EFFECT OF UNCERTAINTY ON OPTIMAL CONTROL POLICIES

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1. Introduction

In this paper, I will provide a method for ascertaining the optimal control policy and the associated welfare cost in a control problem where the welfare cost is quadratic and the econometric model used is linear and the values of its parameters are uncertain. In previous papers (1970, 1972a,b, and c) I have treated the control problem with quadratic welfare and linear model under the assumption that the parameters in the model are given for certain. It would be interesting to relax the assumption of certainty of the parameters, and to examine the effects on the optimal control equations and the associated welfare cost.

As it has been generally recognized, one important use of econometric models is in the design of optimal quantitative economic policies. However, because our knowledge of the economic system is imperfect, one might be led to argue against the use of existing econometric models for policy purposes -- although a Bayesian would not take this position but would rather find an optimal way to utilize his imperfect knowledge. One problem which I have tried to study in the application of econometric knowledge to policy decisions is the measurement of the possible advantage of an optimal policy based on an econometric model over a policy of maintaining constant rates of change for the instruments, under the assumption that the parameters of the model are known for certain. Calculations using a simple macro-econometric model presented in Chow (1972b) have indicated that the welfare costs for the latter policy can be about 40 to 80 per cent higher than for the optimal policy based on an econometric model when the model parameters are assumed to be known constants. A natural second problem is to measure the gain from optimal control when knowledge of the model parameters is uncertain. This and other problems of economic policy can be studied by methods of this paper.

When we assume that the parameters of a linear econometric model are uncertain we can take one of two approaches in deriving the optimal control policies. The first is to assume a given joint density for these parameters which is available at the beginning of the planning horizon and which is not to be modified while the economic process is being controlled. The second is to allow for continuous modification of the joint density of the unknown parameters as more observations become available to the policy maker. To derive optimal control policies from the second approach is more difficult because policies applied to the early periods affect not only the performance of the economy during these and later periods but also the knowledge of the economy which can be utilized to control the economy for later periods. We will point out the mathematical difficulty of this problem in section 2. It is a well-known problem in the literature of control theory, but no truly optimal solution has been obtained, although numerous approximations are available. The present paper is confined to the first, and easier, approach. However, it will provide an upper limit to the measure of the effect of uncertainty since a control policy that utilizes additional observations during the control process will obviously increase the value of optimal control in the face of uncertainty. Furthermore, in many applications, the amount of information on the economic structure available at the beginning of the control process is large compared with additional information to be obtained while the economy is subject to the control rules. The solution given in this paper will then be a reasonable first approximation.

In the literature on optimal control of stochastic systems, e.g., Aoki (1967, pp. 46-47), one can find parts of the solution to the problem of this paper stated in general form, but not a complete solution that will provide numerical answers to the optimal control equations and associated welfare costs. Using the method of this paper, one can obtain numerical answers from observations on the economy. The distribution of the unknown parameters will be derived from historical data from either a Bayesian or a classical point of view. The distribution will then be applied to obtain the optimal feedback control equations and the associated welfare cost.

In section 2, the optimal feedback control equations are derived under the assumption that the expectations involving certain functions of the unknown parameters are known. The basic result of this section is not new, but it is expressed in a more convenient form and the derivation is simpler than what appears to be available in the literature. Section 3 is devoted to simplification of the required expectations, a necessary step in the numerical implementation of the theoretical results of section 2. In section 4, the evaluations of the expectations are explicitly stated, by both Bayesian and classical statistical methods. Section 5 is an attempt to compare the optimal control equations and the associated welfare cost for a system having random parameters with those prevailing when the parameters are reduced to constants.

2. Optimal Control Equations for Linear Systems with Random Parameters

It is assumed that the system is linear,

$$y_t = A_1 y_{t-1} + \dots + A_m y_{t-m} + C_o x_t + \dots + C_n x_{t-n} + B w_t + u_t \tag{2.1}$$

and that $A_1, \dots, A_m$, $C_o, \dots, C_n$, B are unknown parameters which are to be treated as random with a joint density function as yet to be specified. y_t is a vector of p dependent variables and x_t is a vector of q control variables or instruments. w_t is a vector of r exogenous variables not subject to control -- as a special case, it may consist of only the dummy variable equal to one. $B w_t$ will also be denoted by b_t for convenience. u_t is a p-variate random vector, normally distributed with mean zero and covariance matrix V (also unknown), and uncorrelated in time and with the parameters $A_1, \dots, C_n$, B.

