A NOTE ON THE ESTIMATION OF POLYNOMIAL DISTRIBUTED LAGS

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I. Introduction

The technique of using polynomials to approximate distributed lags has become widespread since its initial proposal by Shirley Almon [1] in 1965. A major advantage of the technique is that if a distributed lag is assumed to lie on a polynomial of a specified degree, then the distributed lag can be estimated by standard linear regression methods. A disadvantage of the technique, when polynomials of degree three or higher are estimated, is that one usually believes that the lag distribution is of a more restrictive shape than this. Even the general third degree polynomial, illustrated in Figure 1a, may have too unrestrictive a shape, since one usually does not believe that the lag distribution has both a trough and a peak. It may turn out, of course, that the undesirable peaks and troughs of polynomials occur outside of the estimated range of the lag distribution, but in general this will not happen.

The purpose of this note is to evaluate the estimation of restricted polynomial distributed lags. The shapes of polynomials can be restricted in a variety of ways, and it is of interest to examine how many restrictions have to be placed on a particular polynomial in order to eliminate the possibility that an undesirable shape will be estimated. It turns out that the shapes of fourth and fifth degree polynomials, once they are restricted to exclude undesirable peaks and troughs, are not much different from the shapes of second and third degree polynomials. The use of fourth and higher degree polynomials is therefore not in general recommended, and if a second or third degree polynomial is not considered to be a good enough approximation to the lag distribution, then the Almon technique should probably not be used.

II. The Use of Polynomial Distributed Lags

The Basic Model

The general distributed lag model can be written:

$$(1) \qquad y_t = \sum_{k=0}^{L} w_k x_{t-k} \quad ,$$

where y_t and x_{t-k} are economic time series at time t and t-k respectively, and w_k are the coefficients of the lag distribution. The length of the distributed lag is L+1 periods. The Almon [1] proposal is a means of imposing linear constraints on the estimated coefficients in such a way that the lag coefficients lie on a polynomial of some specified degree, say P.[1] In what has been termed the Direct Method, the specification of the polynomial takes the form:

$$(2) \qquad w_k = \sum_{j=0}^{P} \alpha_j k^j \quad , \qquad k=0,1,\ldots,L \quad .$$

Substituting (2) into (1) then yields:

$$(3) \qquad y_t = \sum_{k=0}^{L} \sum_{j=0}^{P} \alpha_j k^j x_{t-k} = \sum_{j=0}^{P} \alpha_j \left(\sum_{k=0}^{L} k^j x_{t-k} \right) = \sum_{j=0}^{P} \alpha_j Z_j \quad ,$$

[1] See Babb [2] for a development of Almon technique from this viewpoint.

where $Z_j = \sum_{k=0}^{L} k^j x_{t-k}$ are linear combinations of the observed data series x_{t-k}.

Using equation (3) and the constructed Z_j variables, the P+1 α_j coefficients can be estimated using standard linear regression techniques. Then using equation (2) and the estimated α_j coefficients, the w_k weights of the distributed lag can be easily calculated. Because the w_k can be interpreted as linear combinations of the α_j, any standard statistical tests of significance that can be performed on the α_j can also be quite easily performed on the w_k.[2]

End Point Restrictions on the Polynomial

It was recognized with the first use of the polynomial distributed lag technique by Almon [1] and Bischoff [3] that it may be desirable to constrain certain of the w_k polynomial values in (1) to be zero. In particular, they suggested that $w_0 = 0$ or $w_L = 0$ might be useful constraints.[3] The imposition of these constraints on the estimated α_j coefficients is straightforward. For example, setting $w_0 = 0$ in equation (2) yields:

$$(4) \qquad \sum_{j=0}^{P} \alpha_j = 0 \quad ,$$

[2]The same final result can be obtained in a variety of ways using different specifications for the polynomial of equation (2). A common alternative to the Direct Method is the use of Lagrangian interpolation polynomials. A comparison of the Direct Method and the Lagrangian technique is given in Cooper [4], Hall [5], Robinson [8] and Sparks [9]. Robinson also discusses other alternative specifications for the polynomial. It should be stressed, however, that all techniques will lead to the same answer. They differ only in terms of computational convenience in performing the statistical tests and in robustness to rounding error.

