ON THE EXISTENCE OF EQUILIBRIUM IN A SECURITIES MODEL*

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1. Introduction

Since the contributions of Lintner [13], Sharpe [16] and Mossin [14] in the mid-1960's, a great deal of work has been done on the analysis of competitive equilibrium in securities markets. In most cases the analysis has been confined to exchange economies in which individuals own fixed initial endowments of securities. Each individual is assumed to possess probability beliefs about security returns, which are independent of current security prices, and to select a portfolio which maximizes the expected value of a concave utility function subject to a budget constraint, the argument of the utility function being the monetary return from the portfolio.

Typically the model is considerably simplified by assuming first that individuals' probability beliefs are identical, secondly that expected utility is a function only of the mean and variance of the portfolio return, and thirdly that there is a riskless security which may be held in unrestricted amounts. From these assumptions some particularly simple relationships between the equilibrium prices of securities and their means, variances and covariances can be deduced.

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These relationships are obtained on the assumption that an equilibrium exists. Surprisingly, no attempt seems to have been made to establish the existence of equilibrium. The usual existence theorems (see, for example, Debreu [4]) cannot be applied directly, since, as a result of the fact that short-sales of securities are permitted,[1] consumption sets (in this case the sets of portfolios individuals are permitted to hold) are unbounded below. In Section 2, it is shown, in a model considerably more general than the mean-variance model, that an equilibrium exists even if probability beliefs are not identical, as long as there is agreement about the expected returns of securities.

In the general case where individuals disagree about expected returns, an equilibrium may well not exist. Suppose, for example, that one individual believes with certainty that security 1 yields a higher return than security 2 and a second individual believes with certainty that security 2 yields a higher return than security 1. If the price of security 1 is less than or equal to the price of security 2, the first individual will engage in profitable arbitrage operations by buying security 1 and selling security 2, and, if the price of security 1 is greater than the price of security 2, the second individual will engage in profitable arbitrage operations by buying security 2 and selling security 1. Hence no equilibrium exists.

In this example, no Pareto-optimum exists either. The central result of Section 2 is that an equilibrium exists if and only if a Pareto-optimum exists, in the general case when there is disagreement about security returns. Establishing this result is the first step in obtaining necessary and sufficient conditions for the existence of equilibrium in terms of individuals' probability beliefs and attitudes towards risk. The second step, which is carried out in Section 3, is to derive necessary and sufficient conditions for the existence of a Pareto-optimum. This is accomplished by using some recent work of Bertsekas [2].

In Section 4, the results of Section 3 are used to explore the intuitive idea that an equilibrium exists if individuals' probability beliefs are similar in some general sense. A standard metric on probability measures is used to formalize the notion that probability beliefs are similar, and an equilibrium is shown to exist when beliefs are sufficiently close in terms of this metric.

The problem of finding sufficient conditions for the existence of equilibrium when individuals' feasible sets are unbounded below has been analyzed in a somewhat different context by Grandmont [6] and Green [8], [9]. Grandmont and Green consider a situation where consumers make decisions about how much money to borrow or lend, or how much of a good to buy or sell forward, on the basis of uncertain beliefs about future commodity prices. Grandmont's and Green's analysis is more general than ours in that they allow beliefs about future prices to be influenced by current prices, whereas we follow Lintner, Sharpe, and Mossin in assuming that beliefs about security returns are independent of current security prices. The independence assumption simplifies the analysis considerably and, more importantly, permits stronger results, including necessary and sufficient conditions for the existence of equilibrium, to be obtained.

2. The Model and the Equivalence of the Existence of Pareto-Optima and Equilibria

We consider a one-period model in which trading in securities takes place at the beginning of the period and security returns are determined at the end of the period. The return of a security may be interpreted as the total value of one unit of the security at the end of of the period including any dividends received during the period. Individuals are assumed to be interested in the value of their portfolio at the end of the period.

