RANDOM PARAMETERS IN A SIMULTANEOUS EQUATION FRAMEWORK: IDENTIFICATION AND ESTIMATION

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1. Introduction

In recent years a number of articles have been written describing the analytical problems involved in random parameter models.[1] Typically, however, these studies have been concerned with problems of estimation in the context of a single equation model.[2] Swamy (9), Zellner (11), and others have considered the random parameter approach in the context of a system of equations but these equations were of the reduced form variety. It is evident, though, that econometric models typically involve systems of simultaneous equations but yet general results concerning identification and estimation of such systems containing random parameters are virtually nonexistent. The purpose of this paper is to provide some results on such a system. In particular, conditions for identification are given and, when appropriate,

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[1]See, for instance, the references listed in Swamy (9). See also, the classical article by Theil and Mannes (10).

[2]For one such elaborate study see Hildreth and Houck (4). For an exception see Nerlove (8, pp. 34-35, 61-82) who considers random parameter problems in the context of production function models.

a consistent estimation procedure is outlined.[3] In addition, a reducibility condition is given under which the conditions for identification of a simultaneous system of equations containing random parameters are identical to what they would be if the parameters of the system were not random. That is, under this condition, the added complication of random parameters in no way complicates the conditions for identification. It also turns out that under this condition the identified equations of this system can be consistently estimated in the traditional two-stage least squares framework.

2. The Model

Consider the system of M simultaneous equations containing random parameters

(1a) $$Y_t = Y_t\Gamma_t + X_tB_t + U_t, \qquad t=1,\ldots N,$$

(1b) $$\Gamma_t = \Gamma + \Omega_t$$

(1c) $$B_t = B + H_t$$

where Y_t, X_t, and U_t are, respectively, 1xM, 1xG, and 1xM vectors at time t of endogenous variables, predetermined variables, and disturbance terms; Γ_t and B_t are the corresponding MxM and GxM matrices of parameters at time t.

[3]If the parameters are random, identification and estimation refer to the means of the parameters.

We assume that some or all of the non-zero elements Γ_t and B_t are random as described in (1b) and (1c) where Γ and B are constant matrices of orders MxM and GxM, and Ω_t and H_t are random matrices of corresponding orders where $E\,\Omega_t = 0$ and $E\,H_t = 0$. We assume that there are "zero-type" restrictions on the system in (1a) in the sense that certain elements of Γ_t and B_t are known _a priori_ to be zero and hence _not random_. Therefore, the zeroes Γ and B correspond to those of Γ_t and B_t. Ω_t and H_t contain all the zeroes of Γ_t and B_t and, if some parameters are not random, additional zeroes. Imposing a normalization rule, we take the diagonal elements of Γ_t to be zero.

Our stochastic specifications concerning the disturbance vector are that $E[U_t|X_t] = 0$, and $E[U_t'U_{t-s}|X_t] = \delta(s)V_u$ where V_u is the contemporaneous variance-covariance matrix and $\delta(s) = 0$ for $s \neq 0$, and $\delta(0) = 1$. Our assumptions concerning Ω_t and H_t are that they are independent of each other, of X_t, and of U_t. Further, each element of Ω_t and of H_t is assumed to be independent of all the other elements. Finally, we assume that with probability equal to one $(I-\Gamma_t)^{-1}$ exists and

$$(2) \qquad (I-\Gamma_t)^{-1} = \Lambda + \theta_t ,$$

where $E\,\theta_t = 0$ so that the reduced form for Y_t exist. Our primary problem concerns the identification of the parameter matrices Γ and B.

3. The Reducibility Condition

Given these assumptions consider the reduced form for Y_t

$$(3) \qquad Y_t = X_t B_t (I-\Gamma_t)^{-1} + U_t (I-\Gamma_t)^{-1} .$$

Substituting (1c) and (2) into (3) we have

$$(4) \qquad Y_t = X_t \pi + [X_t(B\theta_t + H_t\Lambda + H_t\theta_t) + U_t(I-\Gamma_t)^{-1}]$$

$$= X_t \pi + W_t , \qquad t=1,\ldots,N ,$$

where $\pi = B\Lambda$, and W_t is the term in brackets in (4) and so $E[W_t|X_t] = 0$. Thus, under the usual further assumptions, the least squares estimate of π , say $\hat{\pi}$, defined by (4) is consistent.[4] Therefore, if a condition could be found such that $\Lambda = E(I-\Gamma_t)^{-1} = (I-\Gamma)^{-1}$, the indirect least squares equation $\pi\Lambda^{-1} = B$ could be set up; hence the identification problem concerning Γ and B could be reduced to the classical non-random parameter case. Notice that this conclusion is independent of whether or not B_t is random.[5]

