THE AXIOMATIC CONCEPT OF A SCALE AND THE LOGIC OF PREFERENCES IN THE THEORY OF CONSUMERS EXPENDITURES

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PREFACE

Description of behaviour by a utility function has been the usual first principle of consumer theory. Houthakker[1] has considered conditions for a behaviour to admit such a description. He replaces an hypothesis of Samuelson[2] by a stronger one, also obviously implied by a description in terms of utility, and which — so it is hoped — is to imply such a description.

Here an expenditure system is the concept for representing a behaviour; and the coherence condition, which is the main object of investigation, is the same as Houthakker's hypothesis.

The object is to give the concepts and the main lines of argument for theory of the coherence of expenditure systems, so as to mark out a subject within which there can be a definite resolution of some long-lived dilemmas about the consumer. Especially for Houthakker's problem, and for the much discussed matter of integrability, there seems to be a perfect clarification, though the terms go somewhat beyond those in which the matter has usually been envisaged. Also some theorems are formulated which are of a type entirely new in the subject, such as, for example, those relating to local and global coherence, and total incoherence.

Certain details of the proofs have been stated elsewhere[3]; and others are to be given in due course. These are omitted here, not just for brevity, but so as to show more plainly the essential ideas.

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1. SCALES: A binary relation R between the elements of a set C has <u>negation</u> $\bar{R}$, <u>converse</u> R', <u>symmetric part</u> $\mathring{R} = R \wedge R'$ and <u>symmetric negation</u> $\tilde{R} = \bar{R} \wedge \bar{R}'$. Also it has a <u>chain extension</u> $\vec{R}$, which is the relation determined between the extremities of an R-<u>chain</u>, that is, a chain in which each element has the relation R to its successor. An R-chain with coincident extremities defines an R-<u>cycle</u>. The symmetric part of the chain extension of R is the relation that elements have by their lying on the same R-cycle, by which condition they may be said to be R-<u>encycled</u>.

Let I, D denote the relations of identification and distinction, that is of equality and inequality between the elements. An <u>equivalence</u> is defined by the conditions of reflexivity $(I \Longrightarrow R)$, symmetry $(R \Longrightarrow R')$ and transitivity $(\vec{R} \Longrightarrow R)$; an <u>order</u> by irreflexivity $(R \Longrightarrow D)$ and transitivity; and a <u>scale</u> by antisymmetry $(R \Longrightarrow \bar{R}')$ and negative transitivity (transitivity of the negation). A complete order is an order which is complete $(\bar{R}' \Longrightarrow R)$.

THEOREM. If S is a scale, then it is also an order, and $\tilde{S}$ is an equivalence; and the scale S, applied to C, is represented by a complete order $\mathcal{S}$, applied to the classes Σ of $\tilde{S}$ which partition C:

$$xSy \Longleftrightarrow \sigma_x \mathcal{S} \sigma_y ,$$

where $\sigma_x \in \Sigma$ is the class of $\tilde{S}$ with representative $x \in C$.

2. CHOICES AND PREFERENCES: An element x in a set R determines a choice $[x; R]$, with x as <u>object</u> and R as <u>range</u>. It is equivalent to the set $\{x, R - x\}$ of preferences (x, y), of the

selected element x to every rejected element $y \in R$, $y \neq x$. With a set of choices, there is first formed the set of base preferences Q, obtained from the choices taken separately, and then the derived preferences $P = \vec{Q}$, which span the chains formed by the base preferences. By its construction, a derived system of preferences is transitive; so it is an order if and only if it is irreflexive. Choices are defined to be coherent if their derived preferences form an order.

3. MARKET PURCHASE: A purchase is specified by its commodity composition, given by a vector x of quantities obtained, together with the prices of the commodities, given by another vector p, thus

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}.$$

Now the expenditure in the purchase (x, p), of composition x at prices p, is

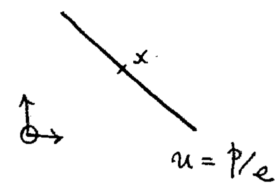
$$e = p_1 x_1 + \dots + p_n x_n = p'x \quad .$$

