CONDITIONS FOR THE GRAPH AND THE INTEGRAL OF A CORRESPONDENCE TO BE OPEN

(Preliminary Version)

Richard R. Cornwall

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Princeton University
Econometric Research Program
207 Dickinson Hall
Princeton, New Jersey

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A correspondence φ from a probability space $(M, \mathcal{M}, \mu)$ to a Banach space S assigns to each element m in M a subset $\varphi(m)$ of S. A measureable (resp., integrable or continuous) function f on M is a <u>measureable</u> (resp., <u>integrable</u> or <u>continuous</u>) <u>selection</u> from φ if $f(m) \in \varphi(m)$ μ-a.e. For any E in $\mathcal{M}$, let

$$\int_E \varphi d\mu = \{\int_E f d\mu : \ f \text{ an integrable selection from } \varphi\}$$

$$\int_E^c \varphi d\mu = \{\int_E f d\mu : \ f \text{ a continuous integrable selection from } \varphi\}.$$

This paper gives conditions under which these integrals are open or, when S is finite dimensional, relatively open. For the integral $\int_E^c \varphi d\mu$, this involves giving conditions for φ to have an open graph. There are two easy corollaries of the openness of these two integrals: The first gives conditions for a relatively-open-set-valued measure on $\mathcal{M}$ to have a Radon-Nikodym derivative. The second gives conditions for $\int_E \varphi d\mu$ to equal $\int_E^c \varphi d\mu$. Finally, we give a result close to the statement that if φ and ψ are two correspondences satisfying $\varphi(m) \subset \psi(m)$, all m, then $\int_E \varphi d\mu$ is open in $\int_E \psi d\mu$.

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1. Some Definitions and Preliminary Results.

The topological dual of S is S'. If $p \in S'$, we shall denote the value of p at a vector x in S by $p \cdot x$ rather than $p(x)$. For any $K \subset S$, $\sup p \cdot K \equiv \sup\{p \cdot x : x \in K\}$. The interior of a subset K of S is denoted $\text{int } K$. The smallest affine subspace of S containing K is $L[K]$. When this notation is used we shall make the additional assumption that S is finite dimensional or that $\text{int } K$ is not empty so $L[K] = S$. The relative interior of K, $\text{ri } K$, is the interior of K as a subset of the topological space $L[K]$. $H[K]$ will denote the linear (or homogeneous) subspace of S gotten by translating $L[K]$ to the origin. The convex hull of an arbitrary subset K of S is denoted $\text{conv}[K]$. An open ball in S of center x and radius ϵ is denoted $B_\epsilon(x)$. The complement of a subset K of S is ${}^cK$. The notation $E \backslash F$ denotes set-subtraction, in $\mathcal{M}$ or for subsets of S.

To distinguish a correspondence from M to S from a function from M to S, we shall sometimes use the notation $\varphi : M \Rightarrow S$ for a correspondence and $f : M \rightarrow S$ for a function. Given a correspondence φ, $\text{int}\,\varphi$ is the correspondence mapping m into $\text{int}\,\varphi(m)$. Similarly, $\text{ri}\,\varphi : m \mapsto \text{ri}\,\varphi(m)$, $\bar{\varphi} : m \mapsto \overline{\varphi(m)}$ (the closure of $\varphi(m)$), and $L[\varphi] : m \mapsto L[\varphi(m)]$. We shall say φ is positively unbounded if there exists a closed, convex cone P with a nonempty interior such that for every m there is some y in $\varphi(m)$ with $y + P \subset \varphi(m)$. (We shall indiscriminately add sets and points: $K_1 + K_2 = \{x \in S: x=x_1+x_2,\ x_i \in K_i,\ i=1,2\}$ and $K_1 + x_1 = K_1 + \{x_1\}$.) A partial order $\subset$ is defined on correspondences by saying $\varphi \subset \psi$ whenever $\varphi(m) \subset \psi(m)$, all m.

We shall suppose throughout this paper that $(M, \mathcal{M}, \mu)$ is a complete probability space. A correspondence $\varphi: M \Rightarrow S$ is measureable whenever its graph, G_φ, is measureable in the product σ-field on $M \times S$ (S has the Borel σ-field generated by open subsets). The following result is a basic tool in this paper:

> Measureable Selection Theorem (MST): If S is a separable Banach space (or even a Polish space) and if $\varphi: M \Rightarrow S$ is measureable and nonempty-valued, then φ has a measureable selection.

This is proven by Aumann [2], for example. When the hypothesis of this theorem is met, we shall say (M, S, φ) satisfies MST. ¶ To state an analogous condition for a correspondence to have a continuous selection, we need some kind of continuity condition on φ; namely φ must be lower semi-continuous (LSC); i.e., for any open G in S, $\{m \in M: \varphi(m) \cap G \neq \emptyset\}$ is open in M; φ is upper semi-continuous if, for any open G in S, $\{m \in M: \varphi(m) \subset G\}$ is open in M.

Continuous Selection Theorem (CST): If M is a Hausdorff, perfectly normal topological space, if S is a separable Banach space, if $\varphi: M \Rightarrow S$ is a convex-nonempty-valued LSC correspondence and if either (1) S is finite dimensional or (2) φ is closed-valued or $\operatorname{int}\varphi$ is nonempty-valued, then φ has a continuous selection.

This is proven by Michael [10]. When the hypothesis of this theorem is met, we shall say (M, S, φ) satisfies CST.

2. Conditions for the Graph of a Correspondence to be Open.

THEOREM 1: If (M, S, φ) satisfies CST and φ is open-valued and positively unbounded, then the graph of φ is open in $M \times S$ (with the product topology).

PROOF: We shall show that the complement of G_φ is closed: If $\{(m^\alpha, y^\alpha);\ \alpha \in A\}$ is a net of elements converging to (m^o, y^o) and satisfying $y^\alpha \notin \varphi(m^\alpha)$, $\alpha \in A$, we want to show that $y^o \notin \varphi(m^o)$.

Define σ_φ from $S' \times M$ to $\mathbb{R} \cup \{+\infty\}$ by

$$\sigma_\varphi(p, m) = \sup p \cdot \varphi(m) .$$

Since $\varphi(m)$ is convex and open, then $y \notin \varphi(m)$ if and only if, for some nonzero p in S', $p \cdot y \geqq \sigma_\varphi(p, m)$. In particular, for each α in A we can choose p^α of norm one in S' such that $p^\alpha \cdot y^\alpha \geqq \sigma_\varphi(p^\alpha, m^\alpha)$. To show that $y^o \notin \varphi(m^o)$, it is enough to find a nonzero p^o in S' such that

(1) $p^{\alpha}\cdot y^{\alpha} \rightarrow p^{o}\cdot y^{o}$

(2) $\sigma_{\varphi}(p^{o},m^{o}) = \lim_{\alpha}\inf \sigma_{\varphi}(p^{\alpha},m^{\alpha})$,

since (1) and (2) imply that $p^{o}\cdot y^{o} \geqq \sigma_{\varphi}(p^{o},m^{o})$.

The unit ball B_1' of S' is weak*-compact [13,5.2 page 141]. Thus there is a subnet of $\{p^{\alpha};\ \alpha \in A\}$ converging to some p^{o} . Without loss of generality, we may ease notation by assuming this subnet is the original net. Thus p^{α} converges weakly to p^{o} . To show $p^{o} \neq 0$, note that $\sup p^{\alpha}\cdot\varphi(m^{\alpha}) \leqq p^{\alpha}\cdot y^{\alpha} < +\infty$ so $p^{\alpha} \in P^{o}$, the polar of P . Because int P is not empty, P^{o} is pointed (i.e., contains no lines; otherwise P would lie in some hyperplane). If $\partial B_1'$ is the surface of B_1' , then we conclude that $0 \notin \text{conv}\,[P^{o} \cap \partial B_1']$. On the other hand, $p^{\alpha} \in \text{conv}\,[P^{o} \cap \partial B_1']$ which is strongly closed and convex and thus is weak*-closed [13, 3.1 page 130]. Thus $p^{o} \in \text{conv}\,[P^{o} \cap \partial B_1']$. In conclusion, $p^{o} \neq 0$.

It is easy to demonstrate (1):

$$|p^{\alpha}\cdot y^{\alpha} - p^{o}\cdot y^{o}| \leqq |p^{\alpha}\cdot(y^{\alpha}-y^{o})| + |(p^{\alpha}-p^{o})\cdot y^{o}|$$

$$\leqq \|y^{\alpha}-y^{o}\| + |(p^{\alpha}-p^{o})\cdot y^{o}|$$

since $\|p^{\alpha}\| = 1$. But the first term on the right converges to zero by assumption and the second converges to zero since p^{α} converges weak* to p^{o} .

