On the Convergence of the Age Structure

by

Martin Golubitsky and Michael Rothschild

I. Introduction and Summary.

Let $\{P(t)\}$, $t = 1,2,\ldots$, be a sequence of non-singular, population matrices. A typical element is

$$(1)\quad P(t) = \begin{pmatrix} b_1(t) & b_2(t) & \cdots & & b_n(t) \\ s_2(t) & 0 & \cdots & & 0 \\ 0 & s_2(t) & \cdots & & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & s_{n-1}(t) & 0 \end{pmatrix}$$

where $b_i(t) \geq 0$ and $s_i(t) > 0$ are birth and survival rates for the i^{th} age group. If the non-negative vector a^o represents the initial age structure of the population (the number of individuals in each age group) then the age structure at time t is given by the normalized vector $a(t)$ where

(2.i) $$a(0) = \frac{a^o}{\|a^o\|}$$

(2.ii) $$a(t) = \frac{P(t)a(t-1)}{\|P(t)a(t-1)\|}$$

and $\| \ \|$ is a norm on $\mathbb{R}^n$. The strong ergodic theorem of stable population theory[1] states that if the projection matrix is constant ($P(t) = P$ for all t) then the age structure of the population converges to a limit which is independent of a^o.

Obviously weaker conditions will ensure the convergence of the age structure. For example, if $P(t) = c(t)P$, where $c(t)$ is any divergent sequence of numbers in the interval $\left(\frac{1}{4}, \frac{3}{4}\right)$, $a(t)$ converges. However conditions which are both necessary and sufficient for the convergence of $\{a(t)\}$ do not seem to be known. This paper is part of an attempt to find some. At this point we report only partial success. We have found necessary and sufficient conditions for convergence but they are not the same. Our sufficient conditions (although weaker than the conditions of the strong ergodic theorem) are clearly not necessary while we have been unable to prove that our necessary conditions are sufficient.

Section II discusses sufficient conditions for the convergence of the age structure. Theorem 1 shows that if the

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ECONOMETRIC RESEARCH PROGRAM
Research Memorandum No. 152
October 1973

Econometric Research Program
PRINCETON UNIVERSITY
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Princeton, New Jersey

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* We are grateful to Michael Shub and Harold Stark for discussions which led to the discovery of the proof of Theorem I. Martin Feldstein, Jane Menken, and Linda Rothschild were also very helpful. Without implication, we thank the National Science Foundation, the Industrial Relations Section of Princeton University and the Institute for Mathematical Studies in the Social Sciences at Stanford University for support.

sequence of projection matrices converges so does the age structure. A by-product of the proof of Theorem 1 is Proposition 2, which states roughly that if the population projection matrices are approximately constant then age structure will also be approximately constant. Although not surprising, this result is important. If something like it were not true, the strong ergodic theorem of stable population would be of little relevance for empirical work in demography. Section II also contains Theorem 3 - a mathematically more pleasing version of Theorem 1.

Section III is concerned with necessary conditions for the convergence of the age structure. The example above shows that the conditions of Theorem 1 (that $P(t) \to P$) are not necessary. We give another example which will be useful in understanding the necessary conditions given in Theorem 4. Suppose $\{P(t)\}$ is a sequence of projection matrices and that $g(P(t))$ is a positive eigenvector of $P(t)$ of unit length. If $g(P(t)) = \bar{a}$ for all t, then, as Proposition 5 shows, the variation in the survival and fertility rates which make up the entries of $P(t)$ is restricted but not eliminated. However if the age structure vector, $a(t)$, is once again defined by (2) with $a^o = \bar{a}$, then $a(t) = \bar{a}$ for all t. The weak ergodic theorem of Coale and Lopez[2] implies that $a(t)$ converges to $\bar{a}$ for any initial age structure. Necessary conditions for convergence are somewhat weaker than those of this example. Theorem 4 shows that if

$a(t) \rightarrow a$ then the sequence $\{g(P(t))\}$ must converge. This result is related to the work by demographers which shows that the age structure can converge even though certain kinds of changes in mortality and fertility persist.[3] Our theorem shows exactly what continuing changes are permitted if the age structure is to converge. The demographic implications of this theorem are stated in more detail in Proposition 5.