Let there be n securities and m individuals. Individual $j(j=1,\ldots,m)$ is assumed to have an initial endowment of $\bar{x}_i^j$ units of security $i(i=1,\ldots,n)$, a von Neumann-Morgenstern utility function $U_j: R \to R$, and beliefs about security returns which are represented by a probability measure P_j defined on the σ-field of Borel sets of

$R_+^n = \{x \in R^n | x \geqq 0\}$.[2] $P_j(A)$ is individual j's probability belief that $(r_1, \ldots, r_n) \in A$, where $r_i (i=1,\ldots,n)$ is the uncertain return of one unit of security i. In confining our attention to $A \subset R_+^n$, we are assuming that the return of each security is non-negative.[3]

We make the following assumptions about tastes and probability beliefs:

A1: U_j is concave $(j=1,\ldots,m)$;[4]

A2: U_j is increasing, that is $U_j(w_1) > U_j(w_2)$ if $w_1 > w_2$ $(j=1,\ldots,m)$;

A3: $P_j(C) = 1$ for some bounded subset C of $R_+^n (j=1,\ldots,m)$.[5]

A_3 says that, for each individual, security returns are bounded with probability 1.

In order to be as general as possible, we assume that each individual is restricted to choosing portfolios from a feasible set, which might, for example, be determined by legal requirements. Let $X^j \subset R^n$ be individual j's feasible set. The vector $x \in X^j$ refers to the portfolio consisting of x_i units of security $i(i=1,\ldots,n)$. We assume:

A4: X^j has the special form $X^j = \{x \in R^n | A^j x \geqq b^j\}$, where $A^j = \{a_{hi}^j\}$ is an $(H \times n)$ matrix and b^j is an H-vector $(j=1,\ldots,m)$.

Assumption A4 appears to include all the interesting cases. For example, if $A^j = 0$ and $b^j = 0$, $X^j = R^n$ and individual j can hold any portfolio; if A^j is the identity matrix and $b^j = 0$, $X^j = R^n_+$ and individual j is prohibited from selling short. The more general case, where individual j can hold some securities in non-negative amounts, some in non-positive amounts, some in zero amounts, and some in unrestricted amounts, is also allowed for by A4.

We make some further assumptions about the feasible sets:

A5: For each $i(i=1,\ldots,n)$, there exists j such that the i^{th} column of A^j is non-negative;

A6: for each $j(j=1,\ldots,m)$, there exists i such that the i^{th} column of A^j is non-negative and $P_j(\{r \epsilon R^n_+ | r_i > 0\}) > 0$.[6]

A5 says that, for each security, there is an individual who can hold that security in unlimited positive amounts (assuming his feasible set is non-empty); A6 says that, for each individual, there is a security which the individual believes yields a positive return with non-zero probability and which he can hold in unlimited positive amounts (assuming his feasible set is non-empty).

Finally, we make a standard assumption which insures that feasible sets are non-empty and, more importantly, that individual demand behaviour is continuous:

A7: There exists $\hat{x}^j \epsilon X^j$ satisfying $\hat{x}^j < \bar{x}^j (j=1,\ldots,m)$.

Assumption A7 is much stronger than necessary, and is made only to simplify the proofs. For a discussion of how this assumption can be weakened, the reader is referred to Debreu [5].

If individual j's portfolio is given by $x \in R^n$, his expected utility is $\int U_j(rx)dP_j$.[7] Define $V_j\colon R^n \longrightarrow R$ by $V_j(x) = \int U_j(rx)dP_j$ $(j=1,\ldots,m)$. It is easy to show that A1 implies that V_j is concave.

Given prices $p = (p_1,\ldots,p_n) \in R^n_+$, where $p_i (i=1,\ldots,n)$ is the price of one unit of security i, individual j selects x maximizing $V_j(x)$ subject to $x \in X^j$ and $px \leqq p\bar{x}^j$.[8]

Equilibrium

Prices $p = (p_1,\ldots,p_n) \in R^n_+$ yield an equilibrium if there exist $x^1,\ldots,x^m$ such that

(I) $x^j \in \{x \in X^j | px \leqq p\bar{x}^j\}$ and $V_j(x^j) \geqq V_j(x)$

for all $x \in \{x \in X^j | px \leqq p\bar{x}^j\}$ $(j=1,\ldots,m)$;

(II) $\sum_j x^j = \sum_j \bar{x}^j$.[9]

(I) is the condition that x^j is optimal for individual j at prices p, and (II) is a market clearing condition.