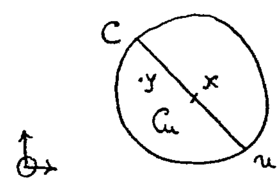
4. BALANCE AND COMPOSITION: The relative prices in a purchase (x, p) are the commodity prices with the expenditure $e = p'x$ as unit of money; they are given by a vector $u = p/e$ which defines the expenditure balance in the purchase. Any composition y is defined to be within, on or over a balance u according as $u'y \lesseqgtr 1$. Now $u'x = 1$ for any purchase, by definition of u; that is, the composition in any purchase is always on the balance in that purchase. In any purchase (x, p), the compositions y in any region C which, at the same prices, could have been obtained instead of x at for no greater expenditure, that is such that $p'y \leq p'x$, and equivalently $u'y \leq 1$, form the set

$$C_u = \{y;\ u'y \leq 1,\ y \in C\}$$

of compositions $y \in C$ within the balance u.

5. EXPENDITURE ALLOCATION AND CHOICE: From a purchase (x, p) involving an absolute money expenditure $e = p'x$, allocated to the different commodities, in different partial expenditures $p_1x_1, \ldots, p_nx_n$, there is derived the representation (x, u) with $u'x = 1$, where $u = p/e$. The fractions $u_1x_1, \ldots, u_nx_n$ give the proportions in which the total expenditure is distributed. Such a derived representation, which characterizes just the distribution of a total, rather than determining that total absolutely, together with its partition among the commodities, will be denoted by $[x; u]$, and called an expenditure allocation. It is equivalent to the representation of the purchase as the choice of $[x; C_u]$ of x from among all the compositions in C_u, that is in C and within the balance u. Again, the choice is equivalent to the set $\{x, C_u - x\}$ of preferences (x, y) of x to every other composition $y \in C_u - x$ in C_u. Thus, in the purchase, the composition x obtained, which is supposed restricted to a region C, is considered chosen from and preferred to every other in the set of compositions which could have been obtained for no greater expenditure at the same prices, that is the set C_u.

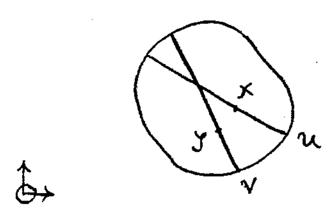




6. EXPENDITURE SYSTEMS AND PREFERENCE RELATIONS: An expenditure system E is a mapping

$$E : B \longrightarrow C \ (u \longrightarrow x;\ u'x = 1)$$

of a region B of balances u into a region C of compositions x subject to the balance condition $u'x = 1$. It is equivalent to the set of choices $[x; C_u]$ $(u \in B)$, with $x = Eu$, that is with x determined on each balance $u \in B$ by E. These choices have base preferences $Q = \bigvee_{u \in B}\{x, C_u - x\}$; and the derived preferences $P = \overrightarrow{Q}$ then define the



the preference relation of the system. The system is called coherent if its choices are coherent, which condition is that P be an order.

7. RESPONSIVITY: An expenditure system E, obtaining compositions $x, y \in C$ on any balances $u, v \in B$, is invertible if $u \neq v \Longrightarrow x \neq y$. Let

$$\lambda_u = \inf_{v \in B} |x - y|/|u - v|, \qquad \lambda_B = \inf_{u \in B} \lambda_u \quad .$$

Then E is called responsive at a point $u \in B$ if $\lambda_u > 0$, and in the region B if $\lambda_B > 0$. Responsivity in a closed region is equivalent to responsivity at each of its points. A system can be responsive only if it is invertible, and the inverse system is continuous. Thus, with a responsive system, there is always a movement of composition in response to a movement of balance, the movement of the one being through a distance which is at least a fixed positive multiple of that of the other.

8. DIFFERENTIABILITY: An expenditure system is differentiable if

$$y - x = (x_u + \rho(u, v))(v - u)$$

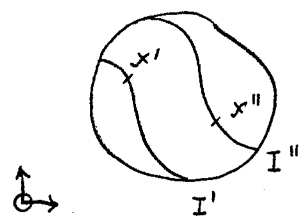
where $\rho(u, v) \longrightarrow 0 \ (v \longrightarrow u)$. Then the partial derivatives $\partial x_i/\partial u_j$ of the elements of x with respect to the elements of u all exist, and form the elements of the matrix x_u. The system is invertible only if the matrix x_u is invertible; and the inverse matrix u_x is the partial derivative matrix of the inverse system, which must then be differentiable also:

$$x_u u_x = u_x x_u = 1 \quad .$$

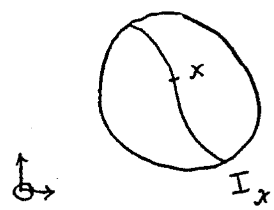
THEOREM. invertible $\wedge$ continuous . $\Longleftarrow$. responsive . $\Longleftarrow$. invertible $\wedge$ differentiable.