The example of the previous paragraph suggests that the fact that the age structure converges has no implications beyond those stated in Theorem 4. Although we strongly suspect that the convergence of $\{g(P(t))\}$ is sufficient as well as necessary for the convergence of $\{a(t)\}$, we have not been able to prove this conjecture.

Notational Conventions

We use capital roman letters for matrices, small roman letters for vectors and identify components by subscripts. By $x \geq 0$ we denote $x_i \geq 0$ all i, $x > 0$ means $x \geq 0$ but $x \neq 0$ while $x \gg 0$ signifies $x_i > 0$, all i.

The letter P is reserved for population matrices of the form (1) while $\{a(t)\}$ is always a sequence of non-negative age structure vectors, as in (2). We shall always assume that P is non-singular and primitive (there is an integer k such that $P^k \gg 0$). It follows that P has a unique, positive eigenvector of unit length; we refer to this vector as $g(P)$. Much of formal demography (and the theory of non-negative matrices) is concerned with what is required for these conditions to hold and with the consequences of P's failure to be so well behaved. These matters are both well studied and ancillary to our main concern; the modifications which weaker assumptions would require are both tedious and obvious.

The proof of Theorem 1 depends on careful choice of a norm. We use $\|\ \|$ to refer both to this norm, which is constructed in the next section, and to norms in general. Other norms on $\mathbb{R}^n$ and its subspaces are distinguished by subscripts. Recall that if $\|\ \|$ is a norm on $\mathbb{R}^n$, there is a natural norm also, denoted $\|\ \|$, on linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ (and the matrices which represent them) defined by $\|L\| = \sup_{\|x\|=1} \|Lx\|$. It follows that

$$\|Lx\| \leq \|L\|\|x\| \quad \text{for all} \quad x \in \mathbb{R}^n .$$

The notation $\{x(t)\}$ indicates a sequence whose t^{th} term is $x(t)$. Script letters refer to subsets of $\mathbb{R}^n$. In particular $\underset{\sim}{S}$ is the unit sphere in the $\|\ \|$ norm, that is $\underset{\sim}{S} = \{x \mid \|x\| = 1\}$. By int $\underset{\sim}{C}$ we denote the interior of the set $\underset{\sim}{C}$.

II. Sufficient Conditions for the Convergence of the Age Structure.

<u>Theorem 1</u>. If $P(t) \to P$, then $Q(t) \to g(P)$.

Theorem 1 is a consequence of the following result.

<u>Lemma</u>. Let $\{B(t)\}$ be a sequence of non-singular matrices converging to a non-singular matrix B with a real eigenvalue λ greater in modulus than any other eigenvalue of B. Let $\underset{\sim}{E}_\lambda$ be the 1-dimensional eigenspace associated with λ and $\underset{\sim}{V}$ be the sum of the generalized eigenspaces associated with the other eigenvalues of B. Assume

$$x(t) = \frac{B(t)x(t-1)}{\|B(t)x(t-1)\|},$$

where $\| \ \|$ is some norm on $\mathbb{R}^n$. Then if $\underset{\sim}{C}$ is a closed cone such that $\underset{\sim}{E}_\lambda - \{0\} \subset \text{int } \underset{\sim}{C}$ and $\underset{\sim}{C} \cap \underset{\sim}{V} = \{0\}$, there is a T such that if $x(t) \in \text{int } \underset{\sim}{C}$ for $t \geq T$, $x(t)$ converges to a vector in $\underset{\sim}{E}_\lambda \cap \underset{\sim}{S}$.

