s of the state variable in that period, and on the values of future control variables $x_{s+1}, \ldots, x_T$. The expectation in (4.2) is conditional on y_s.

Let $W^\circ(s, y_s)$ denote the maximum welfare obtainable given the initial value y_s in period s, with x_t determined by the optimal control parameters G_t° and g_t° $(t=s+1, \ldots, T)$ according to the equation (4.3). It will first be observed that W° must take the form

$$(4.4) \qquad W^\circ(s, y_s) = -y_s' H_s y_s + 2h_s' y_s - p_s$$

for some symmetric matrix H_s, some vector h_s and some scalar p_s. To see this, we note that each y_t can be written as

$$\begin{aligned}
(4.5) \qquad y_t &= (A + CG_t^\circ) y_{t-1} + Cg_t^\circ + b + u_t \\
&= R_t y_{t-1} + Cg_t^\circ + b + u_t \\
&= u_t + R_t u_{t-1} + R_t R_{t-1} u_{t-2} + \cdots + (R_t R_{t-1} \cdots R_{s+2}) u_{s+1} \\
&\quad + (Cg_t^\circ + b) + R_t (Cg_{t-1}^\circ + b) + \cdots + (R_t R_{t-1} \cdots R_{s+2})(Cg_{s+1}^\circ + b) \\
&\quad + (R_t R_{t-1} \cdots R_{s+1}) y_s ,
\end{aligned}$$

that the expectation of $-y_t' K_t y_t + 2k_t' y_t$ must be quadratic in y_s, and that the welfare function (4.2) is a sum of these quadratic functions, thus having the form (4.4).

Once the form of $W^\circ(s,y_s)$ is ascertained, we can apply the principle of optimality in dynamic programming:

$$(4.6) \qquad W^\circ(s,y_s) = \max_{x_{s+1}} E\{-y_s'K_sy_s + 2k_s'y_s + W^\circ(s+1,y_{s+1})\} .$$

According to this equation, the maximum welfare from period s on is the maximum, with respect to x_{s+1}, of the sum of the welfare contribution of this period and the maximum of welfare from the next period on. The expectation in (4.6) is conditional on y_s; the term $W^\circ(s+1,y_{s+1})$ depends on y_{s+1} which is yet unknown, or is a random vector, from the vintage point of period s. Using the form (4.4) for W°, we can write the term to be maximized as

$$(4.7) \qquad -y_s'K_sy_s + 2k_s'y_s + E(-y_{s+1}'H_{s+1}y_{s+1} + 2h_{s+1}'y_{s+1} - p_{s+1})$$

$$= -y_s'K_sy_s + 2k_s'y_s - (Ay_s + Cx_{s+1} + b)' H_{s+1}(Ay_s + Cx_{s+1} + b)$$

$$- E\, u_{s+1}'H_{s+1}u_{s+1} + 2h_{s+1}' (Ay_s + Cx_{s+1} + b) - p_{s+1} .$$

Maximizing (4.7) with respect to x_{s+1} by differentiation yields

$$(4.8) \qquad -2C'H_{s+1}(Ay_s + Cx_{s+1} + b) + 2C'h_{s+1} = 0 ;$$

$$(4.9) \qquad x_{s+1} = G_{s+1}y_s + g_{s+1} ,$$

where

$$(4.10) \qquad G_{s+1} = -(C'H_{s+1}C)^{-1}C'H_{s+1}A ,$$

$$(4.11) \qquad g_{s+1} = -(C'H_{s+1}C)^{-1}C'(H_{s+1}b - h_{s+1}) .$$

Thus, the optimal control parameters are given by (4.10) and (4.11). To find H_s and h_s, we substitute (4.9) back to (4.7), and equate the result to the left-hand side of (4.6). The maximum of (4.7) is

$$(4.12)\quad - y_s'K_sy_s + 2k_s'y_s - y_s'(A + CG_{s+1})' H_{s+1}(A + CG_{s+1})y_s$$
$$- (Cg_{s+1} + b)' H_{s+1}(Cg_{s+1} + b) - 2(Cg_{s+1} + b)' H_{s+1}(A + CG_{s+1})y_s$$
$$- \operatorname{tr} H_{s+1}V + 2h_{s+1}'(A + CG_{s+1})y_s + 2h_{s+1}'(Cg_{s+1} + b) - p_{s+1}$$

which should be equated to

$$(4.13)\quad W^\circ(s,y_s) = y_s'H_sy_s + 2h_s'y_s + p_s ;$$

yielding the following two difference equations for H_s and h_s :

$$(4.14)\quad H_s = K_s + (A + CG_{s+1})'H_{s+1}(A + CG_{s+1}) ,$$

$$(4.15)\quad h_s = k_s - (A + CG_{s+1})'H_{s+1}(Cg_{s+1} + b) + (A + CG_{s+1})' h_{s+1}$$
$$= k_s - (A + CG_{s+1})'(H_{s+1}b - h_{s+1}) .$$

Equations (4.10) and (4.14) are identical with equations (2.10) and (2.11) of section 2; they can be used to determine G_s and H_s. Given G_s and H_s, equations (4.11) and (4.15) can be used to determine g_s and h_s. They are identical with equations (3.11) and (3.10) respectively.