9. GEOMETRIC AND ANALYTIC INTEGRABILITY: If an expenditure system E is invertible, so u is obtained as a function

of x, there can be formed the differential form $u'dx$ and the differential equation $u'dx = 0$ of the system. A surface I of compositions which has at each of its points a unique tangent given by the locus of compositions on the corresponding balance u defines an integral surface of the differential equation, or of the system. The existence of at least one integral surface through any point defines the condition of geometric integrability. The condition of simple geometric integrability is when there exists just one surface through any point. Under this condition, the set $\mathcal{I}$ of integral surfaces of the system gives a partition of C, with every composition $x \in C$ belonging to just one surface $I_x \in \mathcal{I}$. Each surface $I \in \mathcal{I}$ disconnects C into two domains, called its sides, distinguished as over and under I. Also the surfaces form a continuum, completely ordered by a relation $\mathcal{G}$, with the property that $I' \mathcal{G} I''$ if and only if any element x' of I' is over I'', and, equivalently, any element x'' of I'' is under I'. This complete order $\mathcal{G}$ of the surfaces which partition C determines a scale G on the elements of C, with the definition



$$xGy \equiv I_x \mathcal{G} I_y \quad ,$$



which may be called the geometric scale, implicit in the condition of simple geometric integrability.

The differential form $u'dx$ is said to be integrable if it is proportional, by a factor λ called an integrating factor, to the total differential $d\phi$ of a differentiable function ϕ, called an integral; and then the system E is said to satisfy the condition of analytic integrability. There can be at most one functionally independent integral; from which it follows that the scale A determined from any integral ϕ with the definition

$$xAy \equiv \phi(x) > \phi(y)$$

is independent of ϕ: it will be called the analytic scale on the elements, implicit in the condition of analytic integrability.

THEOREM: (i) analytic integrability $. \Longrightarrow .$ simple geometric integrability $\wedge$ A = G

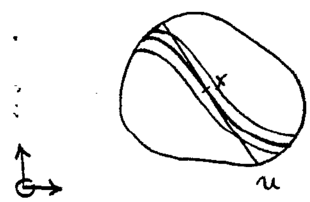
(ii) for a responsive system there can be at most one integral surface through any point.

(iii) responsivity $. \Longrightarrow .$ geometric integrability $\Longrightarrow$ analytic integrability.

Moreover, with geometric integrability implied by analytic integrability, the integral surfaces of the equation are the level surfaces of any integral of the form, these being the surfaces on which the integrals are constant.

Geometric integrability is automatic in two dimensions, but not in more than two.

10. EQUILIBRIUM: If ϕ_x is the vector of partial derivatives of $\phi = \phi(x)$ with respect to the elements of x, then the condition that it be an integral of the expenditure system is

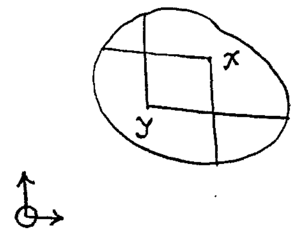


$$u\lambda = \phi_x,$$

where $\lambda = x'\phi_x$ since $u'x = 1$. This is also the condition that ϕ be stationary at x, or the system be in equilibrium under the constraint $u'x = 1$, with λ now appearing as the Lagrangian multiplier belonging to the constraint. Analytic integrability is thus equivalent to the system having the structure determined by equilibrium relative to some differentiable function ϕ.