The condition implying that $E(I-\Gamma_t)^{-1} = (I-\Gamma)^{-1}$ is fairly straight forward. Specifically, because each element of Ω_t has a zero mean and is independent of all of the other elements, we can say that $E[I-\Gamma_t]^{-1} = E[I-\Gamma-\Omega_t]^{-1} = (I-\Gamma)^{-1}$ if $(I-\Gamma-\Omega_t)^{-1}$ does not contain a nonlinear form of any element of

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[4] See Dhrymes (1, pp. 176-180) and Goldberger (3, pp. 299-302).

[5] As is evident from (4), however, the randomness of B_t affects the efficiency of our estimates because of its effect on the heteroskedasticity of W_t .

Ω_t . For instance, if $(I-\Gamma-\Omega_t)^{-1}$ does not involve any of the elements of Ω_t in a nonlinear form, then

$$(5) \qquad (I-\Gamma-\Omega_t)^{-1} = (I-\Gamma)^{-1} + L(\Omega_t)$$

where $L(\Omega_t)$ is the MxM matrix whose elements are linear in <u>each</u> of the elements of Ω_t;[6] hence $\Lambda = (I-\Gamma)^{-1}$ since $E\,L(\Omega_t) = 0$.

We now note that each element of $(I-\Gamma-\Omega_t)^{-1}$ is a ratio of a co-factor divided by a determinant. Because the co-factors satisfy this linearity condition, our condition for $\Lambda = (I-\Gamma)^{-1}$ is that the determinant of $(I-\Gamma-\Omega_t)$ must not involve any of the elements of Ω_t . This condition can be stated in terms of the structural parameters as

$$(6) \qquad \det(I-\Gamma_t) = f(\Gamma_*)$$

where Γ_* represents the non-random elements of Γ_t . It should be noted that if all of the elements of Γ_t are random, (6) can only hold if $(I-\Gamma_t)$ is triangular so that $f(\Gamma_*) = 1$.[7]

To summarize, we have shown that if the "reducibility" condition (6) holds, the conditions for identification of Γ

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[6] For example, the i,j^{th} term of $L(\Omega_t)$ may involve the product of elements of Ω_t but it may not involve the, say, square or reciprocal of any element of Ω_t .

[7] If Γ_* does contain some elements, trangular systems are not the only systems satisfying (5). For example, the reader may work through the case of a three equation system in which only one element of Γ_t is random.

and B are exactly as they would be in a non-random parameter context. We will show below that if a weaker version of (6) holds, an identified equation of the system may be consistently estimated by a direct application of two stage least squares.

4. The General Case

We now consider the question of identification when $\Lambda \neq (I-\Gamma)^{-1}$. In this case it is clear that the indirect least squares approach can not be taken without modifications. That is, Λ would still be a (different) function of the elements of Γ but it would also involve, in general, parameters whose values depend upon the higher moments of the distributions of the elements of Ω_t. Hence, although Γ and B, or at least some of their elements, may still be identified, general rules concerning their unique solution (identification) in terms of the indirect least squares equation, $\pi = B\Lambda'$, would be cumbersome.

We consider, instead, a somewhat more direct approach. Consider the first equation of (1),

$$(7) \qquad y_{1t} = Y_t\Gamma_1 + X_tB_1 + Y_t\Omega_{1t} + X_tH_{1t} + u_{1t} ,$$

where y_{1t} and u_{1t} are the first elements of Y_t and U_t, and $\Gamma_1, B_1, \Omega_{1t}$ and H_{1t} are the first columns of Γ, B, Ω_t, and H_t. Our technique will now be to replace Y_t and $Y_t\Omega_{1t}$

in (7) by their "reduced form" equations in X_t , and then inquire as to whether or not the regressors in the resulting equation are linearly independent.