<u>Proof of Theorem 1</u>: Since P has a largest eigenvalue μ_o $\{P(t)\}$ and P satisfy the hypotheses of the Lemma. Consider the closed cone $\underset{\sim}{C} = \{x |\ x \geq 0 \text{ or } x \leq 0\}$. Since $g(P) \gg 0$, $g(P) \in \text{int } \underset{\sim}{C}$. Since $P(t) \to P$ and P is primitive, there

is a T_1 such that $\prod_{\tau=1}^{t} P(t) \gg 0$ for all $t \geq T_1$. Thus if $t \geq T_1$, $a(t) \gg 0$ and $a(t) \in \underset{\sim}{C}$.

To complete the proof we only must show that $\underset{\sim}{C} \cap \underset{\sim}{V} = \{0\}$ where $\underset{\sim}{V}$ is the sum of the generalized eigenspaces corresponding to the other eigenvalues. Suppose $x \in \underset{\sim}{C} \cap \underset{\sim}{V}$. Since if $x \leq 0$, $-x \geq 0 \in \underset{\sim}{C}$, we may assume $x \geq 0$. Suppose for simplicity that P is diagonalizable. Then if $x \in \underset{\sim}{V}$ there are k $(\leq n-1)$ eigenvectors, $v_{j_1}, \ldots, v_{j_k}$, such that $x = \sum_{i=1}^{k} v_{j_i}$. Let $\tilde{P} = \mu_o^{-1} P_o$. Then

$$\lim_{t\to\infty} \tilde{P}^t x = \lim_{t\to\infty} \sum_{i=1}^{k} (\mu_{j_i}/\mu_o)^t v_{j_i} = 0 \ .$$

However, it is well known[4] that $\lim_{t\to\infty} \tilde{P}^t = Q$ where Q is a strictly positive matrix. Thus $\lim_{t\to\infty} \tilde{P}^t x = Qx$ so that $Qx = 0$. Since $x \geq 0$, it must be that $x = 0$. This argument goes through with the obvious modifications if P is not diagonalizable.

<u>Proof of the Lemma</u>: It will suffice to prove the Lemma for a particular norm; suppose $\| \ \|_1$ and $\| \ \|_2$ are two norms and $x^1(0) = x^2(0) = x^0$ while

$$x^i(t) = \frac{B(t)x^i(t-1)}{\|B(t)x^i(t-1)\|_i}$$

then it is easy to show by induction that

$$x^1(t) = \frac{B(t)x^2(t-1)}{\|B(t)x^2(t-1)\|_1}$$

so that $x^1(t) \to x^1$ if and only if $x^2(t) \to x^2$ where $x^1 = \alpha x^2$ for some number α. The key to the proof lies in choosing a convenient norm. First note that if $\hat{B}(t) = \gamma B(t)$ for some $\gamma > 0$ and if $\hat{x}(t_o) = x(t_o)$ for some t_o and

$$\hat{x}(t) = \frac{\tilde{B}(t)\tilde{x}(t-1)}{\|\tilde{B}(t)\tilde{x}(t-1)\|}$$

then $x(t) = \tilde{x}(t)$ for all $t > t_o$. Thus we may as well assume that $\lambda > 1$ while the other eigenvalues of B, those associated with $\underset{\sim}{V}$, are in modulus less than 1. Consider $B|_{\underset{\sim}{V}}$, the restriction of B to $\underset{\sim}{V}$. $B|_{\underset{\sim}{V}}$ maps $\underset{\sim}{V}$ into $\underset{\sim}{V}$. Since all the eigenvalues of $B|_{\underset{\sim}{V}}$ are less than 1 in modulus, it is straightforward to show that there is an inner product $(\ ,\ )_1$, an associated norm $\|\ \|_1$, and a number σ $(0 < \sigma < 1)$ such that $\|B|_{\underset{\sim}{V}}v\|_1 \leq \sigma\|v\|_1$ for all $v \in \underset{\sim}{V}$. Let e_λ be any non-zero vector in $\underset{\sim}{E}_\lambda$. Extend $(\ ,\ )_1$ to an inner product on $\mathbb{R}^n$ by requiring that $(v,e_\lambda) = 0$ for all $v \in \underset{\sim}{V}$ and $(e_\lambda,e_\lambda) = 1$. Let $\|\ \|$ be the norm determined by $(\ ,\ )$. That is, $\|x\| = (x,x)^{1/2}$.