11. MONOTONICITY: With the elements of u all positive, the elements of ϕ_x are here all positive or all negative, in which case ϕ is monotone, increasing or decreasing. An adjustment of the

sign of $\o$ will now leave it increasing:



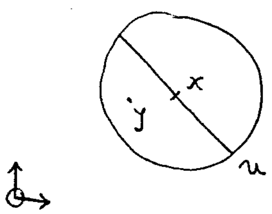
x) y

$$x \,)\, y \Longrightarrow \o(x) > \o(y) \quad ,$$

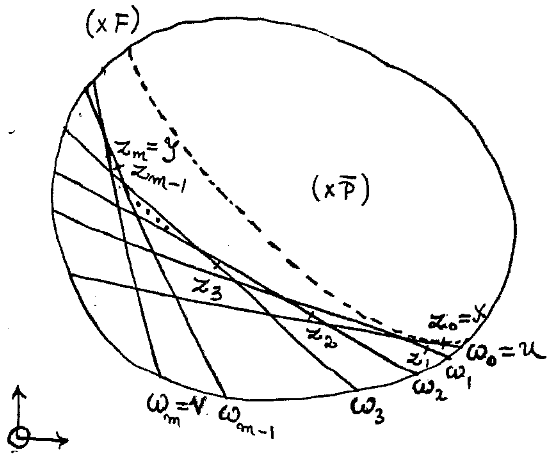
where $x \,)\, y$ means the composition x contains the composition y, in that each of its quantities is at least the corresponding one in y, and not all thus corresponding quantities are equal.

12. PREFERENCE DOMAINS AND FRONTIERS: Let E be an expenditure system, with base preference relations Q, and derived preference relation $P = \vec{Q}$, where

$$xQy \equiv u'y \leq 1 \wedge y \neq x \quad .$$



xQy

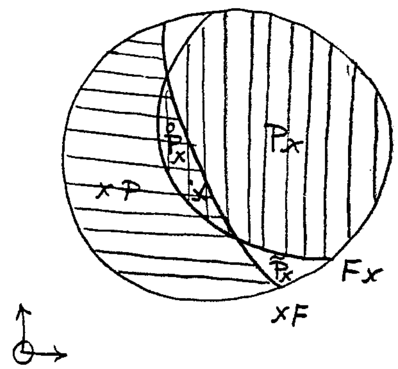


xPy

The sets xP, Px of compositions y which are inferior and superior in preference to a given composition x, that is such that xPy, yPx respectively, define the <u>inferior</u> and <u>superior preference domains</u> of x. They are the domains of extremities of chains descending and ascending from x. They have complements $x\bar{P}$, $\bar{P}x$ defining the <u>non-inferior</u> and <u>non-superior preference domains</u>. Their frontiers xF, Fx define the <u>inferior</u> and

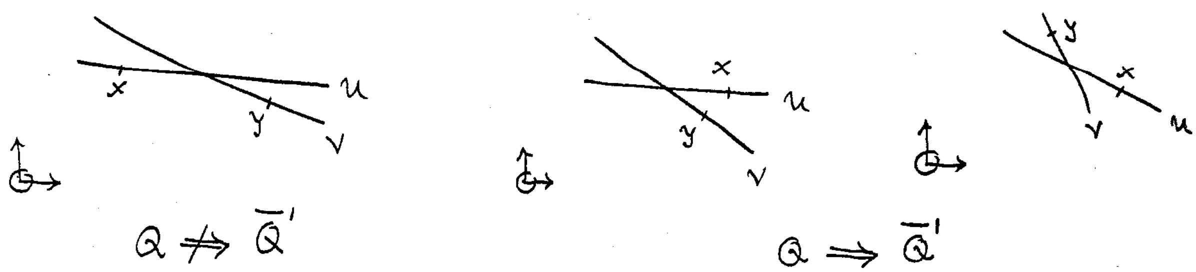
<u>superior preference frontiers</u> of x. The regions $\mathring{P}_x = xP \cap Px$ and $\tilde{P}_x = x\bar{P} \cap \bar{P}x$ define the <u>encyclement</u> and <u>indifference domains</u> of x, formed of elements which are encycled with x by base preference and therefore both inferior and superior to x by derived preferences, and of elements which are neither inferior nor superior in preference to x, respectively.