[3]More specifically, Almon and Bischoff suggested that w_{-1} and w_{L+1} be set to zero. The rationale for their procedure was that if the distribution

and setting $w_L = 0$ in equation (2) yields:

$$(5) \qquad \sum_{j=0}^{P} \alpha_j L^j = 0 \quad .$$

It is apparent that (4) and (5) are both linear constraints (given L) and thus can be easily incorporated into the estimation of (3).[4] In the following discussion, these two constraints will be referred to as "zero head" and "zero tail" constraints.

Recent work has also suggested the desirability of constraining the slope of the polynomial to be zero at $k = L$.[5] This can also be easily accomplished. Differentiating equation (2) with respect to k and evaluating the result at $k = L$ yields:

$$(6) \qquad \frac{\partial w_k}{\partial k} = \sum_{j=1}^{P} \alpha_j L^{j-1} j = 0 \quad .$$

Again, (6) is a linear constraint on the α_j coefficients and it can be directly incorporated in the estimation of (3).

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extends over the range (0,L), then it might be reasonable in specific applications to assume that the polynomial is zero one period outside this range. This approach has been questioned by Robinson [8], who argues that no theoretical justification has been offered for constraining the polynomial to be zero outside of the interval over which it is estimated.

[4]The imposition of linear constraints is an example of another case in which all the polynomial specifications described in footnote 2 must yield the same results, although some specifications may be more convenient than others. In particular, imposing zero restrictions of the type described here is trivial when the Lagrangian interpolation polynomials are used.

[5]This constraint is referred to in Cooper [4] and Sparks [9].

Other Restrictions on the Polynomial

End point restrictions do not, of course, exhaust the kinds of restrictions that can be placed on the shapes of polynomials; other linear constraints on the α_j can be imposed. Each constraint reduces the number of estimated coefficients by one and places some constraint on the shape of the polynomial. It will be seen below, for example, that additional constraints are necessary in order to insure that the fifth degree polynomial has a desirable shape. Also, the shape of the fourth degree polynomial can be easily constrained without having to consider the use of end point restrictions. In the following analysis these additional constraints will take the form of setting certain coefficients of the polynomial equal to zero, but similar conclusions would hold if more general linear constraints on the coefficients were considered.

III. The Shapes of Polynomial Distributed Lags

First and Second Degree Polynomials

The estimation of restricted first and second degree polynomials is only of limited value and requires no extended discussion here. For the first degree polynomial, for example, zero restrictions can be placed at either the head or the tail of the distribution, in which case the single estimated coefficient determines the slope of the distribution. For the second degree polynomial, if either zero head or tail restrictions are imposed, then the two estimated coefficients determine the height and horizontal position of the peak of the lag distribution. Furthermore, if both zero head and tail restrictions are imposed, then because of the symmetry of the second degree polynomial, the peak or trough of the distribution occurs halfway between the head and tail. The single estimated coefficient in this case determines the height of the distribution.

Third Degree Polynomials

The general third degree polynomial is illustrated in Figure 1a. For this polynomial, end point restrictions are sufficient to insure that the polynomial does not have an undesirable shape within the estimated range of the distribution. This can be done by constraining the polynomial and its slope to be zero at the tail of the distribution, as is illustrated in Figure 1b. The resulting polynomial is then a function of two coefficients, where the two coefficients determine the height and the horizontal position of the peak of the distribution. If the head of the distribution is also constrained to be zero, then it can be shown that the peak of the distribution must occur one-third of the way between the head and the tail.[6] The one estimated coefficient in this case merely determines the height of the distribution.

Fourth Degree Polynomials

The general fourth degree polynomial is written as

$$(7) \qquad w_k = \alpha_0 + \alpha_1 k + \alpha_2 k^2 + \alpha_3 k^3 + \alpha_4 k^4$$

and is illustrated in Figure 2a. There are two basic ways to insure that the fourth degree polynomial does not have an undesirable shape. One way

[6] This can be seen by imposing the three restrictions on the polynomial and then solving for the zero derivative points of the polynomial. These points turn out to be independent of the coefficient estimate and occur at 1/3L and L. It should also be noted that imposing the above restrictions on the third degree polynomial does not rule out the possibility of the inverted version of Figure 1b occurring.