To simplify our derivation of optimal control equations, let the system (2.1) be written as

$$
(2.2)\quad
\begin{bmatrix} y_t \\ \hline y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-m+1} \\ \hline x_t \\ \hline x_{t-1} \\ \vdots \\ x_{t-n+1} \end{bmatrix}
=
\left[\begin{array}{cccc|c|cc|c}
A_1 & A_2 & A_3 & \cdots & A_m & C_1 & \cdots & C_n \\ \hline
I & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & I & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & & & & & & & \\
0 & 0 & 0 & \cdots I & 0 & 0 & \cdots & 0 \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline
0 & 0 & 0 & \cdots & 0 & I & 0 & 0 \\
 & & & \cdots & & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots I & 0
\end{array}\right]
\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-m} \\ \hline x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-n} \end{bmatrix}
+
\begin{bmatrix} C_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline I \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_t
+
\begin{bmatrix} b_t \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
+
\begin{bmatrix} u_t \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

or, in a more compact notation (with y_t, b_t and u_t redefined),

$$(2.3)\qquad y_t = Ay_{t-1} + Cx_t + b_t + u_t \quad .$$

The welfare cost is assumed to be quadratic,

$$(2.4)\qquad W = E_0 \sum_{t=1}^{T} (y_t - a_t)'K_t(y_t - a_t)$$

$$= E_0 \sum_{t=1}^{T} (y_t' K_t y_t - 2\, y_t' K_t a_t) + \text{const.}$$

where a_t are prescribed targets, K_t are given symmetric, positive semi-definite matrices -- K_t are usually diagonal, with non-zero elements corresponding only to the two subvectors y_t and x_t in the notation of (2.2) -- and E_0 is expectation conditional on all the information available at the end of period 0 .

The problem of optimal control is to minimize the expected welfare cost (2.4), given the linear system (2.3). The method of dynamic programming of Bellman (1957) will be applied to this problem. By this method, one first solves the problem for the last period, given the information up to the end of period T-1 . Thus one minimizes

(2.5) $$W_T = E_{T-1}(y_T' K_T y_T - 2 y_T' K_T a_T)$$

$$= E_{T-1}(y_T' H_T y_T - 2 y_T' h_T + c_T)$$

where, in anticipation of generalization to the multiperiod problem, we have let

(2.6) $$H_T = K_T ; \quad h_T = K_T a_T ; \quad c_T = 0 .$$

Substituting the system (2.3) for y_T in (2.5), and taking expectations, we have

(2.7) $$W_T = E_{T-1}(Ay_{T-1} + Cx_t + b_T)'H_T(Ay_{T-1} + Cx_T + b_T) + E_{T-1}u_T' H_T u_T$$

$$- 2 E_{T-1}(Ay_{T-1} + Cx_T + b_T)'h_T + E_{T-1} c_T$$

$$= E_{T-1}(Ay_{T-1} + b_T)'H_T(Ay_{T-1} + b_T) + x_T' E_{T-1}(C'H_T C)x_T$$

$$+ 2x_T' E_{T-1} C'H_T(Ay_{T-1} + b_T) + E u_T' H_T u_T$$

$$- 2 E_{T-1}(Ay_{T-1} + b_T)'h_T - 2 x_T'(E_{T-1} C')h_T + E_{T-1} c_T .$$

Minimization of (2.7) with respect to x_T by differentiation

$$(2.8) \qquad \frac{\partial W_T}{\partial x_T} = 2\, E_{T-1}(C'H_TC)x_T + 2\, E_{T-1}(C'H_TA)y_{T-1} + 2\, E_{T-1}(C'H_Tb_T) - 2(E_{T-1}C')h_T = 0$$

yields

$$(2.9) \qquad \hat{x}_T = G_T\, y_{T-1} + g_T$$

where

$$(2.10) \qquad G_T = -(E_{T-1}C'H_TC)^{-1}(E_{T-1}C'H_TA)$$

$$(2.11) \qquad g_T = -(E_{T-1}C'H_TC)^{-1}[(E_{T-1}C'H_Tb_T) - (E_{T-1}C')h_T] .$$

The optimal feedback control equation (2.9) may appear to be a linear function of y_{T-1}, but this in general is not the case because G_T and g_T, insofar as they depend on the conditional expectations as of the end of period $T-1$, are functions of $y_{T-1}, y_{T-2}, \ldots, y_1$, and of $x_{T-1}, x_{T-2}, \ldots, x_1$. However, if one is willing to approximate the joint density of (A,C,B) as of $T-1$ by their density as of the end of period 0, thus ignoring possible revisions of the density by observations on y_t and x_t from period 1 on, the feedback control equation (2.9) can be treated as linear. This is the approximation to be taken in this paper. Since the optimal policy thus derived can be improved upon by better approximation, the result of this paper will provide a lower bound to the value of control, or an upper bound to the loss arising from uncertainty in the parameters.