13. ANTISYMMETRY OF BASE PREFERENCES: The antisymmetry $Q \Longrightarrow \bar{Q}'$ of the base preference relation Q is the condition



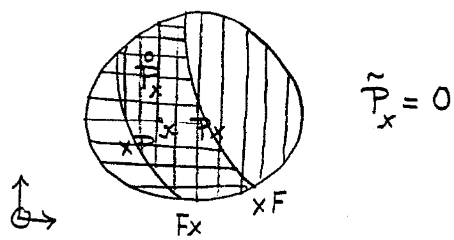
$$u'y \leq 1 \Longrightarrow v'x > 1,$$

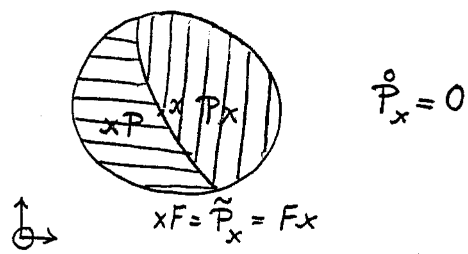
associated with Samuelson,[2] and known as "the Weak Axiom of Revealed Preference." Since $Q \Longrightarrow P$, it is implied by the condition $P \Longrightarrow \bar{P}'$, for the antisymmetry of the derived preferences $P = \vec{Q}$, which is the same as the here-considered coherence condition, and also the so-called "Strong Axiom of Revealed Preference" which was stated by Houthakker.[1]



While, abstractly, the Strong Axiom is stronger than the Weak Axiom, since there are more derived preferences than base preferences, nevertheless in two dimensions they are equivalent; but not in more than two dimensions. In any case, the Strong Axiom, which is coherence, is a fundamental theoretical condition, that the derived preferences form an order; and the Weak Axiom has all its significance in relation to this stronger condition. However, the Weak Axiom does, in the process of developing the consequences of coherence, have some interesting implications on its own, as shown in the following theorem.

> THEOREM: If E is invertible and Q antisymmetric then xP and Px are strictly concave and convex open domains, and their frontiers xF and Fx are integral surfaces of the differential equation $u'dx = 0$. If, moreover, E is responsive, then xF and Fx are either disjoint, in which case $\tilde{P}_x = 0$, or identical, and passing through x, in which case $\mathring{P}_x = 0$ and $xF = \tilde{P}_x = Fx$.





14. COHERENT SYSTEMS: The condition of coherence is that

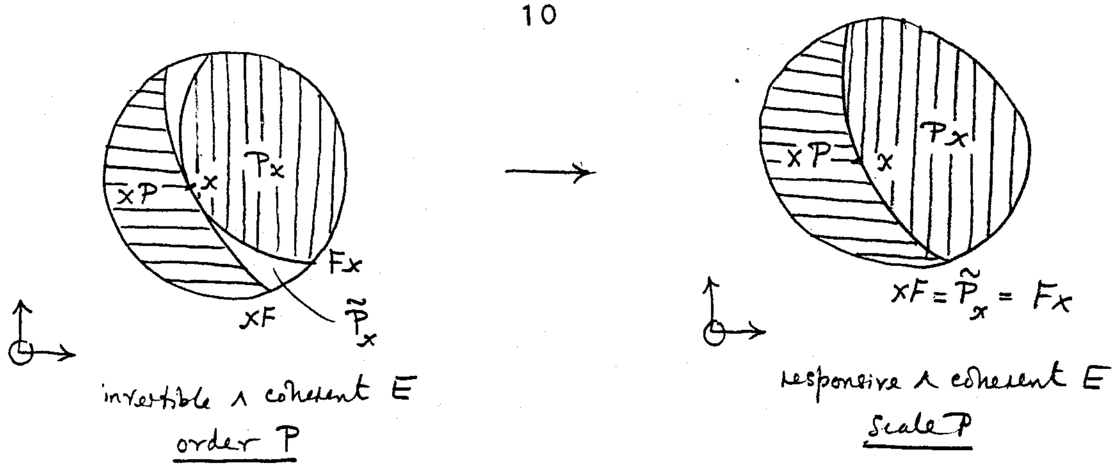
$$\mathring{P}_x = xP \cap Px = 0 \quad ;$$

and, since a composition x must always belong to the closure of its preference domains, this condition requires that it lie on both its preference frontiers:

$$\mathring{P}_x = 0 \Longrightarrow x \in xF,\ Fx \quad .$$

Since, with invertability, these surfaces must then be integral surfaces, there now follows:

> THEOREM: (i) invertability $\wedge$ coherence $. \Longrightarrow .$ geometric integrability
>
> (ii) responsivity $\wedge$ coherence $. \Longrightarrow .$ analytic integrability $\wedge\ P = A$.