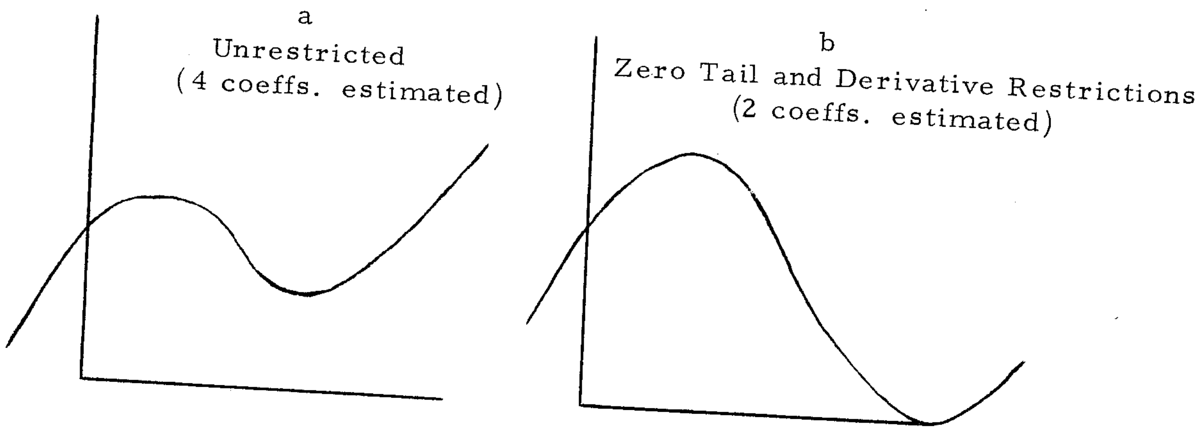

Figure 1 -- THIRD DEGREE POLYNOMIALS

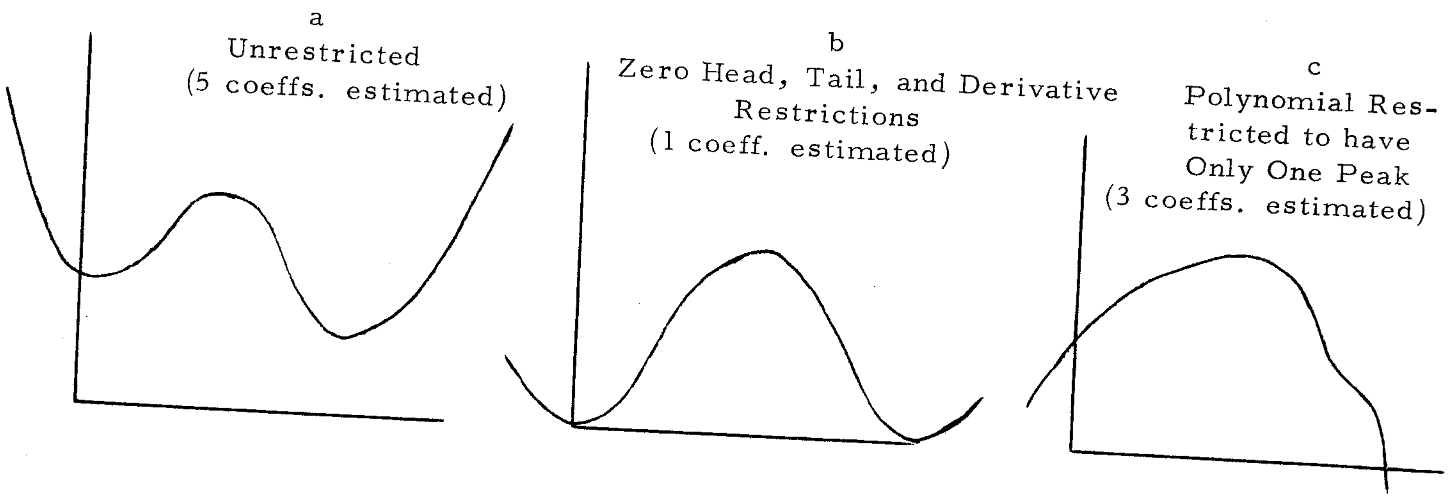

Figure 2 -- FOURTH DEGREE POLYNOMIALS

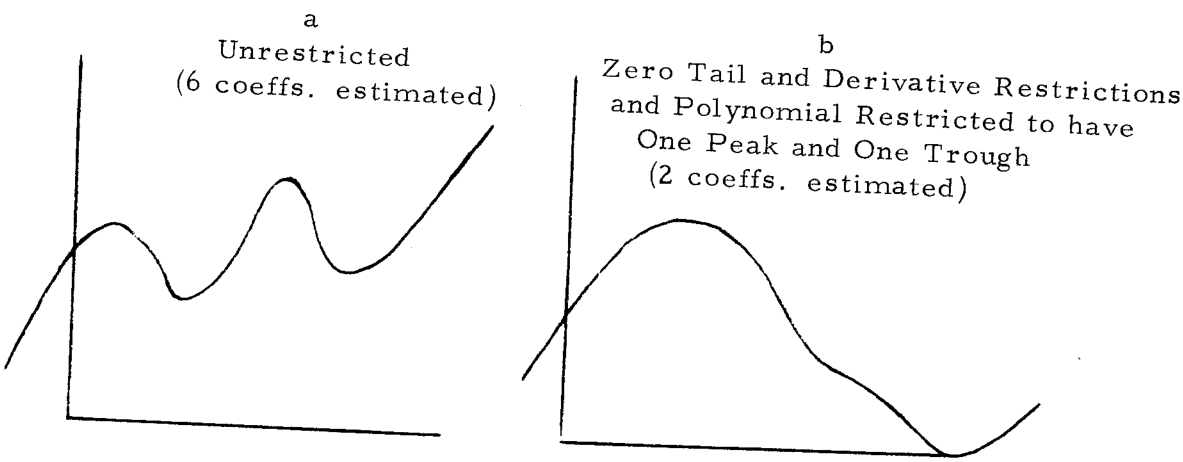

Figure 3 -- FIFTH DEGREE POLYNOMIALS

is to constrain the polynomial and its slope to be zero at both the head and the tail of the distribution, as is illustrated in Figure 2b. In this case the estimated polynomial is a function of only one coefficient, and it can be shown that the peak of the distribution occurs one-half of the way between the head and the tail.[7] The one coefficient estimate determines the height of the distribution.

The fourth degree polynomial can also be constrained to have the general shape illustrated in Figure 2c. To do so, however, requires that restrictions other than end point restrictions be considered. The polynomial in Figure 2c has only one peak, and a necessary and sufficient condition for the polynomial in (7) to have only one peak or trough is[8]

$$(8) \qquad \frac{b^2}{4} + \frac{a^3}{27} > 0 \text{ , where } \quad a = \frac{1}{2}\frac{\alpha_2}{\alpha_4} - \frac{3}{16}\left(\frac{\alpha_3}{\alpha_4}\right)^2$$

$$b = \frac{2}{27}\left(\frac{3\alpha_3}{4\alpha_4}\right)^3 - \frac{1}{3}\left(\frac{3\alpha_3}{4\alpha_4}\right)\left(\frac{2\alpha_2}{4\alpha_4}\right) + \frac{\alpha_1}{4\alpha_4}$$

An easy way to insure that (8) is satisfied (although obviously not the only way) is to set α_2 and α_3 equal to zero. The formula then reduces to $\frac{1}{4}\left(\frac{\alpha_1}{4\alpha_4}\right)^2 > 0$, which is satisfied for all nonzero values of α_1 and α_4.

[7] See footnote 6. The zero derivative points in this case occur at 0, $\frac{1}{2}$ L, and L.