The minimum welfare cost for the last period is obtained by substituting (2.9) for x_T in (2.7),

$$(2.12) \quad \hat{W}_T = E_{T-1}[(A+CG_T)y_{T-1} + b_T + Cg_T]'H_T[(A+CG_T)y_{T-1} + b_T + Cg_T]$$
$$+ E_{T-1}u_T'H_Tu_T - 2\,E_{T-1}[(A+CG_T)y_{T-1} + b_T + Cg_T]'h_T + E_{T-1}c_T$$
$$= y_{T-1}'\,E_{T-1}(A+CG_T)'H_T\,(A+CG_T)y_{T-1} + 2\,y_{T-1}'E_{T-1}(A+CG_T)'(H_Tb_T - h_T)$$
$$+ E_{T-1}(b_T+Cg_T)'H_T(b_T+Cg_T) + E_{T-1}u_T'\,H_T\,u_T - 2\,E_{T-1}(b_T+Cg_T)'h_T$$
$$+ E_{T-1}\,c_T \;.$$

Again, $\hat{W}_T$ can be treated as a quadratic function of y_{T-1} if the conditional density of A, C and B as of $T-1$ is assumed to be independent of $y_{T-1},\ldots,y_1$ and $x_{T-1},\ldots,x_1$.

Now, consider including the period $T-1$ in our optimization problem. By the principle of optimality of Bellman (1957),

$$(2.13) \quad \min_{x_T,x_{T-1}} W_{T-1} = \min_{x_T,x_{T-1}} E_{T-2}(W_T + y_{T-1}'\,K_{T-1}y_{T-1} - 2y_{T-1}'K_{T-1}\,a_{T-1})$$
$$= \min_{x_{T-1}} E_{T-2}(\hat{W}_T + y_{T-1}'K_{T-1}y_{T-1} - 2\,y_{T-1}'\,K_{T-1}a_{T-1}),$$

and, using (2.12) for $\hat{W}_T$, we have

$$\text{(2.14)} \quad \min_{w_T, x_{T-1}} W_{T-1} = \min_{x_{T-1}} E_{T-2}(y'_{T-1} H_{T-1} y_{T-1} - 2 y'_{T-1} h_{T-1} + c_{T-1})$$

where

$$\text{(2.15)} \quad H_{T-1} = K_{T-1} + E_{T-1}(A + CG_T)'H_T(A + CG_T)$$

$$= K_{T-1} + E_{T-1}(A'H_T A) + G'_T(E_{T-1}C'H_T A)$$

$$\text{(2.16)} \quad h_{T-1} = K_{T-1}a_{T-1} + E_{T-1}(A + CG_T)'(h_T - H_T b_T)$$

$$= K_{T-1}a_{T-1} + E_{T-1}(A+CG_T)'h_T - E_{T-1}(A'H_T b_T) - G'_T(E_{T-1}C'H_T b_T)$$

$$\text{(2.17)} \quad c_{T-1} = E_{T-1}(b_T + Cg_T)'H_T(b_T + Cg_T) - 2\, E_{T-1}(b_T + Cg_T)'h_T +$$

$$+ E_{T-1}\, u'_T H_T u_T + E_{T-1}\, c_T\,.$$

The minimization problem of (2.14) is seen to be identical with that of (2.5), with T-1 replacing T. The solution will therefore take the form of (2.9), with supplementary equations (2.10) and (2.11), and with T-1 replacing T in these equations. Once $\hat{x}_{T-1}$ is found, one can evaluate $\hat{W}_{T-1}$ as in (2.12), and proceed to include the additional period T-2 in the optimization problem by

$$(2.18) \quad \min_{x_{T-2}} E_{T-3}(\hat{W}_{T-1} + y'_{T-2}K_{T-2}y_{T-2} - 2\, y'_{T-2}K_{T-2}\, a_{T-2})$$

as in (2.13). The process can be continued in this fashion until the optimal x_1 is found.

To recapitulate, the optimal feedback control equation for each period is linear in y_{t-1} as given by (2.9) _if_ the conditional expectations required in evaluating the coefficients G_t and g_t in (2.10) and (2.11) can be computed independently of y_{t-1}, $y_{t-2}, \ldots, x_{t-1}, x_{t-2}, \ldots$. Under this assumption, we compute G_T, H_{T-1}, $G_{T-1}, \ldots$ backward in time using the pair of equations (2.10) and (2.15), and the initial condition(2.6). Similarly, we compute g_T, $h_{T-1}, g_{T-1}, \ldots$ backward in time using the pair of equations (2.11) and (2.16). In these computations, it is essential to evaluate the conditional expectations. We turn to this subject in the next two sections.