Invertability with coherence establishes the preference frontiers as a pair of integral surfaces through x, obtaining the condition of geometric integrability. Every composition to the one side of one surface is inferior to x, and every composition to the other side of the other is superior, while every composition on or between them is neither superior nor inferior, but indifferent. With responsivity, there can be at most one integral surface through any point; so the preference frontiers coincide in the indifference domain. Integrability becomes strengthened from geometric to simple geometric, and to analytic; and the preference order P becomes identified with the geometric scale G, which is again identical with the analytic scale A.

The coherence condition on an expenditure system E, with preference relation P, is just that P be an order; and this does not necessarily require that P, more stringently, be a scale, which is a rather special kind of order. However, it turns out that for responsive systems, this generally effective distinction, between the condition that P be an order and a scale, vanishes. Thus, though generally

$$\text{scale } P \Longrightarrow \text{order } P$$

but not conversely, the converse becomes true under the condition of responsivity.

COROLLARY: responsive E . $\Longrightarrow$. order P $\Longleftrightarrow$ scale P.

COROLLARY: For a responsive coherent system, the preference order is a scale, whose indifference class with a given composition as representative is the unique integral surface through that composition, everywhere smooth and strictly convex.

15. PREFERENCE GAUGES AND INTEGRALS. Given a numerical function ϕ and a relation R such that

$$\phi(x) > \phi(y) \Longleftrightarrow xRy \quad ,$$

the relation is necessarily a scale; and ϕ defines a gauge, measuring that scale.

In regard to the conclusion P = A, obtained for a responsive and coherent system — that the preference order be identical with the analytic scale, implicit in the analytic integrability condition — its content is that the preference order is a scale for which any integral of the system is a gauge.

THEOREM: A responsive coherent system is analytically integrable and has any integral as a preference gauge.

The direct construction of the preference relation of an expenditure system, by operations following the form of its definition, is plainly impossible: it involves the construction of all possible chains, of all possible lengths. Thus the preference relation between any given compositions is not directly knowable. However, from this theorem one derives an analytic process for the indirect construction of the preference relation of a responsive expenditure system, under precisely that condition by which it has significance as a preference system, the coherence condition.

16. STABILITY. It has been noted that analytic integrability is the same as equilibrium for the system, constrained by the balance condition $u'x = 1$, with any integral ϕ as the potential. To this condition there may be added the further condition that equilibrium be <u>stable</u>: with ϕ stationary under $u'x = 1$ for equilibrium, the stationary value must be an absolute maximum for stability. Equilibrium with stability implies coherence. This is just what Houthakker observed. The integral ϕ which is the potential in terms of which the equilibrium and the stability are defined appears as a gauge for the scale formed by the coherent preferences. But the converse proposition, that coherence implies stable equilibrium, under the balance condition, relative to some potential function, which is presumably what is considered by Houthakker, is not generally true. It does, however, become true under the responsivity condition.

THEOREM: responsivity $: \Longrightarrow :$ coherence $. \Longleftrightarrow .$ equilibrium $\wedge$ stability.

For responsivity and coherence implies analytic integrability, with any integral ϕ as a gauge for P. The stability of equilibrium relative to ϕ now follows because

$$y \neq x \wedge u'y = 1 \;.\Longrightarrow.\; xPy \;.\Longleftrightarrow.\; \phi(x) > \phi(y) \quad .$$

17. CONVEXITY. If an expenditure system has the property of geometric integrability, there may be applied to it a further condition, of <u>convexity</u>: that all the integral surfaces be everywhere strictly convex. This condition has already been deduced for responsive, coherent systems. With analytic integrability, which is the equilibrium property, the further condition of convexity is necessary and sufficient for stability, and for coherence:

THEOREM: responsivity $. \Longrightarrow .$ coherent $\Longleftrightarrow$ integral $\wedge$ convex.