[8] A necessary and sufficient condition for a fourth degree polynomial to have only one peak or trough is for the derivative of the polynomial (which is itself a third degree polynomial) to have only one real root. A necessary and sufficient condition for a third degree polynomial to have only one real root is well known and is given, for example, in Hodgman [6], p. 282. This condition is the condition in (8).

Setting α_2 and α_3 equal to zero leaves three coefficients to be estimated, and the resulting polynomial has either the general shape in Figure 2c or the inverted version of it. If the polynomial in Figure 2c is further restricted to be zero at the head and the tail of the distribution, then the polynomial is a function of only one coefficient. In this case the peak of the distribution occurs at $\frac{1}{\sqrt[3]{4}} L$, or approximately two-thirds of the way between the head and the tail.

Fifth Degree Polynomials

The general fifth degree polynomial is written as

(9) $$w_k = \alpha_0 + \alpha_1 k + \alpha_2 k^2 + \alpha_3 k^3 + \alpha_4 k^4 + \alpha_5 k^5$$

and is illustrated in Figure 3a. In order to insure that the fifth degree polynomial does not have an undesirable shape, it is necessary to consider both end point restrictions and other kinds of restrictions. The most straightforward way to insure a desirable shape is to constrain the polynomial to have only one peak and one trough and to constrain the polynomial and its slope to be zero at the tail of the distribution. The polynomial then has either the general shape illustrated in Figure 3b or the inverted version of it. It can be seen that setting the coefficients α_3 and α_4 equal to zero insures that the polynomial has only peak and one trough.[9]

[9] An easy way to see this is the following. The derivative of (9) is a fourth degree polynomial, and in order for (9) to have only one peak and one trough, its derivative must have only two real roots. With α_3 and α_4 set equal to zero, the derivative is: $\alpha_1 + 2\alpha_2 k + 5\alpha_5 k^4$. Setting this derivative equal to zero and solving for k^4 yields: $k^4 = -\frac{\alpha_1}{5\alpha_5} - \frac{2\alpha_2}{5\alpha_5} k$. Now, k^4, when graphed against k, is simply a u-shaped curve symmetric about the origin, and $-\frac{\alpha_1}{5\alpha_5} - \frac{2\alpha_2}{5\alpha_5} k$, when graphed against k, is a straight line. It is apparent that these two curves can intersect at most twice, and so the derivative can have at most two real roots.

The resulting polynomial is then a function of only two coefficients. If the head of the distribution is also constrained to be zero, then the polynomial is a function of only one coefficient. The coefficient determines the height of the distribution, with the peak occurring at approximately .38L.[10]

Conclusion

The restrictions discussed in this section do not, of course, exhaust the total number of restrictions that can be imposed on polynomials. An attempt has been made, however, to discuss the more important kinds. One of the conclusions of this exercise is that polynomials are not as general as one might hope for in approximating distributed lags. The fourth degree polynomial, for example, appears to be of limited usefulness, since the shape in Figure 2b can be approximated, at least roughly, by a second degree polynomial and since the shape in Figure 2c is not likely to be consistent in most cases with *a priori* views about the shape of the lag distribution. Likewise, polynomials of degree six and higher are likely to be of limited usefulness, since once they are restricted in the appropriate manner, their shapes are not likely to differ much from the shapes in Figures 1, 2, and 3.

For many purposes the restricted third degree polynomial in Figure 1b may be sufficient for approximating lag distributions. Another leading candidate is the restricted fifth degree polynomial in Figure 3b, but the principle of Occam's razor would dictate using the restricted third degree

[10] Imposing the five restrictions and setting the derivative of (9) equal to zero yields: $3L^4 - 8L^3k + 5k^4 = 0$. The two real solutions to this equation are k equal to L and k approximately equal to .38L.

polynomial unless one had sufficient *a priori* information for doing otherwise. (Both polynomials are a function of two coefficients, and so no added flexibility is gained on this score by using the fifth degree polynomial.) In general, if a second or restricted third degree polynomial is not considered to be a good enough approximation to the lag distribution, then, as mentioned above, the Almon technique should probably not be used.

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