To demonstrate (2), we show σ_{φ} is LSC on $S' \times M$; i.e., for any real λ , the set $\{(p,m) \in S' \times M: \sigma_{\varphi}(p,m) \leqq \lambda\}$ is closed (where S' has the weak*-topology). Suppose $\{(p^\beta, m^\beta);\ \beta \in B\}$ is a net of elements of this set converging to (p^1, m^1), but on the other hand, $\sigma_\varphi(p^1, m^1) > \lambda$. Then there exists $y^1 \in \varphi(m^1)$ such that $p^1 \cdot y^1 > \lambda$. By the CST, we choose a continuous selection f from φ so that $f(m^1) = y^1$. Then, as in (1) above, $p^\beta \cdot f(m^\beta) \to p^1 \cdot f(m^1) > \lambda$ which contradicts the choice of (p^β, m^β), $\beta \in B$. ∎

<u>COROLLARY</u>: If (M,S,φ) and (M,S,ψ) satisfy CST and if φ is open-valued and positively unbounded, then $\{m \in M: \varphi(m) \cap \psi(m) \neq \phi\}$ is open.

<u>PROOF:</u> We show that if $x^o \in \varphi(m^o) \cap \psi(m^o)$, then $\varphi(m) \cap \psi(m)$ is nonempty for m in some neighborhood U of m^o. Since (M,S,ψ) satisfies CST, we can choose a continuous selection f from ψ such that $f(m^o) = x^o$. By Theorem 1, the graph of φ is open, so we can choose an open U_1 containing m^o and an open ball $B_\epsilon(x^o)$ around x^o such that $U_1 \times B_\epsilon(x^o)$ is contained in the graph of φ. Choose U to be an open neighborhood of m^o contained in U_1 such that $\|f(m) - f(m^o)\| < \epsilon$ for m in U. Clearly $f(m) \in \varphi(m) \cap \psi(m)$, $m \in U$. ∎

In case S has finite dimension N, one is tempted to try to derive a conclusion similar to that of the preceding Corollary under the weaker hypothesis that φ be relatively-open-valued. This leads to the need for an extension of Theorem 1 to this case.

For any integer n between 0 and N, define

$$M_\varphi^n = \{m \varepsilon M: \text{dimension of } \varphi(m) \text{ equals } n\}$$

THEOREM 2: If S is finite dimensional, (M,S,φ) satisfies CST and φ is relatively open-valued and if $0 \leqq n \leqq N$, for any $m^o \varepsilon M_\varphi^n$ and $x^o \varepsilon \varphi(m^o)$ there exists $\epsilon > 0$ and a neighborhood U of m^o such that $m \varepsilon U \cap M_\varphi^n$ and $x \varepsilon B_\epsilon(x^o) \cap L[\varphi(m)]$ imply $x \varepsilon \varphi(m)$.

PROOF: We can imitate the proof of Theorem 1 if we first assume that the graph of $H[\varphi]$, restricted to M_φ^n, is closed. This assertion will be proven in the following lemma.

Given $m^o \varepsilon M_\varphi^n$ and $x^o \varepsilon \varphi(m^o)$, there fails to exist such an $\epsilon > 0$ and such a neighborhood U if and only if there is a net $\{(m^\alpha, x^\alpha); \alpha \varepsilon A\}$ converging to (m^o, x^o) and satisfying $m^\alpha \varepsilon M_\varphi^n$ and $x^\alpha \varepsilon L[\varphi(m^\alpha)] \cap {}^c\varphi(m^\alpha)$. The latter condition means that there exists a vector p^α with $\|p^\alpha\| = 1$, $p^\alpha \cdot x^\alpha \geqq \sigma_\varphi(p^\alpha, m^\alpha)$, and $p^\alpha \varepsilon H[\varphi(m^\alpha)]$, the linear subspace parallel to $L[\varphi(m^\alpha)]$. The compactness of the unit sphere in S means we can (without loss of generality) assume that p^α converges to some p^o which also has norm one. By Lemma 1 below, the graph of the correspondence $H[\varphi]$ is closed, so $p^o \varepsilon H[\varphi(m^o)]$. As in the proof of Theorem 1, we conclude that $p^o \cdot x^o \geqq \sigma_\varphi(p^o, m^o)$. Because $p^o \varepsilon H[\varphi(m^o)]$ and $p^o \neq 0$, we have shown that $x^o \notin \varphi(m^o)$.

Remark: When S is finite dimensional, Theorem 2 gives a stronger result than Theorem 1: G_φ is open when (M,S,φ) satisfies CST and φ is open-valued. φ need not be positvely unbounded. It would seem that this should be true for any Banach space S.

LEMMA 1: If S has dimension N, (M,S,φ) satisfies CST and if $0 \leq n \leq N$, then the graphs of $H[\varphi]$ and $L[\varphi]$ restricted to M_φ^n are closed in $M_\varphi^n \times S$.

PROOF: If $\{(m^\alpha, x^\alpha); \alpha \varepsilon A\}$ is a net converging to (m^o, x^o) and satisfying $x^\alpha \varepsilon L[\varphi(m^\alpha)]$ and $m^\alpha \varepsilon M_\varphi^n$, then we want $x^o \varepsilon L[\varphi(m^o)]$. Choose $n+1$ affinely independent points $\{y^i\}_0^n$ in $\varphi(m^o)$ which span $L[\varphi(m^o)]$. By the CST, choose $n+1$ continuous selections f^i from φ satisfying $f^i(m^o) = y^i$, $i=0,\ldots,n$. There exists a neighborhood U of m^o on which $\{f^i(m)\}_0^n$ are affinely independent. Since $m^\alpha \varepsilon M_\varphi^n$, then for α large enough, $L[\varphi(m^\alpha)]$ is spanned by $\{f^i(m^\alpha)\}_0^n$. Thus for some scalars λ_i^α satisfying $\Sigma_0^n \lambda_i^\alpha = 1$ we have $x^\alpha = \Sigma_0^n \lambda_i^\alpha f^i(m^\alpha)$. It is routine to show that if $u^o,\ldots,u^n$ are affinely independent, then the affine coordinates λ_i, $i=0,\ldots,n$, exhibited in $x = \Sigma_0^n \lambda_i u^i$ are continuous functions of $(x, u^o,\ldots,u^n)$ over the set in which $u^o,\ldots,u^n$ are affinely independent and in which x is an affine combination of $u^o,\ldots,u^n$. Thus, as α becomes large, λ_i^α converges to some λ_i^o and these limiting scalars satisfy $\Sigma_0^n \lambda_i^o = 1$ and $x^o = \Sigma_0^n \lambda_i^o f^i(m^o)$. Thus $x^o \varepsilon L[\varphi(m^o)]$ as was to be shown.

To demonstrate that $H[\varphi]$ has a closed graph, we note that the correspondence ψ, defined by $\psi(m) = \varphi(m) - f(m)$ where f

is any given continuous selection from φ, is also LSC and convex-valued. By the preceding paragraph, $H[\varphi] = L[\psi]$ has a closed graph on $M_\psi^n = M_\varphi^n$. $\blacksquare$

The preceding proof also demonstrates how to show that when (M,S,φ) satisfies CST, so do $L[\varphi]$ and $H[\varphi]$: Given any $z^o \in L[\varphi(m^o)]$, there exists a finite subset $\{x^i\}_o^n$ of $\varphi(m^o)$ such that $z^o = \Sigma_o^n \lambda_i x^i, \Sigma_o^1 \lambda_i = 1$. Choose continuous selections f^i from φ for which $f^i(m^o) = x^i$. Then f defined by $f(m) = \Sigma_o^n \lambda_i f^i(m)$ is a continous selection from $L[\varphi]$ through z^o. It is easy to see that this means that $L[\varphi]$ is LSC.

An example which demonstrates that $L[\varphi]$ need not be USC even if φ is LSC and USC is constructed by letting $M = [0,1]$, $S = \mathbb{R}^2$, $\varphi(m) = \{(x_1,x_2) = t(1-m,m): 0 \leqq t \leqq 1\}$. Then $L[\varphi(m)]$ is the line through $(0,0)$ and $(1-m,m)$. For any $\epsilon > 0$ the only m such that $L[\varphi(m)]$ is contained in the open set $\{(x_1,x_2): |x_1| < \epsilon\}$ is $m = 1$.

