PROBLEMS OF ECONOMIC POLICY
FROM THE VIEWPOINT OF OPTIMAL CONTROL

Gregory C. Chow

Econometric Research Program
Research Memorandum No. 139
May 1972

The research described in this paper was supported by NSF Grant GS 32003X.

Econometric Research Program
PRINCETON UNIVERSITY
207 Dickinson Hall
Princeton, New Jersey

PROBLEMS OF ECONOMIC POLICY FROM THE VIEWPOINT OF OPTIMAL CONTROL

Gregory C. Chow

I have been prompted to write this paper by the reading of Robert S. Holbrook's recent article, "Optimal Economic Policy and the Problem of Instrument Instability."[1] It is my purpose to show that the questions raised, and partially studied, by Holbrook, as well as other questions concerning optimal economic policy, can be answered by the techniques of optimal control in a more general setting. It will be convenient to use the problem of instrument instability as a starting and focal point of analysis in the exposition of this paper.

"Instrument instability" was said to exist by Holbrook (1972, p. 57) if "attempts to offset completely the cumulative impact of past changes in the policy instrument may require ever greater changes in the future value of the instrument." An example is provided by the model

$$y_t = .4x_t + .6x_{t-1} + 10$$

where y_t is the endogenous variable to be controlled and x_t is the instrument. If the objective is to make $y_t = 10$, for many future periods, the solution is to set $.4x_t + .6x_{t-1} = 0$, or $x_t = -1.5x_{t-1}$, a situation characterized as one of "instrument instability."

Among the questions to be studied in this paper are the following. Given a linear econometric model in its reduced form,

$$(1)\qquad y_t = A_1 y_{t-1} + \ldots + A_m y_{t-m} + C_o x_t + \ldots + C_n x_{t-n} + b_t + u_t$$

where y_t is a vector of endogenous variables, x_t is a vector of instruments, A_i and C_i are given constant matrices, b_t are given vectors capturing the combined effects of other exogenous variables which are not subject to control, and u_t is a random vector with mean 0, covariance matrix V, and is uncorrelated with u_s $(t \neq s)$, under what conditions will "instrument instability" (as examplified above, but to be specified more precisely later) exist? Secondly, if "instrumental instability" is found to exist, to what extent and in what way can the policy maker trade off this instability by allowing for instability in the endogenous variable(s) to be controlled? Third, if one instrument, say federal expenditures, is found to be unstable when it is regarded as the only instrument (holding as fixed another instrument such as money supply), to what extent will the availability of another instrument alleviate the problem of instrument instability?

As it will be seen, the third question, which was not raised by Holbrook because his framework only allowed him to study one instrument at a time, can be reduced to a special case of the second question. Our framework will be more general than Holbrook's in (a) having possibly more than one endogenous variable to be controlled,

(b) having possibly more than one instrument to be studied, (c) having possibly unequal numbers of endogenous variables and instruments, and (d) having possibly random disturbances in the system. Furthermore, our solution can be generalized to the case of (e) treating the matrices A_i and C_i as random matrices, and (f) of non-linear dynamic models. This paper will deal only with generalizations (a), (b), (c) and (d), leaving (e) and (f) to an appropriate reference. All the questions raised so far can be analyzed fairly easily once the framework of optimal control is set forth. This is the immediate task of section I.

I. The Framework of Optimal Control

Let us begin by rewriting the dynamic system (1) as a first-order system in which the "current endogenous variables" y_t will incorporporate the instruments:

$$(2)\quad \begin{bmatrix} y_t \\ \vdots \\ y_{t-m+1} \\ x_t \\ \vdots \\ x_{t-n+1} \end{bmatrix} = \begin{bmatrix} A_1 \ldots A_m & C_1 \ldots C_n \\ \ldots I \; 0 & 0 \ldots 0 \\ 0 \ldots 0 & 0 \ldots 0 \\ 0 \ldots 0 & \ldots I \; 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-m} \\ x_{t-1} \\ \vdots \\ x_{t-n} \end{bmatrix} + \begin{bmatrix} C_0 \\ \vdots \\ 0 \\ I \\ \vdots \\ 0 \end{bmatrix} x_t + \begin{bmatrix} b_t \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} u_t \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which will be redesignated simply as

$$(3)\quad y_t = Ay_{t-1} + Cx_t + b_t + u_t \quad .$$

Note that the newly defined y_t includes current and (possibly) lagged endogenous variables as well as current and (possibly) lagged instruments, whereas x_t remains the same as before.