Let $\underset{\sim}{B}$ be an orthonormal (relative to this norm) basis for $\mathbb{R}^n$ whose first $n-1$ vectors are in $\underset{\sim}{V}$. We write vectors in terms of this basis for the remainder of the proof. Define the function

$$\rho(x) = \begin{cases} 0 & \text{if } x = 0 \\ |x_n|^{-1}\|x_1,\ldots,x_{n-1}\|_1 & \text{if } x_n \neq 0 \\ \infty & \text{if } x \neq 0,\ x_n = 0 . \end{cases}$$

Consider closed cones of the form

$$\underset{\sim}{C}_\alpha = \{x \mid \rho(x) \leq \alpha\} .$$

It is easy to see that if $\underset{\sim}{C}$ is a closed cone such that $\underset{\sim}{E}_\lambda \subset \underset{\sim}{C}$ and $\underset{\sim}{C} \cap \underset{\sim}{V} = \{0\}$, then $\underset{\sim}{C}$ is contained in a cone of this form. Since $\underset{\sim}{C} \cap \underset{\sim}{S}$ is compact and $\rho(\ )$ continuous on $\underset{\sim}{C} \cap \underset{\sim}{S}$, $\rho(\ )$ attains a finite maximum, α, on $\underset{\sim}{C} \cap \underset{\sim}{S}$. For this α, $\underset{\sim}{C} \subset \underset{\sim}{C}_\alpha$.

To prove the proposition it will suffice to show that for every $\alpha > 0$ there is a T_α such that $x(T_\alpha) \in \text{int } \underset{\sim}{C}_\alpha$ implies $x(t)$ converges to a vector in $\underset{\sim}{E}_\lambda$.

Define the matrix $D(t)$ by $B(t) = B + D(t)$. Then $D(t) \to 0$ and $\|D(t)\| \to 0$. Let δ be any number in the open interval $(\lambda^{-1}, 1)$. Choose $\bar{\epsilon} > 0$ such that if $\bar{\epsilon} > \epsilon > 0$ then

$$(3) \qquad \frac{\alpha + (\alpha+1)\epsilon}{\lambda - (\alpha+1)\epsilon} < \delta\alpha < \alpha \ .$$

For some $\epsilon < \bar{\epsilon}$, let T_α be such that $\|D(t)\| < \epsilon$ for all $t \geq T_\alpha$.

Suppose $x \in \operatorname{int} \underset{\sim}{C}_\alpha$. We now show that if $t \geq T_\alpha$, $\rho(B(t)x) < \alpha$ or that $\frac{B(t)x}{\|B(t)x\|} \in \operatorname{int} \underset{\sim}{C}_\alpha$. Since $\rho(B(t)(\beta x)) = \rho(B(t)x)$ for all $\beta \neq 0$, we may assume $x = (x_1,\ldots,x_{n-1},1)$. Let $z = B(t)x = w + y$ where $w = Bx$ and $y = D(t)x$. In the basis we are using $B = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \lambda \end{array}\right)$ so that $(w_1,\ldots,w_{n-1}) = A(x_1,\ldots,x_{n-1})$ while $w_n = \lambda$. We chose $\| \ \|_1$ so that

$$\begin{aligned} \|w_1,\ldots,w_{n-1}\|_1 &= \|A(x_1,\ldots,x_{n-1})\|_1 \\ &\leq \sigma\|x_1,\ldots,x_{n-1}\|_1 \\ &\leq \sigma\alpha < \alpha \ . \end{aligned}$$