It is also easy to see that $L[\varphi]$ need not have a closed graph in $M \times S$: Let $M = \mathbb{R}_+^1$,

$$\varphi(m) = \begin{cases} \{y \in \mathbb{R}^1: \ m + y < 1, \ y > 0\} & 0 \leqq m < 1 \\ \{0\} & m \geqq 1 \end{cases}$$

$$L[\varphi(m)] = \begin{cases} \mathbb{R}^1 & 0 \leqq m < 1 \\ \{0\} & m \geqq 1 \end{cases}$$

<u>COROLLARY:</u> If S has dimension N, (M,S,φ) and (M,S,ψ) satisfy CST, φ is relatively-open-valued and $L[\psi] \subset L[\varphi]$, then for $0 \leq n \leq N$:

$$\{m \varepsilon M_\varphi^n : \varphi(m) \cap \psi(m) \neq \emptyset\}$$

is open in M_φ^n.

<u>PROOF:</u> If $m^o \varepsilon M_\varphi^n$ and some x^o is in $\varphi(m^o) \cap \psi(m^o)$, by Theorem 2 we can find $\epsilon > 0$ and a neighborhood U_1 of m^o such that $m \varepsilon U_1 \cap M_\varphi^n$ and $x \varepsilon B_\epsilon(x^o) \cap L[\varphi(m)]$ imply $x \varepsilon \varphi(m)$. Choose a continuous selection g from ψ with $g(m^o) = x^o$. But then there is some open subset U of U_1 such that m^o U and $\|g(m) - g(m^o)\| < \epsilon$ when $m \varepsilon U$. Thus $m \varepsilon U \cap M_\varphi^n$ implies $g(m) \varepsilon \varphi(m) \cap \psi(m)$. ∎

3. Openness of the Integral of a Correspondence

To derive conditions for $\int_E \varphi d\mu$ to be open whenever φ is open-valued, we define $d_\varphi(m)$ to be the distance from zero to the complement, ${}^c\varphi(m)$, of $\varphi(m)$:

$$d_\varphi(m) = \sup\{\epsilon \geqq 0: \ B_\epsilon(0) \subset \varphi(m)\} \quad .$$

LEMMA 2: If φ is measureable, so is d_φ.

PROOF: This is proven by showing that for any finite, nonnegative $\epsilon \geqq 0$, $\{m: d_\varphi(m) < \epsilon\}$ and $\{m: d_\varphi(m) = \epsilon\}$ are measureable.

Because φ is measureable, $\{(m,s) \varepsilon M \times S: \ s \ \varepsilon \ B_\epsilon(0) \cap {}^c\varphi(m)\}$ is measureable. But $\{m: d_\varphi(m) < \epsilon\} = \{m: B_\epsilon \cap {}^c\varphi(m) \neq \emptyset\} = \text{proj}_M \{(m,s): s \ \varepsilon \ B_\epsilon(0) \cap {}^c\varphi(m)\}$, so $\{m: d_\varphi(m) < \epsilon\}$ is analytic and hence is measureable since $(\mathcal{M}, \mu)$ is a complete measure space. [6 , 3.4, page 357]. The proof that $\{m: d_\varphi(m) = \epsilon\}$ is measureable follows similarly by noting that $\{m: d_\varphi(m) = \epsilon\} = \{m: B_\epsilon(0) \subset \varphi(m)\} \cap [\bigcap_{n=1}^{\infty} \{m: B_{\epsilon+1/n}(0) \cap {}^c\varphi(m) \neq \emptyset\}]$. By the preceding, it is sufficient to show $\{m: B_\epsilon(0) \subset \varphi(m)\}$ is measureable. But this set equals $[\text{proj}_M \{(m,s): \varphi(m) \cap B_\epsilon(0) \neq \emptyset\}] \setminus \{m: d_\varphi(m) < \epsilon\}$. ∎

THEOREM 3: If φ is measureable and open-valued, then, for any E, $\int_E \varphi d\mu$ is open.

PROOF: By a simple argument, we can let $E = M$. We want to show that if h is any integrable selection from φ, then $\int_M h\, d\mu$ is in the interior of $\int_M \varphi d\mu$. By letting $\psi = \varphi - h$, we have $0 \in \psi(m)$, all m. It is sufficient to show that some neighborhood of 0 is contained in $\int_M \psi\, d\mu$.

Because ψ is open-valued, $d_\psi(m) > 0$, all m. By Lemma 1, we can find $\epsilon > 0$ and a measureable $F \subset M$ such that $\mu(F) > 0$ and $d_\psi(m) \geqq \epsilon$, m in F. But then $B_\epsilon(0) \subset \varphi(m)$, m in F, so $\mu(F)\, B_\epsilon(0) \subset \int_F \psi\, d\mu \subset \int_M \psi\, d\mu$. Since $\mu(F)\, B_\epsilon(0)$ is a neighborhood of 0, we are finished. ∎

LEMMA 3: If G_1 is open, convex and dense in G_2 which is also open in S, then $G_1 = G_2$.

PROOF: Because G_2 is open and G_1 is dense in G_2, $G_2 \subset \text{int}\, \overline{G}_1$. Thus it is enough to show that $\text{int}\, \overline{G}_1 \subset G_1$. If $x \in \text{int}\, \overline{G}_1$, there is some open neighborhood U which is symmetric around x and contained in $\overline{G}_1$. U open implies there exists $y \in U \cap G_1$; U symmetric around x implies $2x-y = x - (y-x) \in U \subset \overline{G}_1$. But then $x = \frac{1}{2} y + \frac{1}{2}(2x-y) \in G_1$ since $y \in G_1$, $2x-y \in \overline{G}_1$ and G_1 is convex [12, Theorem 6.1, page 45]. ∎

COROLLARY: If (M,S,φ) satisfies MST, if $\text{int}\, \varphi(m)$ is dense in $\varphi(m)$, μ-a.e. and if either (1) $\text{int}\, \varphi$ is convex-valued or (2) μ is nonatomic and S is finite-dimensional, then for every E,

$$\text{int} \int_E \varphi\, d\mu = \int_E \text{int}\, \varphi\, d\mu .$$

PROOF: Suppose we knew

(3) $\int_E \text{int}\,\varphi\, d\mu$ is dense in $\int_E \varphi\, d\mu$.

Then $\int_E \text{int}\,\varphi\, d\mu$ and $\text{int} \int_E d\mu$ are both open sets in S (by Theorem 3) and the former is a convex subset of the latter [1]. By Lemma 3, the two sets coincide. (By (3), if one set is empty, so is the other.)

To prove (3), for any $\epsilon > 0$ and any integrable selection f from φ , let ψ be defined by

$$\psi(m) = \{x \in \text{int}\,\varphi(m): \|x - f(m)\| < \epsilon \} .$$

ψ is nonempty-valued since $\text{int}\,\varphi(m)$ is dense in $\varphi(m)$. Clearly ψ is measureable (see proof of Lemma 6) and so by the MST , has a measureable selection g . Since f is integrable, g is also and $\|\int_E (f-g)d\mu\| < \epsilon$. ∎

It is easy to see that the denseness of $\text{int}\,\varphi(m)$ in $\varphi(m)$ is needed for the result stated in the Corollary. Consider the example where $M = [0,1]$, μ is Lebesgue measure and $\varphi(m) = \{x \in \mathbb{R}^1: 0 \leqq x \leqq \frac{1}{2} \text{ or } x = 1\}$, m in M . Then $\text{int} \int_M \varphi\, d\mu = (0,1)$ and $\int_M \text{int}\,\varphi\, d\mu = (0,\frac{1}{2})$. Less obvious is the need for $\text{int}\,\varphi$ to be convex-valued or else for μ to be nonatomic: Suppose $M = \{0,1,2\}$, $S = \mathbb{R}^1$ and for each $i=1,2,3$: $\mu(\{i\}) = \frac{1}{3}$ and $\varphi(i) = [0,\frac{1}{4}] \cup [\frac{3}{4},1]$. Then $\text{int} \int_M \varphi\, d = (0,1)$ but $\int_M \text{int}\,\varphi\, d\mu = (0,\frac{1}{4}) \cup (\frac{1}{4},\frac{1}{2}) \cup (\frac{1}{2},\frac{3}{4}) \cup (\frac{3}{4},1)$.

We derive an analogue of Theorem 1 for $\int^c \varphi\, d\mu$:

THEOREM 3': If (M,S,φ) satisfy CST , φ is positively unbounded, and μ is regular, then for any E , $\int_E^c \varphi\, d\mu$ is open.