The performance of the system will be measured by the deviations of y_t, as defined in (3), from the target vectors a_t $(t=1,\ldots,T)$. Specifically, welfare cost is measured by

$$(4) \qquad W = E_o \sum_{t=1}^{T} (y_t - a_t)' K_t (y_t - a_t)$$

where E_o denotes expectation conditional on the initial condition y_o, again in the notation of (3), and K_t are known, symmetric (usually diagonal), positive semi-definite matrices, with zero elements as a rule corresponding to lagged (endogenous and control) variables.

The problem is to minimize expected welfare cost (4) by setting the time path of the instrument x_t $(t=1,\ldots,T)$, given the model (3). This is one of the simplest and most basic problems in stochastic control theory, and elementary methods of solving it can be found in Chow (1972a, 1972b). I will provide here a solution using the method of dynamic programming of Bellman (1957). The method begins by solving the problem for the last period T, given the initial condition y_{T-1}. The welfare cost is, for $H_T = K_T$, $h_T = k_T \equiv K_T a_T$, and $c_1 = a_T' K_T a_T$,

$$(5) \qquad W_T = E_{T-1} (y_T - a_T)' K_T (y_T - a_T) = E_{T-1} [y_T' H_T y_T - 2 y_T' h_T + c_1] \ .$$

Using the model (3) for y_T and taking expectations, we minimize

$$(6) \qquad W_T = (Ay_{T-1} + Cx_T + b_T)'H_T(Ay_{T-1} + Cx_T + b_T)$$
$$- 2(Ay_{T-1} + Cx_T + b_T)'h_T + E_{T-1}\, u_T'K_Tu_T + c_1$$

by differentiating with respect to the vector x_T ,

$$(7) \qquad \frac{\partial W_T}{\partial x_T} = C'H_T(Ay_{T-1} + Cx_T + b_T) - 2C'h_T = 0$$

or

$$(8) \qquad \hat{x}_T = G_T\, y_{T-1} + g_T$$

where

$$(9) \qquad G_T = - (C'H_TC)^{-1} (C'H_TA) ;$$

$$(10) \qquad g_T = - (C'H_TC)^{-1} C'(H_Tb_T - h_T) .$$

Substituting the solution (8) for x_T in (6), we obtain

$$(11) \qquad \hat{W}_T = y_{T-1}' (A + CG_T)'H_T (A + CG_T)y_{T-1} + 2\, y_{T-1}' (A + CG_T)'H_Tb_T$$
$$- 2\, y_{T-1}' (A + CG_T)'h_T + \text{terms not involving } y_{T-1} .$$

Next, consider the problem for one more period $T-1$. The principle of optimality of Bellman (1957) is to minimize

$$(12) \qquad E_{T-2}[\hat{W}_T + y_{T-1}' K_{T-1}y_{T-1} - 2\, y_{T-1}' k_{T-1} + c_2]$$

with respect to the only unknown x_{T-1} , since the other unknown x_T has already been found and eliminated in (11). Substituting (11) for $\hat{W}_T$ into (12) will yield

(13) $$W_{T-1} = E_{T-2}[y'_{T-1} H_{T-1} y_{T-1} - 2y'_{T-1} h_{T-1} + \text{constant}]$$

where

(14) $$H_{T-1} = K_{T-1} + (A + CG_T)' H_T (A + CG_T)$$

(15) $$h_{T-1} = k_{T-1} + (A + CG_T)' (h_T - H_T b_T) \quad .$$

The solution is complete if one observes that the expression (13) to be minimized with respect to x_{T-1} has the same form as expression (5). One can thus repeat the process from equations (5) to (15), with the subscript T replaced by $T-1$, and so forth. In summary, optimal control consists of choosing the instrument x_t as a linear function $G_t y_{t-1} + g_t$ of the variables y_{t-1} of the previous period, as in equation (8). The matrices of coefficients G_t are determined, together with H_t, by solving equations (9) and (14) alternatively, backward in time from $t = T$, and with initial condition $H_T = K_T$. G_t and H_t having been obtained, the vectors g_t , together with h_t , are determined by solving equations (10) and (15) alternatively, backward in time from $t = T$, and with initial condition $h_T = k_T \equiv K_T a_T$.