Furthermore, it is easy to show that

$$\begin{aligned} \|(y_1,\ldots,y_{n-1})\|_1 &\leq \|D(t)(x_1,\ldots,x_n)\| \\ &\leq \|D(t)\|\,\|x_1,\ldots,x_n\| \\ &\leq \epsilon(\alpha + 1) \end{aligned}$$

and similarly $|y_n| < \epsilon(\alpha + 1)$. Thus

$$\|(z_1,\ldots,z_{n-1})\|_1 \le \|(w_1,\ldots,w_{n-1})\|_1 + \|y_1,\ldots,y_{n-1}\|$$

$$< \alpha + (\alpha + 1)\epsilon$$

while $|z_n| \ge \lambda - (\alpha + 1)$. Thus

$$\rho(z) = |z_n|^{-1}\|(z_1,\ldots,z_{n-1})\|_1 < \frac{\alpha + (\alpha+1)\epsilon}{\lambda - (\alpha+1)\epsilon} < \delta\alpha .$$

Consider the sequence $\beta(t) = \rho(x(t))$. We have shown that for $t > T_\alpha$, $\beta(t) \in [0,\alpha]$ so that $\{\beta(t)\}$ has limit points. We now show that the only limit point that $\beta(t)$ can have is 0. Suppose $\bar{\beta} > 0$ is a limit point. Choose $\hat{\beta}$ such that $\hat{\beta} > \bar{\beta} > \delta\hat{\beta}$ where, as in (3), $\lambda^{-1} < \sigma < 1$. Then, since $\bar{\beta}$ is a limit point of $\{\beta(t)\}$ there is a T_1 such that $\beta(t) < \hat{\beta}$ for all $t \ge t_1$. But then we may use the argument given above to show that $\beta(t) < \delta\hat{\beta} < \bar{\beta}$ for all $t > T_1 + 1$. $\bar{\beta}$ cannot be a limit point of $\{\beta(t)\}$.

This completes the proof of Theorem 1. Notice that our argument has established the following "continuity" result.

<u>Proposition 2</u>. For every $\epsilon > 0$ there are integers T_1 and T_2 and a $\delta > 0$ such that if $\|P(t_1) - P\| < \delta$ for all $t_1 > T_1$ then $\|a(t_2) - g(P)\| < \epsilon$ for all $t_2 > T_2$.

The Lemma can be used to prove a cleaner and mathematically more interesting result.

Theorem 3. Let $\{B(t)\}$, B and $\underset{\sim}{E}_\lambda$ be as in the Lemma. Suppose

$$x(0) = x^o$$

$$x(t) = \frac{B(t)x(t-1)}{\|B(t)x(t-1)\|^{-1}} .$$

Then there is a subspace $\underset{\sim}{U}$ of dimension less than n such that if $x^o \notin \underset{\sim}{U}$ then $\{x(t)\}$ converges to a vector in $\underset{\sim}{E}_\lambda$.

This theorem states that the sequence $\{x(t)\}$ converges to a vector in $\underset{\sim}{E}_\lambda$ for all $x^o \in \mathbb{R}^n$ except perhaps a hyperplane. Since hyperplanes are "small" sets relative to $\mathbb{R}^n$, this means that $x(t)$ converges for almost all initial conditions. This result is of somewhat less demographic interest than Theorem 1 as there seems no direct way of showing that $\underset{\sim}{U} \cap \mathbb{R}^n_+ = \{0\}$. (Of course, the weak ergodic theorem could be used to deduce Theorem 1 from Theorem 3 - or from the Lemma.)

Only a sketch of the proof is given here. Since $\underset{\sim}{S}$ is compact, the sequence $\{x(t)\}$ has a limit point w. If $w \notin \underset{\sim}{V}$ then w is in the interior of some cone $\underset{\sim}{C}$ such that $\underset{\sim}{E}_\lambda \subset \underset{\sim}{C}$ and, $\underset{\sim}{C} \cap \underset{\sim}{V} = \{0\}$. The Lemma implies that $\{x(t)\}$ converges to some $e_\lambda \in \underset{\sim}{E}_\lambda$.

Consider the unnormalized sequence $z(t) = \prod_{\tau=1}^{t} P(t)x^o$. It is possible to show (considering again the case where $\lambda > 1$

and the moduli of all other eigenvalues of B are less than 1) that if $w \in \underset{\sim}{V}$ then $z(t) \to 0$. Clearly the set of all x^o such that $z(t) \to 0$ is a subspace of $\mathbb{R}^n$. It is of dimension less than n since the fact that $\prod_{\tau=1}^{t} P(\tau)$ is non-singular and the Lemma together imply that $x(t) \to e_\lambda$ for some x^o.