In defining an equilibrium, we are allowing individuals to hold portfolios which yield a negative return with non-zero probability. If all economic activity ceases at the end of the period, those individuals holding portfolios with negative values will presumably go bankrupt. We assume, however, that no bankruptcy provisions are taken into account when portfolio decisions are being made, so that $U_j(\bar{r}x^j)$ is individual j's assessment of his utility in the event that $r = \bar{r}$ even if $\bar{r}x^j < 0$. For a discussion of models in which bankruptcy is dealt with explicitly, the reader is referred to Grandmont [6] and Green [9]. An alternative interpretation of the model, in which bankruptcies do not occur, is that securities markets re-open next period with new initial endowments given by the equilibrium portfolios of this period, modified to take account of any dividend payments.

<u>Pareto-Optimum</u>

The set of feasible portfolio allocations, F, is defined by $F = \{(x^1,\ldots,x^m) \mid x^j \in X^j$ for each j and $\sum_j x^j = \sum_j \bar{x}^j\}$. $(x^1,\ldots,x^m)$ is said to be a Pareto-optimum if

(I) $(x^1,\ldots,x^m) \in F$;

(II) $(\hat{x}^1,\ldots,\hat{x}^m) \in F$ and $V_j(\hat{x}^j) \geqq V_j(x^j)$ for each $j \Rightarrow V_j(\hat{x}^j) = V_j(x^j)$ for each j.

In other words, $(x^1,\ldots,x^m)$ is a Pareto-optimum if it is a feasible allocation and there is no other feasible allocation which makes some people better off and nobody worse off. The terms better off and worse off are used only in an ex-ante sense.

Since individuals' feasible sets are not necessarily bounded below, neither an equilibrium nor a Pareto-optimum need exist. Theorem 2.1 states that, under weak assumptions, an equilibrium exists if and only if a Pareto-optimum exists.

THEOREM 2.1: If each U_j is strictly concave, the existence of a Pareto-optimum is a necessary and sufficient condition for the existence of equilibrium.

PROOF: Necessity is obvious since an equilibrium is Pareto-optimal under our assumptions. To establish sufficiency, we consider a sequence of bounded economies converging to the original economy. We show that eventually the equilibria of the bounded economies are also equilibria of the original economy.

Let $\mathbb{E}$ denote the economy described above, which we assume has a Pareto-optimum, and let $\mathbb{E}_t$, $t=1,2,\ldots,$ denote the economy in which there is an added restriction that each individual may hold only portfolios x satisfying $x \geqq -\underset{\sim}{t}$, where $\underset{\sim}{t}$ is an n-dimensional vector, each of whose components is t. We confine our attention to $t > -\bar{x}_i^j$ for all i and j.

Define

$$B^j(p) = \{x \in X^j \mid px \leqq p\bar{x}^j\} ,$$

$$B^j_t(p) = \{x \in X^j \mid px \leqq p\bar{x}^j \quad \text{and} \quad x \geqq -\underset{\sim}{t}\} ,$$

$$D^j(p) = \{\hat{x} \in B^j(p) \mid V_j(\hat{x}) \geqq V_j(x) \quad \text{for all} \quad x \in B^j(p)\} ,$$

$$D^j_t(p) = \{\hat{x} \in B^j_t(p) \mid V_j(\hat{x}) \geqq V_j(x) \quad \text{for all} \quad x \in B^j_t(p)\} .$$

Under our assumptions, the economy $\mathbb{E}_t$ has an equilibrium for each t (see Debreu [4, pp. 83-84]). That is, there exist $p^t \in R^n_+$ and $x^j_t \in D^j_t(p^t)$ $(j=1,\ldots,m)$ with $\sum_j x^j_t \leqq \sum_j \bar{x}^j$ and $p^t(\sum_j x^j_t - \sum_j \bar{x}^j) = 0$. We may assume indeed that $\sum_j x^j_t = \sum_j \bar{x}^j$. For let $u = \sum_j \bar{x}^j - \sum_j x^j_t$. By A5, for each i, there exists j_i such that the i^{th} column of A^{j_i} is non-negative. Define

$$x'^{j_i}_t = x^{j_i}_t + u_i e_i \qquad \text{for} \quad i=1,\ldots,n$$

and

$$x'^j_t = x^j_t \qquad \text{for} \quad j \neq j_1,\ldots,j_n ,$$

where u_i is the i^{th} component of u and e_i is the i^{th} unit vector. By construction, $\sum_j x'^j_t = \sum_j \bar{x}^j$, and, since $u \geqq 0$ and $p^t u = 0$, $x'^j_t \in D^j_t(p^t)$ $(j=1,\ldots,m)$.