The exposition of this section has, in fact, also demonstrated the solution of a non-stochastic control problem by dynamic programming, since we could easily convert the stochastic model (4.1)

into a non-stochastic model by letting $u_t = 0$; the welfare function (4.2) could remain unchanged since the expectation E , though unnecessary in the non-stochastic case, is harmless. We could follow the same development from equation (4.2) on, noting $u_t = 0$ in all places, thus $V = 0$ in (4.12), and still arriving at the same results (4.10), (4.11), (4.14) and (4.15). This exposition is presented because it ties together the theories of stochastic and non-stochastic control, and because, to the author's knowledge, an elementary treatment of stochastic control theory using dynamic programming is not available in the literature.

5. CONCLUDING REMARKS

In this paper, we have replaced the assumptions of a previous paper [3] that the welfare function is constant through time and that the linear econometric system under control is stationary, by the assumptions that the welfare function is a sum for a finite number of periods and that the econometric system under control may be non-stationary. We still believe that the assumptions of the previous paper [3] are relevant for many economic applications, especially when the variables y_t are interpreted as the first differences of certain economic variables, or perhaps of their logarithms. Under these assumptions, the present paper provides the transient solution to supplement the steady-state solution given earlier.

In some economic applications, one may wish to relax these assumptions. For example, if one wishes to have the variables y_t (first differences or not) follow certain target vector a_t , in the

sense of minimizing the sum, over t, of certain positive semi-definite quadratic forms in the deviations $y_t - a_t$, the welfare function can be reformulated as in this paper, namely

$$W = E[-\sum_{t=1}^{T}(y_t - a_t)'K_t(y_t - a_t)]$$

$$= E[\sum_{t=1}^{T}(-y_t'K_ty_t + 2a_t'K_ty_t)] - \sum_{t=1}^{T}a_t'K_ta_t$$

which is equivalent to (1.2). There is also the possibility that the system under control may be non-stationary. Under these circumstances, the solution given in this paper will be applicable.

However, even if the objective is to follow closely certain growing target vector a_t, the assumptions of the previous paper [3] might still be valid. As an example, let a_t satisfy the difference equation $a_t = Da_{t-1}$, where D is a diagonal matrix with some diagonal elements greater than one. We may augment system (1.1) by the above difference equation, redefine a new output vector to include both y_t and a_t, and consider maximizing the welfare function

$$W = E\left[-\sum_{t=1}^{T}(y_t - a_t)'\,K(y_t - a_t)\right]$$

$$= E\left[-\sum_{t=1}^{T}(y_t' \quad a_t')\begin{pmatrix} K & -K \\ -K & K \end{pmatrix}\begin{pmatrix} y_t \\ a_t \end{pmatrix}\right]$$

The parameters of this welfare function are time-invariant. Equations (2.10) and (2.11) might have a stationary solution for G and H, identical with the solution of the previous paper [3].

While relaxing the assumptions of the previous paper, we have applied the same method, that of Lagrangian multipliers, to solve the problem of this paper. We have also provided an exposition of the method of dynamic programming to solve a stochastic and a non-stochastic control problem, thus pointing out a unifying element in optimal control theory for stochastic and non-stochastic systems. It is hoped that the methods of these two papers will be applied to empirically relevant econometric models and welfare functions, so as to contributue to our knowledge of optimal policies for economic stability and growth.

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FOOTNOTES

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1. In the original model of Simon [5], the control variable, production, was assumed to depend on past sales, and sales were the random variables in the model. Simon's formulation was quite natural, but when Theil [6] adopted Simon's model to an econometric system, the variables on which x_t depends became the somewhat unnatural $s_1,\ldots,s_{t-1}$ and $Es_t,\ldots,Es_T$ in equation (1.3).

2. Of course, the inverse of $C'H_TC$ might not exist. In this case, the optimal reaction coefficients G_T are not unique, and we interpret $(C'H_TC)^{-1}$ as a generalized inverse.

3. One fairly obvious approach is to reduce the size of this linear problem by eliminating the variables $\mu_1,\ldots,\mu_T$ and $\lambda_1,\ldots,\lambda_T$. Using the model (2.2), each μ_t becomes a linear function of $g_1,\ldots,g_t$, and $\mu_1,\ldots,\mu_T$ become linear functions of $g_1,\ldots,g_T$. The welfare function (3.7) then becomes a quadratic function in $g_1,\ldots,g_T$, and can be easily maximized with respect to these remaining variables.