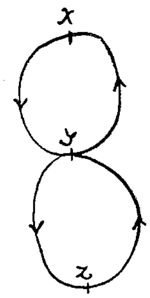
18. SUBSTITUTIONAL SYMMETRY AND NEGATIVITY: For a differentiable expenditure system E, with partial derivative matrix x_u, there may be formed the substitution matrix $s = x_u(1 - ux')$. The conditions of substitutional symmetry and negativity, respectively, are defined by the conditions $s = s'$, for the symmetry of s, and $du'sdu < 0$ ($du \not\parallel u$), for the negativity of s in regard to every direction not parallel to u. A system is responsive if it is differentiable and invertible, and a necessary and sufficient condition for coherence, that is the irreflexivity of the preference relation P, is the condition on the partial derivative given by the substitutional symmetry and negativity conditions. The part of this condition given by substitutional symmetry is equivalent to the part of the coherence condition given by integrability.

THEOREM: (i) invertibility $\wedge$ differentiability $: \Longrightarrow :$ coherence $. \Longleftrightarrow .$ symmetry $\wedge$ negativity.
(ii) integrability $\Longleftrightarrow$ symmetry.

19. ENCYCLEMENT: The double-preference relation $\mathring{P} = P \wedge P'$, the symmetric part of the preference relation P, is transitive, since P is transitive; and its symmetry is immediate in the form of its definition, as the symmetric part of a relation. It is the same as the relation of encyclement by the base preference relation Q, or Q-encyclement, which elements have if they together lie on a Q-cycle; and transitivity is again obtained, since two cycles with a common element determine a third cycle, which describes them both, and crosses itself at that element. Symmetry is again evident in the form of the relation. Encyclement, defined by Q, or, equivalently, double-preference, defined by $P = \vec{Q}$, since symmetric and transitive, is an

equivalence if and only if it is a reflexive relation, which is if and only if P is reflexive: $I \Longrightarrow P$. This reflexivity condition is also that there exists a Q-cycle passing through every element. With $\overset{\circ}{P}$ an equivalence, the compositions are resolved by it into <u>encyclement</u> classes, within each of which every pair of elements lie together on some Q-cycle, while any elements taken from different classes are not so related.

Consider an invertible and differentiable expenditure system, for which the substitional symmetry or negativity conditions fail at every point in a region. The preference relation defined for every neighbourhood of the region is reflexive: in every neighbourhood of an element there exists a cycle passing through that element. The preference relation for the whole region is reflexive, and partition of the compositions into encyclement classes is obtained.



$x\overset{\circ}{P}y \wedge y\overset{\circ}{P}z \Rightarrow x\overset{\circ}{P}z$

THEOREM: $\sim$ symmetry $\vee \sim$ negativity $. \Longrightarrow . \; I \Longrightarrow P.$

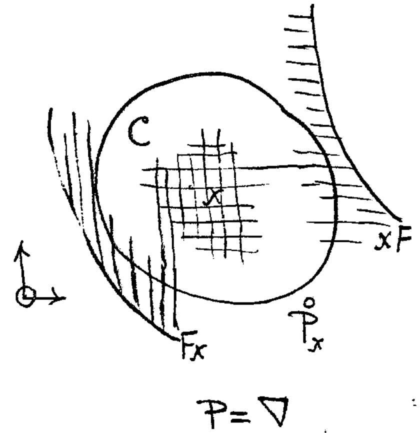
20. TOTAL INCOHERENCE: If every pair of compositions are encycled, so encyclement is an equivalence, and there is just one encyclement class, in which case the derived preference relation is universal, $P = \nabla$, every composition is preferred to every other, the expenditure system will be said to be <u>totally incoherent,</u> this being a condition which is like a complete opposite of coherence, and is much stronger than just incoherence. Now coherence requires both integrability, and the antisymmetry of the base preferences:

$$\text{coherence} \; . \Longrightarrow . \; \text{integrability} \wedge \text{Q-antisymmetry}.$$

Therefore certainly the negation of either integrability or Q-antisymmetry, $Q \Longrightarrow \bar{Q}'$, will imply incoherence. However, if the antisymmetry of Q is affirmed while integrability is denied, much more, in fact total incoherence is obtained.

THEOREM: Q-antisymmetry $\wedge$ non-integrability . $\Longrightarrow$. total incoherence.