Proposition 1 states that p^t is an equilibrium for the original economy $\mathbb{E}$ if the lower bounds on the x^j_t's are not binding.

<u>PROPOSITION 1:</u> Suppose that, for some t, $x^j_t \in D^j_t(p^t)$ $(j=1,\ldots,m)$, $\sum_j x^j_t = \sum_j \bar{x}^j$, and $x^j_t > -\underset{\sim}{t}$ for all j. Then p^t is an equilibrium price vector for $\mathbb{E}$ and the x^j_t's are the equilibrium portfolios.

PROOF: We need only show that $x_t^j \in D^j(p^t)$ $(j=1,\ldots,m)$. This follows immediately from the facts that $x_t^j \in D_t^j(p^t)$, $x_t^j > -\underset{\sim}{t}$, X^j convex, and V_j concave. Q.E.D.

In view of Proposition 1, we may confine ourselves to the case where, for each t, there exist $p^t \in S$ and x_t^j $(j=1,\ldots,m)$ satisfying

$$x_t^j \in D_t^j(p^t) \qquad (j=1,\ldots,m), \tag{1}$$

$$\sum_j x_t^j = \sum_j \bar{x}^j \tag{2}$$

and

$$x_t^j \not> -\underset{\sim}{t} \qquad \text{for some } j\,. \tag{3}$$

Consider the subsequence $\left\{\frac{x_t^j}{\sum_{k=1}^{m}\|x_t^k\|}\right\}$, where, for $a \in R^n$, $\|a\|$ is defined to be $\left\{\sum_{i=1}^{n} a_i^2\right\}^{\frac{1}{2}}$. Choosing a subsequence if necessary, we may assume that $\frac{x_t^j}{\sum_{k=1}^{m}\|x_t^k\|}$ tends to a limit as $t \to \infty$. Let

$$\lim_{t\to\infty} \frac{x_t^j}{\sum_{k=1}^{m}\|x_t^k\|} = x^j \qquad (j=1,\ldots,m). \tag{4}$$

Clearly

$$\sum_j \|x^j\| = 1\,, \tag{5}$$

and, since $\sum_j x_t^j = \sum_j \bar{x}^j$ by (2) and $\lim_{t\to\infty} \sum_{k=1}^{m} \|x_t^k\| \geqq \lim_{t\to\infty} \max_j \|x_t^j\| \geqq \lim_{t\to\infty} t = \infty$ by (3),

$$\sum_j x^j = 0 . \tag{6}$$

The next step in the proof is to show that $rx^j = 0$ with probability 1 for individual j . The notation $x^j \equiv_j 0$ will be used throughout the paper as a short-hand for this.

PROPOSITION 2: $x^j \equiv_j 0$.

PROOF: Let $z = (z^1, \ldots, z^m)$ be a Pareto-optimum, which exists by assumption. Since x^j_t is chosen by j when $\bar{x}^j$ could be chosen,

$$V_j(x^j_t) \geqq V_j(\bar{x}^j) . \tag{7}$$

It follows that the sequence $\{x^j_t\}$, with $\alpha_t = \sum_{k=1}^m \|x^k_t\|$, satisfies the conditions of the following Lemma, which is proved in appendix 1.

LEMMA 1: Suppose that $\{x_t\}$ is a sequence such that $x_t \in X^j$, $V_j(x_t)$ is bounded below, and $\lim_{t\to\infty} \frac{x_t}{\alpha_t} = x$, where $\{\alpha_t\}$ is a sequence satisfying $\lim_{t\to\infty} \alpha_t = \infty$. Then, for any $y \in R^n$ and for all $\mu \geqq 0$, $V_j(y + \mu x) \geqq V_j(y)$ and $y + \mu x \in X^j$ if $y \in X^j$.

Putting $y = z^j$ in Lemma 1, we obtain

$$z^j + \mu x^j \epsilon X^j \tag{8}$$

and

$$V_j(z^j + \mu x^j) \geqq V_j(z^j) \tag{9}$$

for all $\mu \geqq 0$. (6) and (8) imply that $(z^1 + \mu x^1, \ldots, z^m + \mu x^m) \epsilon F$, the set of feasible portfolio allocations, and hence by (9) and the fact that $(z^1, \ldots, z^m)$ is a Pareto-optimum, $V_j(z^j + \mu x^j) = V_j(z^j)$ for all $\mu \geqq 0$. In particular, $V_j(z^j + x^j) = \frac{1}{2} V_j(z^j + 2x^j) + \frac{1}{2} V_j(z^j)$. Putting $\lambda = \frac{1}{2}$, $x = z^j + 2x^j$ and $x' = z^j$ in Lemma 2, which is proved in appendix 1, we obtain $x^j \equiv_j 0$. Q.E.D.