PROOF: As with the proof of Theorem 3 above, it is sufficient to consider only the case where $0 \in \varphi(m)$, all m. We then want to find some neighborhood of 0 which is contained in $\int_E^c \varphi d\mu$. By Theorem 1, for any m^0 there exists $\epsilon > 0$ and an open set U containing m^0 such that $U \times B_\epsilon(0) \subset G_\varphi$. If M were compact, we could then find an ϵ' such that $M \times B_{\epsilon'}(0) \subset G_\varphi$ so that $\mu(E) B_{\epsilon'}(0) \subset \int_E^c \varphi d\mu$. In the general case, since μ is regular, we can choose a closed subset F of $U \cap E$ with positive measure. By Urysohn's Lemma [8,page 146], we can choose a continuous function λ from M to $[0,1]$ such that

$$\lambda(m) = \begin{cases} 1 & m \in F \\ 0 & m \notin U \end{cases} .$$

For any $z \in B_\epsilon(0)$, $\lambda(\cdot)z$ is a continuous selection from φ whose integral on E is $[\int_E \lambda \, d\mu] z$. Thus $[\int_E \lambda \, d\mu] \, B_\epsilon(0) \subset \int_E^c \varphi d\mu$. Since $\int_E \lambda \, d\mu \geq \mu(F) > 0$, this exhibits an open neighborhood of 0 contained in $\int_E^c \varphi \, d\mu$. ∎

COROLLARY: If $(M, S, \text{int}\,\varphi)$ satisfies CST, if μ is regular, if $\text{int}\,\varphi$ is positively unbounded and if, for every m, $\text{int}\,\varphi(m)$ is dense in $\varphi(m)$, then for every E,

$$\text{int} \int_E^c \varphi \, d\mu = \int_E^c \text{int}\,\varphi \, d\mu .$$

The proof of this is similar to the proof of the Corollary to Theorem 3 with continuous selections replacing measureable selections.

When S is finite-dimensional, we can sharpen Theorem 1. We first establish a preliminary result.

LEMMA 4: If (M,S,φ) satisfies MST, if f is an integrable selection from φ and $p \in S'$, then for any E, $p\cdot\int_E f\,d\mu = \sup p\cdot\int_E \varphi\,d\mu$ if and only if $p\cdot f(m) = \sup p\cdot\varphi(m)$ μ-a.e. on E.

PROOF: Sufficiency is clear, since if $p\cdot f(m) = \sup p\cdot\varphi(m)$ μ-a.e. on E, then

$$\begin{aligned} \sup p\cdot\int_E \varphi\,d\mu &\geqq p\cdot\int_E f\,d\mu \\ &= \int_E \sup p\cdot\varphi\,d\mu \\ &\geqq \sup p\cdot\int_E \varphi\,d\mu . \end{aligned}$$

To demonstrate the converse, suppose $p\cdot\int_E f\,d\mu = \sup p\cdot\int_E \varphi\,d\mu$. If we define

$$F = \{m \in E:\ p\cdot f(m) < \sup p\cdot\varphi(m)\},$$

then we want $\mu(F) = 0$.

It is readily checked that F is measureable since the mapping $m \mapsto \sup p\cdot\varphi(m)$ is measureable [7]. Suppose $\mu(F) > 0$. Define a correspondence θ on F:

$$\theta(m) = \{x \in \varphi(m):\ p\cdot x > p\cdot f(m)\} .$$

Then $G_\theta = G_\varphi \cap \{(m,s) \in F\times S:\ p\cdot s > p\cdot f(m)\}$ is measureable so θ is measureable. θ is nonempty-valued on F so there exists a measureable selection $g|_F$ from θ. If $g|_F$ is not integrable, then there is some integer n such that $F_n = \{m \varepsilon F: \|g(m) - f(m)\| \leq n\}$ has positive measure (since $F = \cup F_n$). Define an integrable selection h from φ by

$$h(m) = \begin{cases} g(m) & m \varepsilon F_n \\ f(m) & \text{otherwise} \end{cases} .$$

Then $p \cdot \int_E h\, d\mu > p \cdot \int_E f\, d\mu$. Since h is an integrable selection from φ, we have found that $\mu(F) > 0$ contradicts $p \cdot \int_E f\, d\mu = \sup p \cdot \int_E \varphi\, d\mu$. Thus $\mu(F) = 0$. ▮

The result in Lemma 4 is closely related to the result that if φ is measureable and $\int_E \varphi\, d\mu \neq \emptyset$, then $\sup p \cdot \int_E \varphi\, d\mu = \int_E \sup p \cdot \varphi\, d\mu$ [7],[9].

THEOREM 4: If S is finite dimensional, (M,S,φ) satisfies MST, if φ is relatively-open-valued and either φ is convex-valued or μ is nonatomic, then for every E, $\int_E \varphi\, d\mu$ is relatively open.

PROOF: To show that $\int_E h\, d\mu \ \varepsilon \ \text{ri} \int_E \varphi\, d\mu$ for any integrable selection h from φ, it again suffices to show that $0 \ \varepsilon \ \text{ri} \int_E \psi\, d\mu$, where $\psi = \varphi - h$, since $\text{ri}[\int_E (\varphi - h)\, d\mu] = \text{ri}[\int_E \varphi\, d\mu] - \int_E h\, d\mu$. If L is the smallest affine subspace containing $\int_E \psi\, d\mu$, then $0 \ \varepsilon \int_E \psi\, d\mu$ means L is a linear subspace. By the separating hyperplane theorem and the convexity of $\text{ri} \int_E \psi\, d\mu$ [1], it is enough to show that for any nonzero p in L', $0 < \sup p \cdot \int_E \psi\, d\mu$.

Suppose that for some nonzero p in L', $0 = \sup p \cdot \int_E \psi \, d\mu$. By Lemma 4 (letting $f \equiv 0$), we see that

(4) $$0 = \sup p \cdot \psi(m), \quad \mu\text{-a.e. on } E .$$

Relation (4) together with $0 \in \text{ri } \psi(m)$, μ-a.e. imply that $\psi(m) \subset p^{-1}(0)$, μ-a.e. on E. But then $p^{-1}(0) \cap L$ is a proper subspace of L containing $\int_E \psi \, d\mu$. This contradicts the definition of L. Thus $\int_E h \, d\mu \in \text{ri} \int_E \varphi \, d\mu$. ∎

We note that the proof given above cannot be extended to give the infinite-dimensional result stated in Theorem 3, since the proof given for Theorem 4 would require that the interior of $\int_E \varphi \, d\mu$ be nonempty if S were infinte-dimensional.

<u>COROLLARY:</u> If S is finite dimensional, (M,S,φ) satisfies MST, if $\text{ri } \varphi(m)$ is dense in $\varphi(m)$, μ-a.e., and if either (1) $\text{ri } \varphi$ is convex-valued or (2) μ is nonatomic, then, for every E,

$$\text{ri} \int_E \varphi \, d\mu = \int_E \text{ri } \varphi \, d\mu .$$

The proof of this is anlogous to the proof of the Corollary to Theorem 3 with S replaced by $L[\int_E \varphi \, d\mu]$.

<u>LEMMA 4':</u> If (M,S,φ) satisfies CST, if f is a continuous integrable selection from φ, if μ is tight and if $p \in S'$, then for any E, $p \cdot \int_E f \, d\mu = \sup p \cdot \int_E^c \varphi \, d\mu$ if and only if $p \cdot f(m) = \sup p \cdot \varphi(m)$ μ-a.e. on E.

PROOF: Sufficiency is again clear. To prove necessity, we want to show that the set

$$F = \{m \in M: \ p \cdot f(m) < \sup \ p \cdot \varphi(m)\}$$

satisfies $\mu(E \cap F) = 0$. Since f is continuous and the mapping $m \mapsto \sup p \cdot \varphi(m)$ is LSC, F is open (proof of Theorem 1).

We define a correspondence θ by

$$\theta(m) = \{x \in \varphi(m): p \cdot x > p \cdot f(m)\}.$$

We want to apply the CST to θ on F. Since θ is nonempty-convex-valued on F, we need show only that θ is LSC. Suppose $m_o \in F$ and for some open G in S, $\theta(m_o) \cap G \neq \phi$. By the CST applied to φ, there exists a continuous selection g of φ such that $g(m_o) \in \theta(m_o) \cap G$. But then there exists a neighborhood U of m_o such that $g(U) \subset G$ and $p \cdot g(m) > p \cdot f(m)$, $m \in U$. Thus $m \in U$ implies $\theta(m) \cap G$ is not empty, so θ is LSC. By the CST applied to θ, choose a continuous selection h from θ (h is only defined on F).