If the matrices K_t in the welfare function are all equal to K , the solution for G_t and H_t may reach a steady state, for

t smaller than a certain value, that will satisfy

$$(16) \qquad G = -(C'HC)^{-1}C'HA \ ;$$

$$(17) \qquad H = K + (A+CG)'H(A+CG) \ .$$

Since (17) can be written as an infinite series

$$(18) \quad H = K + (A+CG)'K(A+CG) + (A+CG)'^2K(A+CG)^2 + \ldots$$

the steady state will exist if and only if the series converges, i.e., if and only if all the characteristic roots of the matrix $(A+CG)$ are smaller than one in absolute value. Even when G_t and H_t do reach a steady state, g_t and h_t will usually not do so if $k_{t-1} (\equiv K a_{t-1})$ and b_t are changing through time, as can be seen from equation (15).

II. Economic Policy from the Viewpoint of Optimal Control - Deterministic Systems

The questions raised by Holbrook (1972) are for a deterministic model, i.e., model (1) with $u_t = 0$. It may be of interest to consider this special case first. Note that the above formulation of, and solution to, the optimal control problem applies easily to this case -- simply erase the expectation signs and set u_t equal to zero. The linear feedback control equations $x_t = G_t y_{t-1} + g_t$, and the computations of G_t and g_t remain the same as in the stochastic case.

Before analyzing any questions, one would wish to define "instrument instability." Holbrook (1972) did not give a precise definition. By this term, one might mean (a) that the optimal time paths of x_t are explosive and/or (b) that they contain oscillations. If x_t satisfies a system of linear difference equations, (a) will occur when some roots of the system are larger than unity in absolute value; (b) can occur if some roots are complex and/or negative.[3] I suppose that the degrees of (a) and (b), namely, how explosive and how large the oscillations, should also matter. Thus the absolute values of the roots, be they real or complex, will matter. If one specifics that the first differences of economic variables satisfy a linear model like (1), he might reasonably require, in his definition of stability, that the instruments x_t (also in first differences) be not explosive, or not very explosive. But if the model explains the levels of economic variables which are expected to grow, some explosiveness in the instruments should not be considered unstable. In short, "instrument instability" requires one to examine the explosiveness and/or the extent of oscillations of the optimal time paths of the instruments x_t.

Will instrument instability exist for a particular system? To answer this question, one simply studies the optimal path of x_t, which, in the notation of (3), is imbedded in the vector y_t. When optimally controlled, y_t will satisfy (in the deterministic case)

$$(19) \qquad y_t = Ay_{t-1} + Cx_t + b_t = (A + CG_t)y_{t-1} + b_t + Cg_t$$

and the behavior of such a system can easily be analyzed, especially when G_t reaches the steady state G. For $G_t = G$, and denoting $b_t + Cg_t$ by $\tilde{b}_t$, say, the solution to (19) is

$$(20) \qquad y_t = (A + CG)^t y_o + \tilde{b}_t + (A + CG)\tilde{b}_{t-1} + \ldots + (A + CG)^{t-1} \tilde{b}_1$$

in which the homogenous part $(A + CG)^t y_o$ is explosive or oscillatory, depending on whether the roots are large in absolute value or complex.

One might ask, if the system omitting the influence of x_t, i.e.,

$$(21) \qquad y_t = A\, y_{t-1} + b_t$$

is explosive (damped), will the system including the instruments x_t which are set optimally, i.e., system (19), be explosive (damped) also? Either case can happen, but to dampen an otherwise explosive system by optimal control is more likely than to change an otherwise stable system to an explosive one. After all, a purpose of control may be to dampen an explosive system. From equations (9) and (14) for G_t and H_t (or from equations (16) and (17) for G and H), each column of the optimal feedback control matrix G_t can be interpreted as coefficients of a regression of the corresponding column of the matrix A on columns of the matrix $-C$. In other words, if the first column of A, say A_1, are observations on a dependent variable, and the columns of $-C$ are observations on the explanatory variables, then the first column of G_t , say $G_{t,1} = -(C'H_tC)^{-1} C'H_tA_1$, are coefficients in the regression $A_1 = -C\,G_{t,1} + R_{t,1}$ obtained by the method of Aitken's generalized least squares. The purpose of regression is to make the columns of residuals

$$(R_{t,1}\ R_{t,2}\ldots) = R_t = A + CG_t$$

small -- in fact, for column 1, it is to minimize the weighted sum of squares of the residuals $R'_{t,1}\ H_t\ R_{t,1}$. Therefore, if some roots of A are larger than one in absolute value, the roots of the matrix $R_t = A + C\,G_t$ of regression residuals may not be, in which case the system under control is damped.