<u>LEMMA 2:</u> Assume that U_j is strictly concave and $0 < \lambda < 1$. Then $V_j(\lambda x + (1-\lambda)x') = \lambda V_j(x) + (1-\lambda) V_j(x')$ implies that $x - x' \equiv_j 0$.

If security returns are linearly independent for each individual, $x^j \equiv_j 0$ implies that $x^j = 0$. This contradicts (5) and proves Theorem 2.1. The case of linear dependence requires a little more work. We use Lemma 3, which is proved in appendix 1. It is in this Lemma that the assumption that X^j has the special form $X^j = \{x \epsilon R^n | A^j x \geqq b^j\}$ is important.

LEMMA 3: Suppose that $\{x_t\}$ is sequence such that $x_t \geqq -\underset{\sim}{t}$, $\lim_{t\to\infty} \frac{x_t}{\alpha_t} = x$, $\alpha_t \leqq \gamma t$, where $\gamma > 0$ is independent of t, and $\lim_{t\to\infty} \alpha_t = \infty$. Then there exists T such that $(x_t - x) \in X^j$ and $x_t - x > -\underset{\sim}{t}$ for all $t \geqq T$.

Since $x_t^j \geqq -\underset{\sim}{t}$ for all j and $\sum_j x_t^j = \sum_j \bar{x}^j$ by (2), $x_t^j \leqq (m-1)\underset{\sim}{t} + \sum_{k=1}^{m} \bar{x}^k$ for all j. Hence $\sum_{k=1}^{m} \|x_t^k\| \leqq \gamma t$ for some $\gamma > 0$ independent of t, and we may apply Lemma 3 to $\{x_t^j\}$ with $\alpha_t = \sum_{k=1}^{m} \|x_t^k\|$. It follows that there exists T^j such that

$$(x_t^j - x^j) \in X^j \tag{10}$$

and

$$x_t^j - x^j > -\underset{\sim}{t} \tag{11}$$

for all $t \geqq T^j$. Let $T = \max_j T^j$. Proposition 3 completes the proof of Theorem 2.1.

PROPOSITION 3: p^T is an equilibrium price vector for $\mathbb{E}$ with $(x_T^j - x^j)$ as the equilibrium portfolios $(j=1,\ldots,m)$.

PROOF: We note first that $\sum_j (x_T^j - x^j) = \sum_j x_T^j = \sum_j \bar{x}^j$ by (2) and (6), and

$$V_j(x_T^j - x^j) = V_j(x_T^j) \tag{12}$$

by Proposition 2. If we can establish that $p^T x^j \geqq 0$ for all j, it will follow from (1), (10), (11) and (12) that

$$(x_T^j - x^j) \in D_T^j(p^T) \quad , \tag{13}$$

and therefore by (11) and the concavity of V_j that $(x_T^j - x^j) \in D^j(p^T)$. This proves Proposition 3.

In order to establish that $p^T x^j \geqq 0$ for all j, we suppose the contrary. Then, since $\sum_j p^T x^j = p^T \sum_j x^j = 0$ by (6), $p^T x^{j_o} > 0$ for some j^o, and hence $(x_T^{j_o} - x^{j_o}) \in D_T^{j_o}(p^T)$ and

$$p^T(x_T^{j_o} - x^{j_o}) < p^T \bar{x}^{j_o} \quad . \tag{14}$$

By A6, there exists i such that the i^{th} column of A^{j_o} is non-negative and $P_{j_o}(\{r \in R_+^n | r_i > 0\}) > 0$. Let $x = (x_T^{j_o} - x^{j_o} + \varepsilon e_i)$, where $\varepsilon > 0$ and e_i is the i^{th} unit vector. Clearly, $V_{j_o}(x) > V_{j_o}(x_T^{j_o} - x^{j_o})$. However, by (14), $x \in B_T^{j_o}(p^T)$ for small ε, which contradicts $(x_T^{j_o} - x^{j_o}) \in D_T^{j_o}(p^T)$. Q.E.D.