3. Simplification of Conditional Expectations

Inspection of equations (2.10), (2.11), (2.15) and (2.16) reveals that the expectations required are $E(C'HC)$, $E(C'HA)$, $E(C'Hb_t)$, $E(A'HA)$ and $E(A'Hb_t)$. The time subscript for E is dropped because, by the approximation of this paper, all expectations are conditional on information as of the end of period 0 . The time subscript for H will hereafter be suppressed. Before evaluating these expectations, it will be convenient to express them in terms only of the random parameters in A, C, and B, matrices that contain many known constants. To do so, let us introduce the symbols

(3.1) $$\Pi = (\Pi_1\ \Pi_2\ \Pi_3)$$

for the coefficients of the system (2.1), where

(3.2) $$\Pi_1 = (A_1 \ldots A_m,\ C_1, \ldots C_n)$$

(3.3) $$\Pi_2 = C_o$$

(3.4) $$\Pi_3 = B$$

Using these symbols, one can write the matrices A, B and b_t of (2.3) as

(3.5) $$A = \begin{bmatrix} \multicolumn{4}{c}{\Pi_1} \\ \hline O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix} + \begin{bmatrix} O & O & O & O \\ \hline I & O & O & O \\ O & O & O & O \\ O & O & I & O \end{bmatrix}$$

(3.6) $$C = \begin{bmatrix} \Pi_2 \\ O \\ I \\ O \end{bmatrix}$$

$$(3.7) \qquad b_t = \begin{bmatrix} \Pi_3 w_t \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Following the partitions of (3.5), (3.6), and (3.7), we partition the matrix H as

$$(3.8) \qquad H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}$$

where H_{11}, H_{22}, H_{33}, and H_{44} have, respectively, p, $(m-1)p$, q, and $(n-1)q$ rows or columns.

Multiplications of the above partitioned matrices give

$$(3.9) \qquad C'HC = \Pi_2' H_{11} \Pi_2 + \Pi_2' H_{13} + H_{31} \Pi_2 + H_{33}$$

$$(3.10) \qquad C'HA = \Pi_2' H_{11} \Pi_1 + \Pi_2'(H_{12} \; 0 \; H_{14} \; 0) + H_{31} \Pi_1 + (H_{32} \; 0 \; H_{34} \; 0)$$

$$(3.11) \qquad C'Hb_t = \Pi_2' H_{11} \Pi_3 w_t + H_{31} \Pi_3 w_t$$

$$(3.12)\quad A'HA = \Pi_1' H_{11} \Pi_1 + \Pi_1'(H_{12}\ 0\ H_{14}\ 0) + (H_{12}\ 0\ H_{14}\ 0)'\Pi_1$$

$$+ \begin{bmatrix} H_{22} & 0 & H_{24} & 0 \\ 0 & 0 & 0 & 0 \\ H_{42} & 0 & H_{44} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(3.13)\qquad A'Hb_t = \Pi_1' H_{11} \Pi_3 w_t \ .$$

Our task is to evaluate the expectations of (3.9) to (3.13) in terms of the first two moments of the elements of Π .

Denote the mean of Π by $\bar{\Pi}$. Letting the $s = (pm + qn) + q + r$ columns of Π be $\pi_1 \ldots \pi_s$, we write the ps elements of Π as a column vector π . Denote the covariance matrix of π by Q , so that

$$(3.14)\quad E\,\pi\pi' = E\,\bar{\pi}\bar{\pi}' + Q = \begin{bmatrix} \bar{\pi}_1\bar{\pi}_1' & \cdots & \bar{\pi}_1\bar{\pi}_s' \\ & \cdots & \\ \bar{\pi}_s\bar{\pi}_1' & \cdots & \bar{\pi}_s\bar{\pi}_s' \end{bmatrix} + \begin{bmatrix} Q_{11} & \cdots & Q_{1s} \\ & \cdots & \\ Q_{s1} & \cdots & Q_{ss} \end{bmatrix} .$$

We proceed to evaluate the expectations of the leading terms of (3.9) to (3.13), and note that they are submatrices of $E(\Pi' H_{11} \Pi)$ according to the partition of Π by (3.1). The i-j element of this matrix is

$$(3.15) \quad E(\Pi' H_{11} \Pi)_{ij} = E\, \pi_i' H_{11} \pi_j = E\, \mathrm{tr}(H_{11} \pi_j \pi_i')$$

$$= \mathrm{tr}\, H_{11} E\, \pi_j \pi_i' = \bar{\pi}_i' H_{11} \bar{\pi}_j + \mathrm{tr}\, H_{11} \Omega_{ji} \quad .$$

Thus, the required expectations of (3.9) to (3.13) can be easily computed once the mean and the covariance matrix of π are given.

4. Mean and Covariance Matrix of the Unknown Parameters

In this section, we will provide two methods of evaluating the mean and covariance matrix of π . The first is Bayesian. The second is an approximate method utilizing an asymptotic distribution of the structural parameters from which the reduced-form parameters π are derived.