Suppose $\mu(E \cap F) > 0$. Since μ is tight, there exists a compact set $K \subset E \cap F$ such that $\mu(K) > 0$. Because M is a normal topological space, there exists an open neighborhood F' of K which is also contained in F and on which h and f are bounded. By Urysohn"s Lemma [8, page 146] there exists a continuous function $\lambda: M \to [0,1]$ such that

$$\lambda(m) = \begin{cases} 1 & m \in K \\ 0 & m \notin F' \end{cases}.$$

Define a function e on M by

$$e(m) = \begin{cases} \lambda(m)\, h(m) + (1-\lambda(m))\, f(m) & m \in F' \\ f(m) & \text{otherwise} \end{cases},$$

Thus e is a continuous integrable selection from φ. For every m, $p \cdot e(m) \geqq p \cdot f(m)$, and for $m \in K$, $p \cdot e(m) > p \cdot f(m)$. Thus $p \cdot \int_E e\, d\mu > p \cdot \int_E f\, d\mu$ which contradicts the hypothesis that $p \cdot \int_E f\, d\mu = \sup p \cdot \int_E^c \varphi\, d\mu$. Thus $\mu(E \cap F) = 0$. ∎

<u>THEOREM 4'</u>: If S is finite-dimensional, (M,S,φ) satisfies CST, μ is tight and φ is relatively-open-valued, then for every E, $\int_E^c \varphi\, d\mu$ is relatively open.

The proof of Theorem 4 carries over with Lemma 4 replaced by Lemma 4'.

<u>COROLLARY:</u> If S is finite dimensional, (M,S,φ) satisfies CST, and if μ is tight, then for every E,

$$\text{ri} \int_E \varphi\, d\mu = \int_E \text{ri}\, \varphi\, d\mu .$$

This result is proven analogously to the Corollary of Theorem 3.

4. <u>Application 1: Existence of a Radon-Nikodým Derivative of a Relatively-Open-Set-Valued Measure</u>

A correspondence $\Phi: \mathcal{M} \Rightarrow S$ is a <u>set-valued measure</u> (or countably-additive correspondence) if it is countably additive: for every sequence $\{E_i\}$ of pairwise disjoint elements of $\mathcal{M}$, $\Phi(\cup E_i) = \Sigma\, \Phi(E_i)$, where, for any sequence $\{X_i\}$ of

S, $\Sigma X_i = \{x \in S:$ for each i there exists $x_i \in X_i$ such that $\Sigma_1^n x_i$ converges absolutely to $x\}$. We say that $\varphi: M \Rightarrow S$ (resp., $\Phi: \mathcal{M} \Rightarrow S$) is positive-valued if there exists a closed convex pointed cone P such that $\varphi(m) \subset P$ for all m (resp., $\Phi(E) \subset P$, all E). Φ is μ-continuous if $\mu(E) = 0$ implies $\Phi(E) = \{0\}$.

If $\Phi(E) = \int_E \varphi \, d\mu$ for all E, we say φ is a Radon-Nikodým derivative of Φ. The very basic work of Debreu and Schmeidler [7] characterizes those set-valued measures which have closed-convex-positive-valued measureable Radon-Nikodým derivatives. In this section we show that a similar characterization of those set-valued measures having relatively-open-convex-positive-valued measureable derivatives follows immediately from Theorem 4 of this paper and the work of Debreu and Schmeidler.

THEOREM 5: If S has finite dimension, then Φ is a countably additive, μ-continuous, positive-convex-relatively-open-set-valued measure if and only if it has a positive-convex-relatively-open-valued measureable Random-Nikodým derivative.

PROOF: The "if" implication is an easy corollary of Theorem 4. To prove the converse, define a partial ordering on the correspondences from $\mathcal{M}$ to S by $\Gamma_1 \subset \Gamma_2$ whenever $\Gamma_1(E) \subset \Gamma_2(E)$ for all E in $\mathcal{M}$. The conditions assumed on Φ ensure that there exists a set-valued measure $\hat{\Phi}$ which is maximal for the partial order $\subset$ in the collection of set-valued measures Γ on S which satisfy $\Phi \subset \Gamma \subset \bar{\Phi}$ (by Theorem 1 of Debreu and

Schmeidler [7]). By Theorem 2 of Debreu and Schmeidler, $\hat{\Phi}$ has a closed-convex-positive-valued measureable Radon-Nikodým derivative $\hat{\varphi}$. Let $\varphi = \text{ri}\,\hat{\varphi}$. By Theorem 4, for any E, $\int_E \varphi\, d\mu = \text{ri} \int_E \hat{\varphi}\, d\mu = \Phi(E)$. $\blacksquare$

5. Application 2: Every Integrable Selection can be Replaced by a Continuous Selection

In this section we use the results of Section 3 to give conditions under which it is possible to assume that any vector in $\int_E \varphi\, d\mu$ is actually the integral over E of a continuous integrable selection from φ.

THEOREM 6: If (M,S,φ) satisfies CST, if μ is regular and has compact support and either (1) φ is open-valued and positively unbounded or (2) S is finite-dimensional and φ is relatively-open-valued, then for every E,

$$\int_E \varphi\, d\mu = \int_E^C \varphi\, d\mu .$$

PROOF: Since μ has compact support, we may assume without loss of generality that M is compact. We shall consider first the case where S is finite-dimensional and φ is relatively-open-valued. Suppose we knew

(5) $\int_E^C \varphi\, d\mu$ is dense in $\int_E \varphi\, d\mu$.

Then $L[\int_E \varphi\, d\mu] = L[\int_E^c \varphi\, d\mu]$ so by (5), Theorems 4 and 4' and Lemma 3 we get the desired equality.

To establish (5), we consider only the case where $\int_E \varphi\, d\mu$ is not empty. Choose any integrable selection f from φ and $\epsilon > 0$. We want to find a continuous integrable selection whose integral over E is within ϵ of $\int_E f\, d\mu$.

Because φ is nonempty-valued, we can choose from it a continuous selection e. e is integrable since μ has compact support. This compactness also means that μ is tight on the Borel subsets of M, so, by Lusin's Theorem [11, page 69], we can choose a closed subset F of E such that $f|_F$ is continuous and the integrals $\int_{E\setminus F} \|f\| d\mu$ and $\int_{E\setminus F} \|e\| d\mu$ are both less than $\epsilon/2$. By the CST, there is a continuous, integrable selection g from φ which extends $f|_F$. By the regularity of μ on M, we can choose a sequence $\{F_n\}$ of closed subsets of $E\setminus F$ such that $\mu((E\setminus F)\setminus F_n) \to 0$. By Urysohn's Lemma [8, page 146], there exists a continuous function $\lambda_n: M \to [0,1]$ satisfying

$$\lambda_n(t) = \begin{cases} 1 & t \in F \\ 0 & t \in F_n \end{cases} .$$

Define

$$h_n(t) = \lambda_n(t)\, g(t) + (1-\lambda_n(t))\, e(t) .$$

Since M is compact, h_n is a continuous, integrable selection from φ and $h_n|_F = f|_F$, $h_n|_{F_n} = e|_{F_n}$. Finally, $\|h_n(t)\| \leq \|g(t)\| + \|e(t)\|$, so the sequence $\{h_n\}$ is uniformly integrable. Then

$$\int_E \|h_n - f\| d\mu \leq \int_{E\setminus F} \|h_n - f\| d\mu$$

$$\leq \int_{E\setminus F} (\|f\| + \|e\|) d\mu + \int_{(E\setminus F)\setminus F_n} \|h_n\| \, d\mu \quad .$$

From the choice of F , the uniform integrability of $\{h_n\}$ and $\mu((E\setminus F)\setminus F_n) \to 0$ we conclude that eventually $\int_E \|h_n - f\| d\mu < \epsilon$.

In the case where S is infinite-dimensional, we have int $\int_E^C \varphi \, d\mu$ is not empty by Theorems 3' ; so we have $L[\int_E \varphi \, d\mu] = L[\int_E^C \varphi \, d\mu] = S$. The rest of the proof goes through unchanged.

<u>Remark:</u> The assumption that μ has a compact support can be replaced by the assumptions that μ be tight and that there exist a continuous bounded selection from φ . This selection would serve the role played by e in the preceding proof. In this case it would also be possible to show that g could be chosen to be a continuous, bounded extension of $f|_F$.

We give an example to show that $\int_E^c \varphi \, d\mu$ need not equal $\int_E \varphi \, d\mu$ unless φ is relatively-open-valued. Define φ_1: $[0,1] \Rightarrow \mathbb{R}^1$ by

$$\varphi_1(m) = \begin{cases} [0,\frac{1}{2}] & 0 \leqq m \leqq \frac{1}{2} \\ [0,1] & \frac{1}{2} < m \leqq 1 \end{cases} .$$

Clearly, φ_1 is LSC and convex-compact-valued. Let μ be Lebesgue measure on $[0,1]$ and define an integrable selection from φ_1 :

$$f_1(m) = \begin{cases} \frac{1}{2} & 0 \leqq m \leqq \frac{1}{2} \\ 1 & \frac{1}{2} < m \leqq 1 \end{cases} .$$

Then $$\int_M f_1 \, d\mu = \frac{3}{4} \in [\int_M \varphi_1 \, d\mu] \setminus [\int_M^c \varphi_1 \, d\mu] .$$

This example can be extended easily to give an unbounded correspondence: Define φ_2: $[0,1] \Rightarrow \mathbb{R}^2$ by

$$\varphi_2(m) = \begin{cases} \{(x_1, x_2): \ 0 \leqq x_1 \leqq \frac{1}{2}, \ x_2 \geqq 0\} & 0 \leqq m \leqq \frac{1}{2} \\ \{(x_1, x_2): \ 0 \leqq x_1 \leqq 1, \ x_2 \geqq 0\} & \frac{1}{2} < m \leqq 1 \end{cases}$$

If

$$f_2(m) = \begin{cases} (\frac{1}{2}, 0) & 0 \leqq m \leqq \frac{1}{2} \\ (1, 0) & \frac{1}{2} < m \leqq 1 \end{cases}$$

then $\int_M f_2 \, d\mu = (\frac{3}{4}, 0) \in [\int_M \varphi_2 \, d\mu] \setminus [\int_M^c \varphi_2 \, d\mu]$.