<u>Remark:</u> The assumption in Theorem 2.1 that the U_j's are strictly concave is stronger than necessary. It can be replaced by the assumption that the U_j's are strictly concave for large values: that is, there exists K such that $U_j(\lambda w_1 + (1-\lambda)w_2) > \lambda U_j(w_1) + (1-\lambda)U_j(w_2)$ if $w_1 \neq w_2$, $\|w_1\| + \|w_2\| > K$, and $0 < \lambda < 1$ $(j=1,\ldots,m)$. The assumption cannot be dispensed with entirely, however.

In Theorem 2.2, which states that an equilibrium exists if there is agreement about expected security returns, the strict concavity assumption is not required.

Before stating this theorem, we make some definitions. We define, for each j,

$$E_x^{+j} = \int_{rx \geq 0} rx \, d\, P_j \,,$$

$$E_x^{-j} = \int_{rx < 0} rx \, d\, P_j \,,$$

$$E_x^{j} = \int rx \, d\, P_j \,,$$

$$S_j^{+} = \lim_{w \to \infty} \frac{dU_j(w)}{dw} \,,$$

and

$$S_j^{-} = \lim_{w \to -\infty} \frac{dU_j(w)}{dw} \,.$$

Clearly $E_x^{+j} \geqq 0$, $E_x^{-j} \leqq 0$ and $E_x^j = E_x^{+j} + E_x^{-j}$. S_j^+ and S_j^- are well defined since a concave function mapping R into R has a derivative except at a countable number of points. S_j^+ is finite and S_j^- is finite or $+\infty$.

For each i and j, define

$$E_i^j = \int r_i \, d\, P_j \quad .$$

$E_i^j = E_{e_i}^j$ where e_i is the i^{th} unit vector. We say that there is agreement about expected security returns if, for each i, E_i^j is the same for all j.

The following Lemma, which is useful in the proofs of Theorem 2.2 and several other theorems in this paper, follows directly from a result of Bertsekas [2, Proposition 2]. The convention that $\infty . 0 = 0$, $\infty . a = -\infty$ if $a < 0$, and $-\infty + a = -\infty$ if a is finite, is adopted.

<u>LEMMA 4:</u> If $y, x \epsilon R^n$, $V_j(y + \mu x) \geqq V_j(y)$ for all $\mu \geqq 0$ if and only if $S_j^+ E_x^{+j} + S_j^- E_x^{-j} \geqq 0$.

<u>THEOREM 2.2:</u> If there is agreement about expected security returns, an equilibrium exists.

PROOF: The proof is identical to the proof of sufficiency in Theorem 2.1, except that we replace Proposition 2, which assumes the existence of a Pareto-optimum and the strict concavity of the U_j's , by a weaker proposition which assumes only that there is agreement about expected returns.

PROPOSITION 2': $V_j(x_t^j - x^j) = V_j(x_t^j)$.

PROOF: Applying Lemma 1 to $\{x_t^j\}$, as in the proof of Proposition 2, we obtain $V_j(y + \mu x^j) \geqq V_j(y)$ for all $\mu \geqq 0$. By Lemma 4, it follows that

$$S_j^+ E_{x^j}^{+j} + S_j^- E_{x^j}^{-j} \geqq 0 . \qquad (15)$$

Since U_j is concave and increasing,

$$S_j^+ \leqq S_j^- \qquad (16)$$

and

$$S_j^- > 0 . \qquad (17)$$

Therefore,

$$S_j^- E_{x^j}^j = S_j^-(E_{x^j}^{+j} + E_{x^j}^{-j}) \geqq S_j^+ E_{x^j}^{+j} + S_j^- E_{x^j}^{-j} \geqq 0 , \qquad (18)$$

and, hence, by (17),

$$E_{x^j}^j \geqq 0 . \qquad (19)$$

Since individuals agree on the expected returns of securities,

$\sum_j E^j_{x^j}$ = the expected return of $\sum_j x^j = 0$ by (6). Therefore, by (19),

$$E^j_{x^j} = 0 . \tag{20}$$

From (18) it now follows that

$$S_j^- E^{+j}_{x^j} + S_j^- E^{-j}_{x^j} = S_j^+ E^{+j}_{x^j} + S_j^- E^{-j}_{x^j} = 0 ,$$

and therefore, by (17), either $S_j^+ = S_j^-$ or $E^{+j}_{x^j} = E^{-j}_{x^j} = 0$.