We assume that, prior to the control process, N observations on the system (2.1) have been available. The data are arranged in two matrices. Y is an $N \times p$ matrix consisting of columns of N observations on the p dependent variables. Z is an $N \times s$ matrix consisting of columns of observations on all the explanatory variables. The system (2.1) is written as

$$(4.1) \qquad Y = Z\Pi' + U .$$

We will first take a Bayesian approach to derive the joint density of Π .[1] Assume that each row of U in (4.1) has a p-variate normal density with mean zero and covariance matrix $V = R^{-1}$, and that the N rows of U are uncorrelated. The pdf of Y will be[2]

$$(4.2) \qquad p(Y|\Pi,R) \propto |R|^{\frac{1}{2}N} \exp\{-\tfrac{1}{2}\operatorname{tr} R(Y'-\Pi Z')(Y-Z\Pi')\} \quad .$$

Assume also that the prior density of the parameters (Π,R) is diffuse, and is represented by

$$(4.3) \qquad p(\Pi,R) \propto |R|^{-\frac{1}{2}(p+1)} \quad .$$

The posterior density of $(\Pi.R)$ will then be

$$(4.4) \qquad p(\Pi,R|Y) \propto |R|^{\frac{1}{2}(N-p-1)} \exp\{-\tfrac{1}{2}\operatorname{tr} R(Y'-\Pi Z')(Y-Z\Pi')\} \quad .$$

Using the identity

$$(4.5) \qquad (Y'-\Pi Z')(Y-Z\Pi') = S + (\hat{\Pi}-\Pi)Z'Z(\hat{\Pi}-\Pi)'$$

where

$$(4.6) \qquad \hat{\Pi}' = (Z'Z)^{-1} Z'Y$$

and

$$(4.7) \qquad S = (Y'-\hat{\Pi}Z')(Y-Z\hat{\Pi}') ,$$

we rewrite (4.4) as

$$(4.8) \qquad p(\Pi,R|Y) \propto |R|^{\frac{1}{2}s} \exp\{-\tfrac{1}{2}\operatorname{tr} R(\Pi-\hat{\Pi})Z'Z(\Pi-\hat{\Pi})'\}$$

$$\times \; |R|^{\frac{1}{2}(N-p-1-s)} \exp\{-\tfrac{1}{2}\operatorname{tr} R\, S\}$$

which is a normal-Wishart density, being the product of a normal density and a Wishart density.[3] (4.8) implies that $p(\Pi|R,Y)$ is normal; the mean of Π is $\hat{\Pi}$ and the covariance matrix of the s columns $\pi_1 \ldots \pi_s$ of Π is $(Z'Z)^{-1} \otimes R^{-1}$. It also implies that $p(R|Y)$ is a Wishart density with parameter set $(S, N-s)$.

The mean and covariance matrix of the elements of Π can be obtained by using the density (4.8) and integrating first with respect to Π. The mean of Π is simply $\hat{\Pi}$, since integration with respect to R using the Wishart density does not involve Π. For the covariance matrix, we have

$$(4.9) \quad \text{Cov}\, \pi = \int [(Z'Z)^{-1} \otimes R^{-1}]\, p(R|Y)\, dR = k(Z'Z)^{-1} \otimes S$$

since $E\,R^{-1}$ is known to be kS, with $k = (N-s-p-1)^{-1}$, if R is a $p \times p$ matrix having a Wishart distribution with parameter set $(S, N-s)$.[4]

By the use of (4.6) for $\bar{\Pi}$ and (4.9) for Q in (3.14), we can simplify the expression for $E(\Pi' H_{11} \Pi)$. Let $k(Z'Z)^{-1} = (c_{ij})$. (4.9) then implies that $Q_{ij} = c_{ij} S$, so that (3.15) becomes

$$(4.10) \quad E(\Pi' H_{11} \Pi)_{ij} = \hat{\pi}_i' H_{11} \hat{\pi}_j + (\text{tr}\, H_{11} S) c_{ji} ,$$

or,

$$(4.11) \quad E(\Pi' H_{11} \Pi) = \hat{\Pi}' H_{11} \hat{\Pi} + k(\text{tr}\, H_{11} S)\, (Z'Z)^{-1}$$

which contains submatrices for the expectations of the leading terms of (3.9) to (3.13).