How would it be possible for the matrix A to be stable, but the matrix $A + CG$ (omitting subscript t) to be not? It is possible, when the weighting matrix H , which is derived from the matrix K of welfare weights by equation (14), is uneven in assigning weights to different residuals. As an example, consider the simple system given in the second paragraph of the introduction of this paper. In the notation of (2) and (3), this system is

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} 0 & .6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} .4 \\ 1 \end{bmatrix} x_t + \begin{bmatrix} 10 \\ 0 \end{bmatrix} .$$

The matrix A has multiple roots of zero. If the purpose of control is to steer y_t to target, but to ignore the behavior of x_t , the

2×2 matrix K is diagonal with unity as its leading element, and

$$(23) \qquad G_T = -[(.4 \quad 1)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}.4\\ 1\end{pmatrix}]^{-1}(.4 \quad 1)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & .6\\ 0 & 0\end{pmatrix}$$

$$= -(.16)^{-1}(0 \quad .24) = (0 \quad -1.5)$$

$$(24) \qquad A + CG_t = \begin{pmatrix}0 & .6\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -.6\\ 0 & -1.5\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -1.5\end{pmatrix}$$

$$(25) \qquad H_{T-1} = K + (A + CG_T)'K\,(A + CG_T) = K = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

Therefore, the values for G_T and H_{T-1} given in (23) and (25) are steady-state values, and we have an explosive root -1.5 in the residual matrix $A + CG_T$.

One can now state several relevant propositions concerning optimal control that are based on the theory of section I. First, it is possible to steer certain endogenous variable(s) exactly to target, under appropriate conditions to be specified below, by using the quadratic welfare function (4) and assigning positive weights only to the deviations of the variable(s) selected. Second, if the number of variables to be controlled (the number of non-zero diagonal elements in K) is equal to the number of instruments, the minimization problem of equation (5) will be solved with $(y_T - a_T)'K\,(y_T - a_T) = 0$, and so on for equation (13) etc., and the variables will be on target exactly -- except when the rank of C is smaller than the above number, in which case $(C'H_tC)$ does not have an inverse and the optimal control matrix G_t cannot be found. Third, if the number of variables to be controlled is larger than the number of instruments, the former variables will not reach targets exactly, and their deviations from targets will depend on the welfare weights in K assigned to them.[4]

Fourth, "instrument instability," insofar as it pertains to some characteristic of the time path x_t, can exist no matter whether the welfare weights are assigned only to the originally endogenous variables, or also to the instruments (imbedded in y_t). Holbrook (1972) treated only the former case, in fact, a rather special case of the former when the numbers of target variables and of instruments are both equal to one. Fifth, once the last point is recognized, as it is from the theory of section I, one can trade off the stability characteristics not only between the truly endogenous variables and the instruments, but also among the instruments themselves (and among the endogenous variables themselves, of course). All these can be accomplished by changing the diagonal elements of the matrix K. For example, if total government expenditures and money supply are two instruments, giving more weight to the former (and specifying its target path to be smooth) will mean more stability for it, as compared with the latter instrument, when the weights for the endogenous variables remain unchanged.

Once the basic ideas of optimal control are understood, one might not wish to take the concept of "instrument instability" very seriously. What matter are the dynamic characteristics of the time path in question, be it a dependent variable or an instrument. Moreover, within the framework of section I, a good measure of the dynamic performance of the variable is the sum of its squared deviations from the target path. If the variable fluctuates violently, and the target path is smooth, its performance is poor by this measure. If the variable increases rapidly, and the target calls for no increase or a small increase per period, its performance is poor by this measure. Having solved the problem of optimal control, one can easily compute the optimal path of y_t (including x_t as a subvector) using equation (19) and thus the sums of squared deviations of all variables from targets, and, in fact, a weighted sum of these sums as a measure of welfare cost.

III. Economic Policy from the Viewpoint of Optimal Control - Stochastic Systems

For a stochastic system, one retains the random disturbance u_t in equation (3) and the expectation sign in equation (5), etc. The objective of control, in the framework of section I, is to minimize the expectation of a weighted sum of squared deviations of selected variables from targets. Because of the random disturbances, one cannot hope to achieve the targets exactly, even when the numbers of target variables and of instruments are equal. As in section II, one may wish to measure certain dynamic characteristics of the (stochastic) time series under control. Space will allow only a brief discussion of this problem.