III. Necessary Conditions for the Convergence of the Age Structure.

<u>Theorem 4</u>. If $a(t) \rightarrow \bar{a}$, then $g(P(t)) \rightarrow \bar{a}$.

<u>Proof</u>: Consider the normalized sequence of matrices $Q(t) = \frac{P(t)}{\|P(t)\|}$. Since $g(Q(t)) = g(P(t))$ it will suffice to show that $g(Q(t)) \rightarrow a$. Let $\bar{Q}$ be a limit point of $\{Q(t)\}$, and suppose $\{t_j\}$ is a subsequence such that $\lim_{j\to\infty} Q(t_j) = \bar{Q}$. Note that

$$(4) \qquad a(t_j) = \frac{Q(t_j)a(t_j - 1)}{\|Q(t_j)a(t_j - 1)\|} .$$

Taking limits on both sides of (4), we obtain $\bar{a} = \gamma\bar{Q}\bar{a}$ where $\gamma = \|\bar{Q}\bar{a}\|^{-1}$. Thus $\bar{a} > 0$ is an eigenvector of $\bar{Q}$. Since $\bar{Q}$ has only one positive eigenvector of unit length, $g(\bar{Q}) = a$ if $\bar{Q}$ is any limit point of $\{Q(t)\}$. Since the $Q(t)$ are bounded this implies $g(Q(t)) = g(P(t)) \rightarrow \bar{a}$.

<u>Proposition 5</u>. If $a(t) \rightarrow \bar{a}$ and P and P are limit points of $\{P(t)\}$ then there are positive numbers γ and γ such that

$$\gamma \Sigma b_i\bar{a}_i = \hat{\gamma} \Sigma \hat{b}_i\bar{a}_i$$

and

$$\gamma s_j = \hat{\gamma}\hat{s}_j \qquad j = 1,\ldots,n-1$$

where b_i, s_j and $\hat{b}_j$, $\hat{s}_j$ are the age specific birth and survival rates associated with P and $\hat{P}$ respectively.

<u>Proof</u>: Let $\gamma = \|P\bar{a}\|^{-1}$ and $\hat{\gamma} = \|\hat{P}\bar{a}\|^{-1}$. Then, we have shown in the course of proving Theorem 4 that

$$\bar{a} = \hat{\gamma}\hat{P}\bar{a} = \gamma P\bar{a} \; .$$

Writing out this equation component by component yields,

$$\bar{a}_1 = \hat{\gamma} \, \Sigma \, \hat{b}_i \bar{a}_i = \gamma \, \Sigma \, b_i a_i$$

$$\bar{a}_{j+1} = \hat{\gamma}\hat{s}_j \bar{a}_j = \gamma s_j \bar{a}_j$$

which completes the proof.

The demographic meaning of these results are clear. An age structure will approach a constant only if the crude birth rates ($\Sigma \, b_i a_i$) and each age specific survival rate approach constants or if in the limit these rates vary proportionately and simultaneously. The fact that the age structure converges imposes no other restrictions on the asymptotic behavior of the entries of $P(t)$.

We strongly suspect that the conditions of Theorem 4 are necessary as well as sufficient. We have not however been able to prove this.

Footnotes

1. See Parlett [3] and the references cited there.

2. See Parlett [3, pp. 200-202].

3. See, for example, the work of Coale [1].

4. See, for example, Nikaido [2, p. 110].

References

1. Ansley J. Coale, "The Growth and Structure of Human Populations: A Mathematical Investigation," Princeton University Press, Princeton, N.J., 1972.

2. Hukukane Nikaido, "Convex Structures and Economic Theory," Academic Press, New York, N.Y., 1968.

3. Bresford Parlett, Ergodic properties of Populations 1: the one sex model. Theoretical Population Biology 1 (1970), 191-207.