For instance, substituting (4) into (7), we have

$$(8) \qquad y_{1t} = (X_t\pi)\Gamma_1 + X_tB_1 + Y_t\Omega_{1t} + q_{1t} ,$$

where $q_{1t} = X_tH_{1t} + u_{1t} + W_t\Gamma_1$, and so $E[q_{1t}|X_t] = 0$. We now note that some elements of Γ_1 , B_1 and Ω_{1t} are known, <u>a priori</u>, to be zero. Imposing these restrictions on Γ_1 and B_1 , (8) can be rewritten as

$$(9) \qquad y_{1t} = (X_t\pi_1)\Gamma_{1_*} + X_{1t}B_{1_*} + Y_t\Omega_{1t} + q_{1t} ,$$

where Γ_{1_*} and B_{1_*} are the M_1x1 and G_1x1 subvectors of nonzero elements of Γ_1 and B_1 , and π_1 and X_{1t} are the corresponding GxM_1 and $1xG_1$ sub-matrix and vector of π and X_t . If now, for example, $\Omega_{1t} = 0$ or $E[Y_t\Omega_{1t}|X_t] = 0$, the condition for identification of Γ_{1*} and B_{1_*} would be given by the classical condition that the elements of $(X_t\pi_1)$ and X_{1t} be linearly independent.[8] The order condition for this is, obviously, that this first equation exclude at least M_1 predetermined variables.

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[8]For example, under these conditions Γ_{1_*} and B_{1_*} could be consistently estimated via two stage least squares. The regressor matrix would be $X_t\hat{\pi}_1$ and X_{1t} where $\hat{\pi}_1$ is the ordinary least squares estimate of π_1 obtained from (4). It can be shown that $X_t\pi_1$ and X_{1t} are linearly independent if the rank of the sub-matrix of π_1 corresponding to those elements of X_t which are <u>not</u> included in X_{1t} is equal to M_1 - see Fisher (2, pp. 52-56). Notice, that a formal consideration of this rank condition involves specifying the distributions of the elements of Ω_t so that $\pi = B\Lambda$ can be evaluated.

We will now give conditions under which $E[Y_t\Omega_{1t}|X_t] = 0$. We will also give conditions for identification when this term is not zero. As a preview, it turns out that if this term is not zero, certain sets of regressors from other equations must be added to our first equation, (9), in order to account for $Y_t\Omega_{1t}$.

In order to simplify the presentation, we assume that Ω_{1t} has only one non-zero term say the second, Ω_{12t}. As will become evident the results can easily be generalized.

If the second element of Ω_{1t}, say Ω_{12t}, is the only nonzero element, then

(10) $$E[Y_t\Omega_{1t}|X_t] = E[y_{2t}\Omega_{12t}|X_t] \quad ,$$

where y_{2t} is the second element of Y_t. Using the reduced form equation for y_{2t}, given by (4), and recalling that H_t and U_t are independent of Ω_t, we have

(11) $$E[y_{2t}\Omega_{12t}|X_t] = X_t\, B\, E\,[\theta_{2t}\Omega_{12t}]$$

where θ_{2t} is the second column of θ_t. Let $(I-\Gamma_t)_2^{-1}$ be the second column of $(I-\Gamma_t)^{-1}$. Then, recalling that $(I-\Gamma_t)^{-1} = \Lambda + \theta_t$, we have

(12) $$X_t B\, E[\theta_{2t}\Omega_{12t}] = X_t\, B\, E[(I-\Gamma_t)_2^{-1}\ \Omega_{12t}]$$

$$= X_t\, B\, E[(I-\Gamma-\Omega_t)_2^{-1}\ \Omega_{12t}] \quad .$$

It is clear from (11) and (12) that $E[y_{2t}\Omega_{12t}|X_t] = 0$ if the <u>second column</u> of $(I-\Gamma-\Omega_t)^{-1}$ does <u>not</u> involve Ω_{12t}.

Now the elements in this second column are simply the co-factors of the elements in the second row of $(I-\Gamma-\Omega_t)$ divided by the determinant of $(I-\Gamma-\Omega_t)$. Since these co-factors can not contain Ω_{12t} because this element is in the second row, the condition we seek is that the determinant of $(I-\Gamma-\Omega_t)$ must not be a function of Ω_{12t} . In brief, if this condition holds, $E[y_{2t}\Omega_{12t}|X_t] = 0$, and so the conditions for identification of our equation are the standard ones.