The Bayesian result (4.11) is exact, but it does not take into account the non-linear restrictions on the elements of Π induced by the over-identifying restrictions, if any, on the parameters $(\mathbb{B}\ \Gamma)$ of the structure

$$(4.12) \qquad Y\mathbb{B}' = Z\Gamma' + E$$

from which the system (2.1) might have been derived, with

$$(4.13) \qquad \Pi = -\mathbb{B}^{-1}\Gamma \ .$$

The following approach is approximate, but it does incorporate the over-identifying restrictions. Let $(\tilde{\mathbb{B}}\ \tilde{\Gamma})$ be consistent and asymptotically unbiased estimates of $(\mathbb{B}\ \Gamma)$, and let the $(p+s)$ columns of these estimates have an asymptotic covariance matrix W. Then, using Theorem 1 of Goldberger, Nagar and Odeh (1961), the columns of the reduced-form estimates $\tilde{\Pi} = -\tilde{\mathbb{B}}^{-1}\tilde{\Gamma}$ will have an asymptotic covariance matrix which may be approximated by

$$(4.14) \quad \tilde{Q} = \{[-\tilde{\Pi}'\ \ I_s] \otimes \tilde{\mathbb{B}}^{-1}\}\ W\ \{[-\tilde{\Pi}'\ \ I_s] \otimes \tilde{\mathbb{B}}^{-1}\}'$$

where I_s is an identity matrix of order s. If one is willing to interpret $\tilde{Q}$ in (4.14) as a covariance matrix of the random parameter Π around the constant $\tilde{\Pi}$, rather than as a covariance matrix of the random estimates $\tilde{\Pi}$, then using $\tilde{\Pi}$ for Π and $\tilde{Q}$ for Q in (3.15) will provide an approximate solution to the evaluation of expectations required in our optimal control problem.

5. Comparison with the Certainty Case

In this section we will attempt to compare the optimal control solution for the random parameter case with the solution prevailing under the assumption of known parameters. What will happen to the solution when the random parameters degenerate into constants? Or, to put it in the opposite way, what will happen when randomness is introduced into the otherwise constant parameters? Two parts of the optimal solution will be compared. They are the optimal feedback equation and the optimal welfare cost.

If the random parameters in system (2.3) are reduced to their means for the certainty case, we simply replace A, C, and B in our solution of section 2 by the mean values. The optimal feedback control coefficients G_t and g_t, as given by (2.10) and (2.11) with t replacing T, can thus be compared with the corresponding coefficients in the certainty case when A becomes $\bar{A}$, etc. Similarly, the optimal welfare cost $\hat{W}_1$, as given by (2.12) with 1 replacing T, can also be compared with the corresponding cost in the certainty case. Clearly, the analytical results of the previous sections can be used to compute the solutions in both the certainty and the uncertainty situations for the purpose of comparison. In the remainder of this section, we ask whether some qualitative results in such a comparison can be ascertained.

To facilitate comparison, we first rewrite the optimization problem of section 2 and its solution in slightly simplified forms. This involves introducing new variables in the quadratic welfare function to eliminate its linear terms, and to make the optimal control equation linear homogenous, eliminating the intercept g_t . Letting

$$(5.1) \qquad z_t = \begin{bmatrix} y_t \\ a_t \\ w_{t+1} \end{bmatrix} \quad Q_t = \begin{bmatrix} K_t & -K_t & 0 \\ -K_t & K_t & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

one can rewrite the welfare cost (2.4) as

$$(5.2) \qquad W = E_0 \sum_{t=1}^{T} z_t' Q_t z_t .$$

Writing, without loss, $a_t = P_t a_{t-1}$ and $w_t = D_t w_{t-1}$, and letting

$$\alpha_t = \begin{bmatrix} A & 0 & B \\ 0 & P_t & 0 \\ 0 & 0 & D_t \end{bmatrix} \quad \Gamma = \begin{bmatrix} C \\ 0 \\ 0 \end{bmatrix} \quad \nu_t = \begin{bmatrix} u_t \\ 0 \\ 0 \end{bmatrix} ,$$

we have, in place of (2.3),

$$(5.4) \qquad z_t = \alpha_t z_{t-1} + \Gamma x_t + \nu_t .$$

Following the dynamic programming approach of section 2, one easily finds, analogous to (2.9), (2.10), (2.12), (2.15) and (2.17) respectively,

$$(5.5) \qquad \hat{x}_t = G_t z_{t-1}$$

(5.6) $$G_t = -(E_{t-1}\,\Gamma' H_t \Gamma)^{-1}\,(E_{t-1}\Gamma' H_t \alpha_t)$$

(5.7) $$\hat{w}_t = z'_{t-1}\,E_{t-1}(\alpha_t + \Gamma G_t)' H_t\,(\alpha_t + \Gamma G_t) z_{t-1} + c_{t-1}$$

(5.8) $$H_{t-1} = Q_{t-1} + E_{t-1}(\alpha_t + \Gamma G_t)' H_t\,(\alpha_t + \Gamma G_t)$$

(5.9) $$c_{t-1} = E_{t-1}\,\nu'_t H_t \nu_t + E_{t-1} c_t$$

with initial conditions $H_T = Q_T$ and $c_T = 0$.