Given a correspondence $\varphi: M \Rightarrow S$, Theorem 6 appears to be useful in studying continuity properties of a mapping $\lambda \to \int \varphi \, d\lambda$ defined on some collection of probabilities on $(M, \mathcal{M})$ with the weak topology. Another application of Theorem 6 is to provide conditions under which the correspondence on $\mathcal{M}$, which takes E into $\int_E^c \varphi \, d\mu$, is countably additive. To see that $\int_E^c \varphi \, d\mu$ is not always countably additive (unlike $\int_E \varphi \, d\mu$), suppose g_1 and g_2 are two continuous, integrable functions from M to $\mathbb{R}^1$ such that $g_1(m) < g_2(m)$ everywhere. Let $\varphi(m) = \{g_1(m), g_2(m)\}$. Then $\int_E^c \varphi \, d\mu$ is clearly not countably additive.

6. Openness of One Correspondence Relative to Another Correspondence

If φ and ψ are two correspondences from M to S such that, for every m, $\varphi(m)$ is an open subset of $\psi(m)$, we say φ is <u>open in</u> ψ. In this section we give conditions under which, if φ is open in ψ, then $\int \varphi \, d\mu$ is open in

$\int \psi \, d\mu$. Our procedure to show $\int \varphi \, d\mu$ is open in $\int \psi \, d\mu$ will be to prove something a little stronger: $\int \varphi \, d\mu$ is open in $L[\int \psi \, d\mu]$. This will be done by showing that $L[\int \psi \, d\mu] = L[\int \varphi \, d\mu]$ and using Theorem 3 or 4. Thus, in effect, we shall reduce the more general problem posed here to the problem solved in Section 3.

<u>LEMMA 5:</u> If (M,S,φ) satisfies MST, if $L[\int_E \varphi \, d\mu]$ is closed and if $0 \in \varphi(m)$, μ-a.e. on E , then

$$L[\varphi(m)] \subset L[\int_E \varphi \, d\mu] \; , \quad \mu\text{-a.e. on } E \; .$$

<u>PROOF:</u> It suffices to show that μ-a.e. on M , $\varphi(m)$ is a subset of the linear subspace $L_1 \equiv L[\int_M \varphi \, d\mu]$. If this were false, then the graph G_φ of φ would not be contained in $M \times L_1$. In fact, if we let $H = G_\varphi \setminus (M \times L_1)$, then we would have $\mu(\text{proj}_M(H)) > 0$. (For the measureability of $\text{proj}_M(H)$, see [6 , (3.4) page 357].)

Since S is separable and L_1 is closed, we can choose a countable collection $\{C_k\}_1^\infty$ of open, convex sets in S whose union is $S \setminus L_1$. (The sets C_k need not be disjoint.) Let $H_k = (M \times C_k) \cap G_\varphi$ so $H = \cup H_k$. Then $\text{proj}_M(H) = \bigcup_k \text{proj}_M(H_k)$, so for some k_o , $\mu(\text{proj}_M(H_{k_o})) > 0$. Let $M_o = \text{proj}_M(H_{k_o})$.

Define a correspondence $\varphi_o \colon M_o \Rightarrow C_{k_o}$ by

$$\varphi_o(m) = \varphi(m) \cap C_{k_o} \; .$$

Then φ_o is nonempty-valued and measureable and so has a measureable selection h . Clearly

$$\int_{M_o} h \, d\mu \ \varepsilon \ \int_{M_o} \varphi_o \, d\mu \subset \int_M \varphi \, d\mu \subset L_1$$

where the first inclusion uses the assumption that $0 \varepsilon \varphi(m)$ μ-a.e.

We obtain the desired contradiction by showing that $\int_{M_o} h \, d\mu$ is also in $\mu(M_o)C_{k_o}$ which is disjoint from L_1. If $\int_{M_o} h \, d\mu \not\subset \mu(M_o) C_{k_o}$ (a nonempty, open, convex set), then there exists nonzero p in S' such that

$$p \cdot \int_{M_o} h \, d\mu \geqq k > p \cdot x , \qquad x \ \varepsilon \ \mu(M_o) C_{k_o} .$$

But then we have the contradiction:

$$p \cdot \int_{M_o} h \, d\mu = \int_{M_o} p \cdot h \, d\mu < k \leqq p \cdot \int_{M_o} h \, d\mu ,$$

where the strict inequality follows from the fact that $p \cdot h(m) < \frac{k}{\mu(M_o)}$, m in M_o , since $h(m) \ \varepsilon \ C_{k_o}$ for m in M_o . ∎

It is easily seen that the assumption that $0 \ \varepsilon \ \varphi(m)$ μ-a.e. cannot be omitted: suppose φ is a positive-single-valued-correspondence (i.e., a positive function).

THEOREM 7: Suppose (M,S,φ) satisfies MST . Then for any E, if $L[\int_E \varphi d\mu]$ is closed and nonempty, we have $L[\int_E \varphi \, d\mu] = \int_E L[\varphi] \, d\mu$.

PROOF: We may assume $E = M$. Let $L_1 = L[\int_M \varphi \, d\mu]$ and $L_2 = \int_M L[\varphi] \, d\mu$.

To show $L_1 \subset L_2$, we show first that L_2 is an affine subspace: If $x^i \ \varepsilon \ L_2$ and t_i is a scalar, $i=1,2$, and $t_1 + t_2 = 1$, then there are integrable selections f^i from $L[\varphi]$ such that $\int_M f^i d\mu = x^i$. If $f = t_1 f^1 + t_2 f^2$, then f is also an integrable selection from $L[\varphi]$ and $\int_M f\, d\mu = t_1 x^1 + t_2 x^2$. Thus L_2 is affine. Since $\varphi \subset L[\varphi]$, then $\int_M \varphi\, d\mu \subset L_2$. Since L_1 is defined as the smallest affine subspace containing $\int_M \varphi\, d\mu$, then $L_1 \subset L_2$.

We show $L_2 \subset L_1$ first for the case where $0 \varepsilon \varphi(m)$, μ-a.e. In this case, L_1 is a (closed) linear subspace. Thus it suffices to show $L[\varphi(m)] \subset L_1$ μ-a.e. But this was established in Lemma 5. Thus if $0 \varepsilon \varphi(m)$, μ-a.e., then $L[\int_M \varphi\, d\mu] = \int_M L[\varphi] d\mu$.

In the general case, since $\int_M \varphi\, d\mu$ is not empty, there exists an integrable selection f from φ . Define ψ by $\psi = \varphi - f$. Then $\int_M \psi\, d\mu = \int_M \varphi\, d\mu - \int_M f\, d\mu$, $L[\int_M \psi\, d\mu] = L[\int_M \varphi\, d\mu] - \int_M f\, d\mu$ and $L[\psi] = L[\varphi] - f$. By the two preceding paragraphs we have

$$\begin{aligned} L[\int_M \varphi\, d\mu] - \int_M f\, d\mu &= L[\int_M \psi\, d\mu] \\ &= \int_M L[\psi] d\mu \\ &= \int_M L[\varphi] d\mu - \int_M f\, d\mu \end{aligned}$$

so $L[\int_M \varphi\, d\mu] = \int_M L[\varphi] d\mu$. ▮

We remark that the inclusion $L[\int_M \varphi\, d\mu] \subset \int_M L[\varphi] d\mu$ required no assumptions. The opposite inclusion can easily be seen to be false without the condition that (M,S,φ) satisfy MST. Let f^1 be an integrable, real-valued function on M and let f^2 be a nonmeasureable, real-valued function on M such that

$f^2(m) \neq f^1(m)$ for all m. If $\varphi(m) = \{f^1(m), f^2(m)\}$, then $L[\int_M \varphi \, d\mu] = \int_M f \, d\mu$ but $\int_M L[\varphi]d\mu = \mathbb{R}^1$ since $L[\varphi(m)] = \mathbb{R}^1$ for all m. It is also easy to construct an example where (M,S,φ) satisfies MST, where φ has no integrable selections, but $L[\varphi(m)] = \mathbb{R}^1$ for all m. Then $\int_M L[\varphi]d\mu = \mathbb{R}^1$ and $L[\int_M \varphi \, d\mu]$ is empty.