Consider, now the case in which $\det(I-\Gamma-\Omega_t)$ is a function of Ω_{12t} . Under this assumption every term in the second column of $(I-\Gamma-\Omega_t)^{-1}$ that is not zero,[9] will have, in general, a non-zero covariance with Ω_{12t} . Thus from (11) and (12) we have

$$(13) \qquad E[y_{2t}\Omega_{12t}|X_t] = (X_{1t}B_{1_*} \; X_{2t} \; B_{2_*} \cdots X_{Mt}B_{M_*}) \; K$$

where X_{1t} is the vector of predetermined variables appearing in the i^{th} equation, B_{i_*} is the vector of corresponding coefficients, and K is the $M\times 1$ vector $K = E(I-\Gamma_t)^{-1}\Omega_{12t}$. We see from (13) that if the i^{th} element of K , K_i, is not zero, $X_{it}B_{i*}K_i$ must be introduced into the first equation of our system in order to help account for $y_{2t}\Omega_{12t}$. Assume for the moment that K_i is the <u>only</u> non-zero element of K .

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[9] The i^{th} element in the second column of $(I-\Gamma-\Omega_t)^{-1}$ will be zero if the co-factor of the i^{th} element in the second row is zero - see Hohn (5, Chapter 3). Since $(I-\Gamma-\Omega_t)$ is assumed, with probability equal to one to have an inverse, all of the elements of the second column of its inverse can not be zero.

Then equation (9) could be rewritten as

$$(14) \qquad y_{1t} = (X_t\pi_1)\Gamma_{1*} + X_{1t}\, B_{1*} + X_{it}\,(B_{i*}\, K_i) + r_t$$

where $E[r_t|X_t] = 0$. From our previous results it follows that (14) is identified if the elements of $(X_t\pi_1)$, X_{1t}, and X_{it} are linearly independent. It is clear that if (14) is identified, it could be consistently estimated by two stage least squares where π_1 is estimated by ordinary least squares in the first stage via the reduced form equation (4). In brief, the essential problem is that if $K_i \neq 0$, the list of predetermined variables in our first equation is made larger.

Generalizations are now straight forward. Assume that K has p nonzero elements. That is, assume that p co-factors of the elements in the second row of $(I-\Gamma-\Omega_t)$ are not zero. Then, from (13) we see that p sets of predetermined variables must be used to account for $y_{2t}\Omega_{12t}$ in our first equation as described in (9). Let Z_{1t} be the union of these sets of regressors. Then, Γ_{1*} and B_{1*} are identified if the elements of $(X_t\pi_1)$, X_{1t} , and Z_{1t} are linearly independent.[10]

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[10]Special cases, of course, can be worked out. For instance, assume that each element of $(X_t\pi_1)$ is linearly independent of X_{1t} and Z_{1t} but X_{1t} and Z_{1t} have some elements in common. Then, Γ_{1*} and all of the elements of B_{1*} except those corresponding to these variables in common are identified. Since such examples are obvious they need not be multiplied.

We are now in a position to formalize and generalize the above in terms of simple rules for identification. Without loss of generality we still focus our attention on the first equation.

Assume that y_{it} appears in the first equation as one of the independent variables and assume that its coefficient is random. Let the i^{th} element of Ω_{1t} be Ω_{1it}. Then, if $\det(I-\Gamma-\Omega_t)$ does <u>not</u> contain Ω_{1it}, the random parameter problem as it relates to the coefficient of y_{it} in the first equation can be <u>ignored</u>. Assume now that $\det(I-\Gamma-\Omega_t)$ involves Ω_{1it}. Let c_{ij} be the co-factor of the i,j^{th} element of $(I-\Gamma-\Omega_t)$. Then, the set of predetermined variables in the j^{th} equation, X_{jt}, will have to be added to the existing set of predetermined variables in the first equation if $c_{ij} \neq 0$, for any j. These predetermined variables must be added in order to account for $y_{it}\Omega_{1it}$ - see (9). In a similar manner, if y_{gt} also appears in our first equation with a random coefficient, and if $\det(I-\Gamma-\Omega_t)$ involves Ω_{1gt}, then the predetermined variables in the v^{th} equation must be added to those of the first if $c_{gv} \neq 0$, etc.. Then, as before, if Z_{1t} is the union of these added predetermined variables, our first equation is identified (and may be estimated by two stage least squares) if the elements of $(X_t\pi_1)$, X_{1t}, and Z_{1t} are linearly independent.

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