Consider the stochastic dynamic system (3) under optimal control, with G_t reaching the steady state G, and with $\tilde{b}_t$ denoting $b_t + Cg_t$:

$$(26) \qquad y_t = (A + CG)y_{t-1} + \tilde{b}_t + u_t = Ry_{t-1} + \tilde{b}_t + u_t .$$

By repeated substitutions of y_{t-1} in (26) by $Ry_{t-2} + \tilde{b}_{t-1} + u_{t-1}$, and y_{t-2} by $Ry_{t-3}+\ldots$, etc., one easily finds that

$$(27) \qquad y_t = R^t y_o + (\tilde{b}_t + R\tilde{b}_{t-1}+\ldots+R^{t-1}\tilde{b}_1)$$
$$+ \; u_t + Ru_{t-1}+\ldots+ R^{t-1}u_1 .$$

The first line on the right-hand side of (27) is the mean of the process, which is identical with the time path (20) of the deterministic system of section II,. The second line of (27) is the deviation from mean, or the random part of the process, as generated by the system

$$(28) \qquad y_t = R y_{t-1} + u_t .$$

Somewhat analogous to the deterministic situation, the stochastic time path can be characterized by its explosiveness and its cyclical fluctuations.[5] If some roots of $R = (A + CG)$ is greater than unity in absolute value, the stochastic time paths explode, with both means and variances increasing through time. If some roots of R are complex, and their absolute values not too small, the time series will have important cycles as defined by local peaks in their spectral density functions (if the time series have variances and covariances constant through time, or if they are "covariance stationary"). Even when some roots of R are greater than unity, it is possible to detrend the realizations of y_t so that the remaining diviations from trends will behave like covariance-stationary series, and thus subject to characterization by spectral techniques.[6] In short, there are convenient ways to characterize a stochastic time series and thus to judge the performance of the system under control. Again, if one takes the welfare function (4) seriously, it provides a summary measure of the performance of the system, endogenous variables and instruments included, but this summary measure can be supplemented by some of the characterizations described above.

Not only is the method of obtaining optimal feedback control equations for the stochastic system identical with the method for the deterministic system, but the trade-offs in terms of the expected sums of squared deviations from targets among different variables can be treated in the same way as in section II where mathematical expectations were not involved. Note that each expected sum of squares can be decomposed into (A) the sum of squares of the deviations of the mean of a variable, given by (20) or the first line of (27), from target, plus (B) the expected sum of squared deviations of the variable from its mean, to be computed from the stochastic system (28).[7]

IV. Welfare Tradeoffs from A Macro-Econometric Model

To provide some illustrative calculations of the welfare trade-offs, I have employed the simple macro-econometric model of Chow (1967).[8] This model consists of four stochastic structural equations explaining total consumption expenditures C, gross private investment expenditures I_1, new construction I_2, and the interest rate on 20-year corporate bonds R, in terms of first differences and using annual data of the United States economy for the periods 1931-1940 and 1948-1963. The instruments are government expenditures G and money supply M. In run (1) reported below, only two variables, private expenditures, $Y_1 = C + I_1 + I_2$ and government expenditures G, are subject to control, with equal weights of unity in the welfare matrix K. In run (2), the rate of interest will be the third variable subject to control, with a weight making a deviation of one percentage point per year from target as costly as a deviation of 10 billion dollars for an expenditures variable. The targets for the expenditures variables are to grow by 5 per cent per year from their initial values as of 1964, and the target for the interest rate is to remain at 4.33 per cent, its value in 1964. The time horizon T is 10 years.[9]

Let us first examine the deterministic part (A) of the expected sum of squares of each variable from target for these two runs. The average, over ten periods, of the squared deviations of the mean of each variable from target, is

	Y_1	G	R	Sum
Run (1)	0	0	9.124	9.124
Run (2)	.684	.752	5.898	7.334

where the variable R is in percentage points times 10, to make 1 percentage point equivalent to 10 billion dollars, and the expenditure variables are in billions of dollars. For example, 5.898 for R would mean a standard deviation of 2.43, or of .243 percentage points. Since, in run (1), the number of target variables equals the number of instruments, the deviations of Y_1 and G from targets are zero in the deterministic part of the welfare cost. The above calculations apply to the deterministic model obtained by ignoring the random disturbances of our econometric model, and thus illustrate the trade-offs discussed in section II. Note that the inclusion of R in the welfare function in Run (2) reduces the average deviation of this variable from target and increases the average deviations of the other two variables.