In the first case, j is risk neutral and $E^j_{x^j} = 0$ by (20).

In the second case, $x^j \equiv_j 0$. In either case, $V_j(x_t^j - x^j) = V_j(x_t^j)$. Q.E.D.

The remainder of the proof of Theorem 2.1 now applies.

3. Necessary and Sufficient Conditions for the Existence of a Pareto-optimum and an Equilibrium

In this section, we derive necessary and sufficient conditions for the existence of a Pareto-optimum. As a consequence of Theorem 2.1, these are also necessary and sufficient conditions for the existence of equilibrium.

We begin with some definitions from Rockafellar [15] (see also Bertsekas [2]). Let $X \subset R^n$ be a closed convex set. x is said to be a direction of recession of X if $y + \mu x \in X$ for every $y \in X$ and for all $\mu \geqq 0$.[10] It is to be noted that if $y + \mu x \in X$ for some $y \in X$ and all $\mu \geqq 0$, the same is true for every $y \in X$.

As in the proofs of Theorems 2.1 and 2.2, we use the notation $x \equiv_j 0$ to mean $rx = 0$ with probability 1 for individual j.

THEOREM 3.1: If each U_j is strictly concave, a necessary and sufficient condition for the existence of a Pareto-optimum is that there do not exist $\hat{x}^1,\ldots,\hat{x}^m$ such that

(I) $\sum_j \hat{x}^j = 0$;

(II) $\hat{x}^j$ is a direction of recession of X^j satisfying

$$S_j^+ E_{\hat{x}^j}^{+j} + S_j^- E_{\hat{x}^j}^{-j} \geqq 0 \quad (j=1,\ldots,m) ;$$

(III) for some j, $\hat{x}^j \not\equiv_j 0$.

PROOF:

Necessity

Let $(z^1,\ldots,z^m)$ be a Pareto-optimum, and suppose there exist $\hat{x}^1,\ldots,\hat{x}^m$ satisfying (I) and (II). By Lemma 4, $V_j(z^j + \mu\hat{x}^j) \geqq V_j(z^j)$ for all $\mu \geqq 0$ and each j. Hence, since $(z^1,\ldots,z^m)$ is a Pareto-optimum and $(z^1 + \mu\hat{x}^1,\ldots,z^m + \mu\hat{x}^m) \in F$ by (I) and (II), $V_j(z^j + \mu\hat{x}^j) = V_j(z^j)$ for all $\mu \geqq 0$ and each j. Applying Lemma 2, as in the proof of Proposition 2, we obtain $\hat{x}^j \equiv_j 0$ $(j=1,\ldots,m)$, which contradicts (III).

Sufficiency

Suppose that there are no $\hat{x}^1,\ldots,\hat{x}^m$ satisfying (I), (II) and (III), and a Pareto-optimum does not exist. Let

$\hat{F} = \{(x^1,\ldots,x^m) \in F | V_j(x^j) \geqq V_j(\bar{x}^j)$ for all $j\}$ and $\hat{F}_t = \{(x^1,\ldots,x^m) \in \hat{F} | x^j \geqq -\underset{\sim}{t}$ for all $j\}$. If $\underset{\sim}{t} > -\bar{x}^j$ for all j, $\hat{F}_t$ is (a) non-empty by A4, A5 and A7, (b) closed since each V_j is concave and therefore continuous, and (c) bounded since $x^j \geqq -\underset{\sim}{t}$ for all j and $\sum_j x^j = \sum_j \bar{x}^j \Rightarrow x^j \leqq (m-1)\underset{\sim}{t} + \sum_{k=1}^{m} \bar{x}^k$ for all j. It follows from Weierstrass' theorem that the problem: maximize $\sum_j V_j(x^j)$ subject to $(x^1,\ldots,x^m) \in \hat{F}_t$ has a solution if $t \geqq -\bar{x}^j$ for all j. Let $(x_t^1,\ldots,x_t^m)$ be a solution to this problem. We show that

$$x_t^j \not> -\underset{\sim}{t} \qquad \text{for some } j. \tag{21}$$

If not, since the V_j's are concave and the X_j's are convex, $(x_t^1,\ldots,x_t^m)$ is a solution of the problem: maximize $\sum_j V_j(x^j)$ subject to $(x^1,\ldots,x^m) \in \hat{F}$. This in turn implies that $(x_t^1,\ldots,x_t^m)$ is a Pareto-optimum, which contradicts the assumption that no Pareto-optimum exists.