Having rewritten our solution, we will try to compare the optimal feedback coefficients G_t and the optimal expected welfare cost $\hat{W}_0$ with the corresponding results for the certainty case. In the certainty case, the random coefficients α_t and Γ in (5.3) are assumed to reduce to their mean values $\bar{\alpha}_t$ and $\bar{\Gamma}$. Let α^* and Γ^* denote the deviations of α and Γ from their means. Some elements of α^* and Γ^* are obviously zero. Consider the problem for the last period T . From (5.6), we have, for the uncertainty case,

(5.10) $$G_T = -(\bar{\Gamma}' H_T \bar{\Gamma} + E\Gamma^{*\prime} H_T \Gamma^*)^{-1}\,(\bar{\Gamma}' H_T \bar{\alpha}_T + E\Gamma^{*\prime} H_T \alpha^*)$$

whereas, in the certainty case, the above reduces to

(5.11) $$G_T = -(\bar{\Gamma}' H_T \bar{\Gamma})^{-1}\,(\bar{\Gamma} H_T \bar{\alpha}_T)\ .$$

What can be said about the relative magnitudes of $\bar{G}_T$ and G_T ?

We can begin by making two elementary observations. First, as it is well-known, if the coefficients α_t and Γ are subject to uncertainty, the principle of "certainty equivalence" does not apply.

According to this principle, one would set the instruments by the rule obtained from replacing α_t and Γ by $\bar{\alpha}_t$ and $\bar{\Gamma}$. Clearly, the contrast of (5.10) and (5.11) shows that this is not the case. Secondly, if the number of instruments q is greater than the number of target variables (which equals the rank of $H_T = Q_T$), one can select a subset of instruments to achieve an optimal policy in the certainty case. This can be observed from equation (5.11) when the rank of $\bar{\Gamma}'H_T\bar{\Gamma}$, a q by q matrix, is less than q ; we would use a generalized inverse of this matrix, but the optimal $\bar{G}_T$ would not be unique and would have rank smaller than q . As Brainard (1967) has suggested, more and possibly all instruments will be required in the uncertainty case even if there are more instruments than targets. This point can be easily seen by noting that the rank of $\bar{\Gamma}'H_T\bar{\Gamma} + E\Gamma^{*}{}'H_T\Gamma^{*}$ in (5.10) will in general be greater than the rank of $\bar{\Gamma}'H_T\bar{\Gamma}$ (5.11) when the latter is smaller than q .

After making these two elementary observations, one may ask whether uncertainty will call forth smaller policy responses to recent economic data as manifested by the smaller magnitudes of the elements of G_T than $\bar{G}_T$. For example, can one say that the squared length of each column of $\bar{G}_T$ in the certainty case is necessarily larger than that of the corresponding column of G_T ? This assertion would mean more policy response to each observed variable in the certainty case. To answer this question, one notices that the model for (5.11) is mathematically identical with the multivariate regression model

(5.12) $\bar{\alpha}_T = \bar{\Gamma}(-\bar{G}_T) + \bar{R}$

where the columns of $\bar{\alpha}_T$ are the dependent variables, the columns of $\bar{\Gamma}$ are explanatory variables, and the columns of $-\bar{G}_T$ are the regression coefficients obtained by generalized least squares. α^* and Γ^* can be regarded as measurement errors, yielding α_T and Γ as observed variables, and $-G_T$ as regression coefficients given by (5.10). Consider the very special case of only one dependent variable and one piece of datum for the instrument to respond to (so that α_T becomes a scalar). Let $H_T = 1$. The optimal feedback coefficient $\bar{G}_T$ under certainty will be larger than G_T if the errors Γ^* and α^* are uncorrelated. However, the deduction of a smaller policy response to economic data under uncertainty may not even be valid for this very special case if Γ^* and α^* are correlated. If $\bar{\alpha}_T > 0$, $\bar{\Gamma} > 0$, and thus $\bar{G}_T = -(\bar{\alpha}_T/\bar{\Gamma}) < 0$, a positive covariance between Γ^* and α^* can make G_T bigger than $\bar{G}_T$.

If the system (5.4) has p dependent variables, the p elements of each column of $\bar{\alpha}_T$ will be explained by the q columns of $\bar{\Gamma}$, with q regression coefficients given by the corresponding column of $-\bar{G}_T$. In this multiple regression situation, if the explanatory variables $\bar{\Gamma}$ are measured with errors Γ^*, and the dependent variable is also measured with errors (the corresponding column of α^*), it is not true in general that each column of $-\bar{G}$

will have greater (or equal) length than the corresponding column of $-G_T$. Even if we assume that the columns of Γ^* are uncorrelated with those of α^*, i.e., $E\,\Gamma^{*\prime}H_T\alpha^* = 0$, we still cannot deduce a greater or equal length for the columns of $\bar{G}_T$. If $E\Gamma^{*\prime}H_T\alpha^* = 0$, (5.10) and (5.11) imply