<u>COROLLARY 1:</u> If (M,S,ψ) satisfies MST, if $\varphi: M \Rightarrow S$ satisfies $L[\varphi] \subset L[\psi]$ and if $L[\int_E \psi \, d\mu]$ is closed and nonempty, then

$$L[\int_E \varphi \, d\mu] \subset L[\int_E \psi \, d\mu] .$$

<u>PROOF:</u> From Theorem 7 and the remarks following it we have

$$L[\int_E \varphi \, d\mu] \subset \int_E L[\varphi] \, d\mu \subset \int_E L[\psi]d\mu = L[\int_E \psi \, d\mu] . \blacksquare$$

<u>COROLLARY 2:</u> If (M,S,φ) and (M,S,ψ) satisfy MST, $L[\varphi] = L[\psi]$ and if the sets $L[\int_E \varphi \, d\mu]$ and $L[\int_E \psi \, d\mu]$ are closed and nonempty, then

$$L[\int_E \varphi \, d\mu] = L[\int_E \psi \, d\mu] .$$

<u>LEMMA 6:</u> If (M,S,φ) and (M,S,ψ) satisfy MST, if φ is open in ψ, if ψ is convex-valued and $\operatorname{int} \psi$ is nonempty-valued, then for any E, either $\int_E \varphi \, d\mu$ is empty or has a nonempty interior (and then so does $\int_E \psi \, d\mu$).

<u>PROOF:</u> If $\int_E \varphi \, d\mu$ is not empty, there exists an integrable selection f from φ. Because φ is open in ψ, for any m

there exists δ such that $0 < \delta \leqq 1$ and such that $U(m) \equiv B_\delta(f(m)) \cap \psi(m)$ is open in $\psi(m)$, contains $f(m)$ and is contained in $\varphi(m)$. Because $\psi(m)$ is convex and $\operatorname{int} \psi(m)$ is not empty, then $\operatorname{int} \psi(m)$ is dense in $\psi(m)$ [12, Theorem 6.1, page 34]. Thus $U(m) \cap \operatorname{int} \psi(m)$ is not empty. Further, $[U(m) \cap \operatorname{int} \psi(m)] \subset \operatorname{int} \varphi(m)$.

If we define a correspondence θ by

$$\theta(m) = B_1(f(m)) \cap \operatorname{int} \varphi(m) ,$$

then θ is open-nonempty-valued. To check that θ is measureable, let $\{s_n\}$ be a dense subset of S. Following the procedure at the start of Section 3, we define

$$d^n_\varphi(m) = \sup\{\epsilon \geqq 0: \ B_\epsilon(s_n) \subset \varphi(m)\} .$$

By Lemma 2, d^n_φ is measureable for each n. Thus for $\epsilon \geqq 0$, $\{m \in M: \ B_\epsilon(s_n) \subset \varphi(m)\} = (d^n_\varphi)^{-1}([\epsilon, +\infty))$ is measureable. The graph of $\operatorname{int} \varphi$ equals $\bigcup_{r,n} \{m: B_{1/r}(s_n) \subset \varphi(m)\} \times B_{1/r}(s_n)$ which is clearly measureable. Since the graph of θ equals $G_{\operatorname{int} \varphi} \cap \{m,s): s \in B_1(f(m))\}$, θ is measureable.

By Theorem 3, $\int_E \theta \, d\mu$ is open in S. By the MST, there exists a measureable selection g from θ. Since $\|g(m) - f(m)\| < 1$ for all m and f is integrable, so is g. Thus $\int_E \theta \, d\mu$ is a nonempty open subset of $\int_E \varphi \, d\mu$, so $\int_E \varphi \, d\mu$ has a nonempty interior. ∎

<u>THEOREM 8:</u> If (M,S,φ) and (M,S,ψ) satisfy MST, if φ is open in ψ, if ψ is convex-valued and if $\operatorname{int} \psi$ is nonempty-valued or S is finite-dimensional, then for any E, either $L[\int_E \varphi \, d\mu]$ is empty or it equals $L[\int_E \psi \, d\mu]$.

PROOF: Suppose $L[\int_E \varphi \, d\mu]$ is not empty. If S is not finite-dimensional, then by Lemma 6 $\int_E \varphi \, d\mu$ and $\int_E \psi \, d\mu$ have nonempty interiors so $L[\int_E \varphi \, d\mu] = S = L[\int_E \psi \, d\mu]$. If S is finite-dimensional, then $L[\int_E \varphi \, d\mu]$ and $L[\int_E \psi \, d\mu]$ are closed. Because ψ is convex-valued and φ is open in ψ, $L[\varphi] = L[\psi]$. Thus by Corollary 2 of Theorem 7, $L[\int_E \varphi \, d\mu] = L[\int_E \psi \, d\mu]$. ∎

COROLLARY 1: If (M,S,φ) and (M,S,ψ) satisfy MST, if φ is open in ψ, if ψ is convex-valued and if $\text{int}\,\psi$ is nonempty-valued or S is finite-dimensional, then for any E, $\text{ri} \int_E \varphi \, d\mu$ is open in $L[\int_E \psi \, d\mu]$ and hence open in $\int_E \psi \, d\mu$.

This Corollary follows at once from Theorem 8 and the definition of relative interiors. This can be combined with the Corollaries of Theorems 3 and 4 to give:

COROLLARY 2: If (M,S,φ) and (M,S,ψ) satisfy MST, if φ is open in ψ, if φ and ψ are convex-valued, if φ is relatively-open-valued, and if $\text{int}\,\varphi$ is nonempty-valued or S is finite dimensional, then for any E, $\int_E \varphi \, d\mu$ is open in $L[\int_E \psi \, d\mu]$ and hence open in $\int_E \psi \, d\mu$.

The results above have not made essential use of the condition that φ be open in ψ. This is seen by noting that the conclusions of the Corollaries above remain valid in the case where S is finite dimensional when the conditions φ open in ψ and ψ convex-valued are weakened to $\varphi \subset \psi$ and $L[\varphi] = L[\psi]$. In our next result we provide the basis for a substantial extension of the results gotten so far by showing that when φ is open in ψ, then the only points in $[\int_E \varphi \, d\mu] \setminus [\mathrm{ri} \int_E \varphi \, d\mu]$ are also in the relative boundary of $\int_E \psi \, d\mu$. The condition that φ be open in ψ is important for this result.

<u>THEOREM 9:</u> If (M,S,φ) and (M,S,ψ) satisfy MST, if φ and ψ are convex-valued, if φ is open in ψ and if either $\mathrm{int}\,\varphi$ is nonempty-valued or S is finite dimensional, then for any E,

$$\mathrm{ri} \int_E \varphi \, d\mu = [\int_E \varphi \, d\mu] \cap [\mathrm{ri} \int_E \psi \, d\mu] .$$

<u>PROOF:</u> We assume $\int_E \varphi \, d\mu$ is not empty. From Theorem 8 we have $L[\int_E \varphi \, d\mu] = L[\int_E \psi \, d\mu]$. Hence the inclusion $\mathrm{ri} \int_E \varphi \, d\mu \subset [\int_E \varphi \, d\mu] \cap [\mathrm{ri} \int_E \psi \, d\mu]$ is obvious.

To derive the opposite inclusion, we show that if $x^o \in [\int_E \varphi \, d\mu] \setminus [\mathrm{ri} \int_E \varphi \, d\mu]$, then $x^o \notin \mathrm{ri} \int_E \psi \, d\mu$. We shall show first that if we have any hyperplane supporting $\int_E \varphi \, d\mu$ at x^o, then it also supports $\int_E \psi \, d\mu$ at x^o: Suppose p is a nonzero element of S' and $p \cdot x^o = \sup p \cdot \int_E \varphi \, d\mu$. By Lemma 4, if f^o is an integrable selection from φ whose integral over E is x^o, then

(6) $$p \cdot f^o(m) = \sup p \cdot \varphi(m) \qquad \mu\text{-a.e. on } E .$$

We shall show that

(7) $$p \cdot f^o(m) = \sup p \cdot \psi(m) \qquad \mu\text{-a.e. on } E .$$

If m_1 is an element of E for which there exists some y^1 in $\psi(m_1)$ such that $p \cdot y^1 > p \cdot f^o(m_1)$, then we could choose a sequence of points y^n from the line segment $\text{conv}\{y^1, f^o(m_1)\}$ which converged to $f^o(m_1)$ and such that $p \cdot y^n > p \cdot f^o(m_1)$ for all n. Since $f^o(m_1)$ is in $\varphi(m_1)$ which is open in $\psi(m_1)$, then for large enough n, y^n is in $\varphi(m_1)$. By (6) this is possible only for a null set of such points m_1. Thus (7) is valid. By Lemma 4 again, $p \cdot x^o = \sup p \cdot \int_E \psi \, d\mu$.