The stochastic part (B) is the average, over 10 years, of expected squared deviations of the time series from its mean:

	Y_1	G	R	Sum
Run (1)	388.4	0	118.8	507.2
Run (2)	396.2	8.4	82.4	487.0

Note the large mean squared deviations here. For example, in run (1), Y_1 has a standard deviation of $\sqrt{388.4} = 19.7$ billions, and the rate of interest has a standard deviation of 1.09 percentage points, due to the random disturbances of the model. In run (1), G grows at 5 per cent, exactly as its target specifies, and it is not subject to any random disturbance. The inclusion of R in the welfare function of run (2) reduces its variance, but increases the variances of Y_1 and G.

V. Conclusion

It has been shown that, using the problem of instrument instability of Holbrook (1972) as a starting point of discourse, one can study interesting problems of economic policy by the techniques of optimal control. Some other interesting problems, including the measurement of welfare gains from employing an optimal control policy as compared with a policy of maintaining a constant rate of growth for each instrument, and the generalizations to non-linear systems with known parameters, or to linear systems with estimated and random parameters A and C, can be found in a related paper.[10] These analyses have demonstrated that the framework of optimal control is extremely useful in the study of economic policies.

REFERENCES

R.E. Bellman, Dynamic Programming, Princeton University Press, Princeton, New Jersey (1957).

G.C. Chow, "Multiplier, Accelerator, and Liquidity Preference in the Determination of National Income in the United States," The Review of Economics and Statistics, Vol. XLIX, No. 1, (February 1967), pp. 1-15.

__________, "The Acceleration Principle and the Nature of Business Cycles," Quarterly Journal of Economics, LXXXII, 3, August, 1968, pp. 403-418.

__________, "Optimal Stochastic Control of Linear Economic Systems," Journal of Money, Credit, and Banking, August, 1970, pp. 291-302.

__________, "Optimal Control of Linear Econometric Systems with Finite Time Horizon," International Economic Review, Vol. 13, No. 1 (February 1972), pp. 16-25 (a).

__________, "How Much Could be Gained by Optimal Stochastic Control Policies," Research Memorandum No. 138, Econometric Research Program, Princeton University, April 1972 (b).

G.C. Chow and R.E. Levitan, "Spectral Properties of Non-Stationary Systems of Linear Stochastic Difference Equations," Journal of the American Statistical Association, Vol. 64 (June, 1969), pp. 581-590 (a).

__________________________, "Nature of Business Cycles Implicit in a Linear Economic Model," The Quarterly Journal of Economics, Vol. LXXXIII, (August, 1969), pp. 504-517 (b).

R.S. Holbrook, "Optimal Economic Policy and the Problem of Instrument Instability," The American Economic Review, Vol. LXII, No. 1 (March, 1972), pp. 57-65.

FOOTNOTES

1. See Holbrook (1972). I would like to acknowledge, with thanks, the financial support from the National Science Foundation through Grant NSF GS 32003X.

2. In Holbrook (1972), the dependent variable(s) (our y_t) is denoted by Y_t , the control or policy variable(s) (our x_t) is denoted by P_t , the "net impact of current and lagged values of all policy instruments (other than P) and other truly exogenous variables as well as lagged values of endogenous variables" (p. 58) is included in one symbol X -- in the last case, we use, respectively, other components of the vector x_t than the component in question, b_t and y_{t-1}, etc.

3. Proofs of these wellknown propositions can be found in Chow (1968).

4. If the number of target variables is smaller than the number of instruments, there will be more than one way to achieve the targets exactly, and the computation of the optimal control matrices G_t can be performed by using a generalized inverse for $(C'H_tC)$, where the rank of H_t and thus of the entire matrix, is smaller than the number of instruments.

5. This remark, and later statements to qualify it, have been developed and discussed at length in Chow (1968), Chow and Levitan (1969a, 1969b), and Chow (1970).

6. See Chow and Levitan (1969a).

7. For the computation of part (B) , see Chow (1972a) or(1972b).

8. This is the model consisting of equations (25), (28), (30) and (31) of Table 1, Chow (1967), p. 9.

9. Calculations using the same econometric model and the same setup, but for different purposes, were reported in Chow (1972b).

10. See Chow (1972b).