$$(5.13)\quad \bar{G}_T'\bar{G}_T = G_T'[I + (\bar{\Gamma}'H_T\bar{\Gamma})^{-1}(E\Gamma^{*\prime}H_T\Gamma^*)]'[I + (\bar{\Gamma}'H_T\bar{\Gamma})^{-1}(E\Gamma^{*\prime}H_T\Gamma^*)]G_T$$

$$= G_T'G_T + G_T\{(\bar{\Gamma}'H_T\bar{\Gamma})^{-1}(E\Gamma^{*\prime}H_T\Gamma^*) + (E\Gamma^{*\prime}H_T\Gamma^*)(\bar{\Gamma}'H_T\bar{\Gamma})^{-1}$$

$$+ (E\Gamma^{*\prime}H_T\Gamma^*)(\bar{\Gamma}'H_T\bar{\Gamma})^{-1}(\bar{\Gamma}'H_T\bar{\Gamma})^{-1}(E\Gamma^{*\prime}H_T\Gamma^*)\}G_T\ .$$

The matrix in curly brackets is not in general positive semidefinite, so that the diagonal elements of $\bar{G}_T'\bar{G}_T$ is not necessarily greater than or equal to the diagonal elements of $G_T'G_T$, although this may often turn out to be the case for specific applications. The deduction would be valid if, in addition to $E\Gamma^{*\prime}H_T\alpha^* = 0$, both $\bar{\Gamma}'H_T\bar{\Gamma}$ and $E\Gamma^{*\prime}H_T\Gamma^*$ were diagonal, but this is a very special case indeed. Intuitively, one reason for a possibly larger policy response in the uncertainty situation is that, while the variances in Γ^* per se may lead to reduction in the magnitudes of G_T, the covariances between Γ^* and α^* can be exploited in the design of active control policies.

Next to be studied is the optimal welfare cost $\hat{W}_T$. By (5.7) and (5.9), it is a quadratic form in the variables z_{T-1} (the given data of the feedback control equation), plus a constant. If uncertainty is to imply no reduction in expected welfare cost for the last period, given any initial conditions z_{T-1}, the matrix

$$(5.14)\quad E(\alpha_T + \Gamma G_T)'H_T(\alpha_T + \Gamma G_T) - (\bar{\alpha}_T + \bar{\Gamma}_T\bar{G}_T)'H_T(\bar{\alpha}_T + \bar{\Gamma}_T\bar{G}_T)$$

$$= E\alpha^{*\prime}H_T\alpha^* + G_T'(\bar{\Gamma}'H_T\bar{\Gamma} + E\Gamma^{*\prime}H_T\Gamma^*)G_T - \bar{G}_T'\bar{\Gamma}'H_T\bar{\Gamma}\bar{G}_T$$

has to be positive semi-definite. (5.14) can be identified as the difference between the covariance matrix of the (weighted) multivariate regression residuals when measurement errors exist and the covariance matrix when the errors are absent. Errors in the dependent variables α_T^* alone will make the former matrix bigger by $E\alpha_T^{*\prime}H_T\alpha_T^*$. The remaining two matrices on the right-hand side of (5.14) are the covariance matrices of the explained parts of the regressions for the error and no-error cases. There is no guarantee that the difference between these two matrices are positive semi-definite. As the related study of Cochran (1970) suggests, without very special assumptions, it is difficult to ascertain a net increase in the variance of the residuals of a multiple regression as a result of measurement errors, although errors in the dependent variable alone will tend to cause such an increase. However, if uncertainty does increase expected welfare cost in period T through the positive semi-definiteness of (5.14), the effect will tend to accumulate backwards to the total expected welfare cost computed in period 1, the process of accumulation being given by equations (5.7) and (5.8).

In summary, this paper has provided an analytical solution of the optimal feedback control equation and the associated expected (quadratic) welfare cost when the parameters of the linear econometric model employed are uncertain. The solution can be used to study the effects of uncertainty by comparison with the result when all parameters are reduced to constants. However, it seems difficult to ascertain *a priori* qualitative results concerning such a comparison, although the partial effects of certain factors have been pointed out. By ignoring the possibility of reducing uncertainty through observations during the control process, this study exagerates, and thus sets an upper limit to, the effect of uncertainty on the optimal control policy and the associated welfare cost.

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FOOTNOTES

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1. The density of Π from a Bayesian point of view is well-known and can be found in Zellner (1971), for example, although the second moment of Π is not so well-known. For the latter, see Chow (1971).

2. Here, and in what follows, the initial conditions of the system and all values of the exogenous variables are assumed to be given in specifying the density of Y .

3. See Anderson (1958), p. 182 and p. 154 respectively, for definitions of these two densities.

4. See Kaufman (1967), p. 14.