We have assumed that x^o is in the relative boundary of $\int_E \varphi \, d\mu$ (which is convex). By assumption, either $\int_E \varphi \, d\mu$ has a nonempty interior (Lemma 6), or S is finite dimensional. Thus by the Separating Hyperplane Theorem [13, page 64], there exists a nonzero element p in the dual of $H[\int_E \varphi \, d\mu] = H[\int_E \psi \, d\mu]$ such that $p \cdot x^o = \sup p \cdot \int_E \varphi \, d\mu$. By the preceding paragraph, $p \cdot x^o = \sup p \cdot \int_E \psi \, d\mu$. Since p is a nonzero element of the dual of $H[\int_E \psi \, d\mu]$, x^o is in the relative boundary of $\int_E \psi \, d\mu$. ∎

<u>COROLLARY 1:</u> If (M, S, φ) and (M, S, ψ) satisfy MST, if φ and ψ are convex-valued, if φ is open in ψ and if either $\text{int}\,\varphi$ is nonempty-valued or S is finite dimensional, then for any E, $[\int_E \varphi \, d\mu] \cap [\text{ri} \int_E \psi \, d\mu]$ is open in $L[\int_E \psi \, d\mu]$ and hence open in $\int_E \psi \, d\mu$.

This result is immediate from Theorem 9 and Corollary 1 of Theorem 8.

<u>COROLLARY 2:</u> Suppose (M,S,φ) and (M,S,ψ) satisfy MST, φ and ψ are convex-valued, φ is open in ψ and either $\operatorname{int}\varphi$ is nonempty-valued or S is finite dimensional. If $x \in \int_E \varphi \, d\mu$ and $y \in \operatorname{ri} \int_E \psi \, d\mu$, then some proper convex combination of x and y is in $\operatorname{ri} \int_E \varphi \, d\mu$.

<u>PROOF:</u> The result is trivial unless x is different from y. Suppose no point of $\operatorname{conv}[\{x,y\}]$ is in $\int_E \varphi \, d\mu$. By the preceding corollary, x is in the relative boundary of $\int_E \psi \, d\mu$. Let L be the line spanned by x and y. Suppose z in L is on the side of x opposite y (i.e. $z = \lambda_1 x + \lambda_2 y$, $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 > 1$). Then $z \notin \int_E \psi \, d\mu$ since if $z \in \int_E \psi \, d\mu$, then $x \in (z,y)$ so $x \in \operatorname{ri} \int_E \psi \, d\mu$. Thus $z \notin \int_E \varphi \, d\mu$. If $z = \lambda_1 x + \lambda_2 y$, $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 < 0$, then $z \notin \int_E \varphi \, d\mu$, since otherwise y, which is a convex combination of x and z, is in the convex set $\int_E \varphi \, d\mu$. In summary, the line L is disjoint from $\operatorname{ri} \int_E \varphi \, d\mu$. By the Separating Hyperplane Theorem, there is a (closed) hyperplane H containing L and disjoint from $\operatorname{ri} \int_E \varphi \, d\mu$. In particular, H supports $\int_E \varphi \, d\mu$ at x.

It was demonstrated in the proof of Theorem 9 that H must therefore support $\int_E \psi \, d\mu$ at x. But $y \in H \cap \operatorname{ri} \int_E \psi \, d\mu$, so $\int_E \psi \, d\mu \subset H$. This would mean $\int_E \varphi \, d\mu \subset H$ which is impossible since we chose H to be disjoint from $\operatorname{ri} \int_E \varphi \, d\mu$. ∎

Corollaries 1 and 2 have been derived by the author in a very different way in an earlier paper [4, Theorems 3 and 4]. The usefulness of these results in mathematical economics has been shown in [3].

Corollary 1 gives a partial answer to the question raised in this section. A complete answer has so far only been given under very restrictive assumptions on ψ. (For example $\psi(m) = P$ for all m where P is a polyhedral cone). In order to demonstrate that some additional conditions must be met in order for $\int \varphi \, d\mu$ to be open in $\int \psi \, d\mu$, we consider an example.

Let $M = [0,1]$ and let μ be Lebesgue measure on M. Define ψ by

$$\psi(m) = \{(x_1,x_2) \in \mathbb{R}^2 : (x_1,x_2) = t(1-m,m), \quad 0 \leqq t \leqq 1\} .$$

Let g be a selection from ψ defined by $g(m) = \frac{1}{2}(1-m,m)$. Define φ by

$$\varphi(m) = \{x \in \psi(m): \ x \geqq g(m) \ \text{ and } \ x \neq g(m)\} .$$

For every m, $\psi(m)$ is a subset of the convex set $\{(x_1,x_2) \geqq 0 : x_1 + x_2 \leqq 1\}$. Thus $\int_M \psi \, d\mu$ is a subset of this set. The point $(\frac{1}{2},\frac{1}{2})$ is in the boundary of this set and is the integral of the selection f from φ defined by $f(m) = (1-m,m)$. (Note that $(\frac{3}{8},\frac{1}{8}) = \int_0^{1/2} f \, d\mu$ so $(0,0)$, $(\frac{3}{8},\frac{1}{8})$ and $(\frac{1}{2},\frac{1}{2})$ are in $\int_E \psi \, d\mu$. Thus $\int_E \psi \, d\mu$ has a nonempty interior in $\mathbb{R}^2$). We shall show that $\int_M f \, d\mu$ is not in the interior of $\int_M \varphi \, d\mu$ relative to $\int_M \psi \, d\mu$ by showing that no other boundary point of $\int_M \psi \, d\mu$ is in $\int_M \varphi \, d\mu$ and that there exists a sequence of points in the boundary of $\int_M \psi \, d\mu$ which converges to $\int_M f \, d\mu$.

We first describe the boundary of $\int_M \psi \, d\mu$. A point x is in the boundary of the convex set $\int_M \psi \, d\mu$ if and only if $p \cdot x = \sup p \cdot \int_M \psi \, d\mu$ for some nonzero p in $\mathbb{R}^2$. By Lemma 4 this happens exactly when there is an integrable selection h^p from ψ such that $p \cdot h^p(m) = \sup p \cdot \psi(m)$, μ-a.e. and $\int_M h^p \, d\mu = x$. But this determines h^p uniquely μ-a.e.:

$$\text{If } p \geqq 0 , \quad \text{then } h^p(m) = (1-m, m) \qquad 0 \leqq m \leqq 1$$

$$\text{If } p < 0 , \quad \text{then } h^p(m) = (0,0) \qquad 0 \leqq m \leqq 1$$

$$\text{If } p_1 > 0,\ p_2 < 0, \quad \text{then } h^p(m) = \begin{cases} (0,0) & m > \dfrac{p_1}{p_1 - p_2} \\ (1-m, m) & \text{otherwise} \end{cases}$$

$$\text{If } p_1 < 0,\ p_2 > 0, \quad \text{then } h^p(m) = \begin{cases} (0,0) & m < \dfrac{p_2}{p_2 - p_1} \\ (1-m, m) & \text{otherwise} \end{cases}$$

Thus if $p_1 < 0$ or $p_2 < 0$, then $h^p(m)$ equals zero on a nonnull subset of M . Thus h^p is a selection from φ only when $p \geqq 0$. In this case $h^p = f$. We conclude that $\int_M f \, d\mu$ is the only boundary point of $\int_M \psi \, d\mu$ which is in $\int_M \varphi \, d\mu$. Finally, if we fix $p_1 > 0$ and let $p_2 \uparrow 0$, then $\int_M h^p \, d\mu \longrightarrow \int_M f \, d\mu$. Thus $\int_M f \, d\mu$ can be approximated by points which are in $\int_M \psi \, d\mu$ but not in $\int_M \varphi \, d\mu$.

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This paper gives conditions under which a correspondence φ has an open graph and has an integral which is an open set. There are two easy corollaries of these results: The first gives conditions for an open-set-valued measure to have an open-valued Radon-Nikodým derivative. The second gives conditions under which any point in the integral of φ is the integral of some continuous integrable selection from φ. Finally, the paper gives results on the problem of showing that if φ and ψ are two correspondences such that $\varphi(m)$ is open in $\psi(m)$ everywhere, then the integral of φ is open in the integral of ψ.

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