For, with antisymmetry of base preferences, the preference frontiers, of any compositions x will be integral surfaces. But, with non-integrability in C, the surfaces cannot cut C, which must therefore be contained entirely between them, and be part of the double-preference domain $\mathring{P}_x$.



In the case of a differentiable system the criterion for non-integrability is the non-symmetry of the substitution matrix.

21. LOCAL AND GLOBAL PROPERTIES: Consider a property π definable in respect to every neighbourhood B_0 in an open region B, including the region B itself. If B itself has the property it is said to have it <u>globally</u>. In contrast to the direct, global definition, the property is defined <u>locally</u> for B if, for every element x_0 in B, there exists some neighbourhood B_0 of x_0 in B which has the property. Obviously the global property implies the local property, since the region itself is a neighbourhood of each of its points, so if the property holds globally, it holds in some neighbourhood of each of its points, that is to say locally. But, abstractly, a property holding locally in a region does not necessarily hold globally in that region. Nevertheless, for certain properties, under certain conditions, the inference from the local property to the global can be made. For example, the convexity of a surface is a property with both a local and a global definition; and they are equivalent. Again, in certain problems in dynamics, local stability of equilibria under a class of constraints is equivalent global stability.

Now coherence for an expenditure system, having application to any region within the region of definition of the system, is a property which can be possessed in a local and a global sense; and while, in general, the local property does not imply the global, the implication is valid, however, for responsive systems.

THEOREM: responsivity . $\Longrightarrow$. local coherence $\Longrightarrow$ global coherence.

Thus, for responsive systems, coherence in any region is equivalent to simple geometric integrability together with the strict convexity of the integral surfaces. Since both these conditions extend from the local to the global, it follows that so does their conjunction, which is equivalent to coherence.

22. DUALITY. In the case of an invertible expenditure system

$$E : B \longleftrightarrow C \; (u \longleftrightarrow x; \; u'x = 1) \quad ,$$

mapping balances into compositions there is a perfect symmetry in the appearance of balance and composition, by which, in any definition or proposition, their roles may be interchanged, to obtain a <u>dual</u>, formally similar definition or proposition. The duality which is thus founded is a logical instrument in the derivation of some theorems. Also it reflects the duality which exists in the analysis of supply and demand, there being, in form, just one analysis, which can be interpreted for one side or the other.

An invertible expenditure system E has, as its dual, an <u>inverse expenditure system</u> F, mapping compositions into balances, with base preferences N given by

$$uNv \equiv x'v \leq 1 \wedge v \neq u \quad ,$$

defining the dual base preferences of the direct system. The derived preferences of N define the <u>dual preference relation</u>

$M = \vec{N}$ of the system. Since $u \neq v \Longleftrightarrow x \neq y$, it follows that

$$uNv \Longleftrightarrow yQx \quad ,$$

and therefore that

$$uMv \Longleftrightarrow yPx \quad .$$

Accordingly, if P^* is the relation between balances induced by the relation P between compositions, with the definition

$$vP^*u \equiv yPx \quad ,$$

then $M' = P^*$, where M' is the converse of M.

> THEOREM: The induced preference relation of an invertible expenditure system is the converse of the dual preference relation.

The two forms of coherence, applicable to an invertible system and its dual, are thus equivalent. An ascending preference chain for the system corresponds to a descending chain for the inverse system. Any proposition concerning an inferior preference domain has corresponding to it a proposition concerning the superior domain; and reversely. The propositions concerning inferior and superior preferences are in pairs, and it is only necessary to prove half of them.

In case the invertability involved in this duality should seem an extra restriction, beyond the customary, it should be noted that, along with the dependence of x on u, which is the first principle of consumer theory, the inverse dependence of u on x has almost always been granted, even if not explicitly. For example, the classical principle that x is determined from u as giving the absolute maximum of a differentiable function ϕ under the constraint $u'x = 1$ gives $u = \phi_x/\lambda$ where $\lambda = x'\phi_x$ and ϕ_x is the vector of partial derivatives of ϕ with respect to x. Also the question of "integrability" is a very standard topic in discussions; and whenever a precise definition is given, which is not always, it involves u as a function of x. Again, if the phenomenon of saturation, which is against the invertability concept, is to be admitted, it is always possible to replace strict saturation by near saturation — as near as no matter.