Consider the sequence $\{(x_t^1,\ldots,x_t^m)\}$, where t takes on integral values greater than $\max_{i,j} -\bar{x}_i^j$. Choosing a subsequence if necessary, we may assume that $\dfrac{x_t^j}{\sum_{k=1}^{m} \|x_t^k\|}$ tends to a limit as $t\to\infty$. Let $\lim_{t\to\infty} \dfrac{x_t^j}{\sum_{k=1}^{m} \|x_t^k\|} = \hat{x}^j$ $(j=1,\ldots,m)$. Clearly

$$\sum_j \|\hat{x}^j\| = 1, \tag{22}$$

and, since $(x_t^1,\ldots,x_t^m) \in F$ and $\lim_{t\to\infty} \sum_{k=1}^{m} \|x_t^k\| = \infty$ by (21),

$$\sum_j \hat{x}^j = 0 . \tag{23}$$

Noting that $V_j(x_t^j) \geqq V_j(\bar{x}^j)$, we may apply Lemma 1 to $\{x_t^j\}$ with $\alpha_t = \sum_{k=1}^{m} \|x_t^k\|$ to obtain

$$\hat{x}^j \text{ is a direction of recession of } X^j \tag{24}$$

and $V_j(y+\mu\hat{x}^j) \geqq V_j(y)$. Hence, by Lemma 4,

$$S_j^+ E_{\hat{x}^j}^{+j} + S_j^- E_{\hat{x}^j}^{-j} \geqq 0 . \tag{25}$$

(23), (24) and (25) imply that $\hat{x}^1,\ldots,\hat{x}^m$ satisfy (I) and (II), and, since $\hat{x}^1,\ldots,\hat{x}^m$ do not satisfy (I), (II) and (III) by assumption, it follows that

$$\hat{x}^j \equiv_j 0 \qquad (j=1,\ldots,m) . \tag{26}$$

Applying Lemma 3 to $\{x_t^j\}$ with $\alpha_t = \sum_{k=1}^{m} \|x_t^k\|$, we obtain

$$(x_T^j - \hat{x}^j) \in X^j \qquad (j=1,\ldots,m) \tag{27}$$

and

$$x_T^j - \hat{x}^j > -\underset{\sim}{T} \qquad (j=1,\ldots,m) . \tag{28}$$

(23), (26), (27) and (28) imply that $(x_T^1 - \hat{x}^1,\ldots,x_T^m - \hat{x}^m)$ is a solution of: maximize $\sum_j V_j(x^j)$ subject to $(x^1,\ldots,x^m) \in \hat{F}_t$. However, $(x_T^j - \hat{x}^j) > -\underset{\sim}{T}$ for all j , which contradicts the argument leading to (21). Q.E.D.

<u>THEOREM 3.2:</u> If each U_j is strictly concave, a necessary and sufficient condition for the existence of equilibrium is that there do not exist $\hat{x}^1,\ldots,\hat{x}^m$ such that

(I) $\sum_j \hat{x}^j = 0$;

(II) $\hat{x}^j$ is a direction of recession of X^j satisfying

$$S_j^+ E_{\hat{x}^j}^{+j} + S_j^- E_{\hat{x}^j}^{-j} \geqq 0 \quad (j=1,\ldots,m) ;$$

(III) for some j , $\hat{x}^j \neq_j 0$.

<u>PROOF:</u> Apply Theorems 2.1 and 3.1. Q.E.D.

<u>Remark 1:</u> If $S_j^+ = 0$ or $S_j^- = \infty$ for each j , the necessary and sufficient condition is that there do not exist $\hat{x}^1,\ldots,\hat{x}^m$ such that $\sum_j \hat{x}^j = 0$, where, for each j , $\hat{x}^j$ is a direction of recession of X^j satisfying $r\,\hat{x}^j \geqq 0$ with probability 1 for individual j , and some $\hat{x}^j \neq_j 0$. This is the general equilibrium version of a partial equilibrium result obtained by Leland in [12].

<u>Remark 2:</u> The results of Sections 2 and 3 have been proved under the assumption that the r_i's are bounded with probability 1 for each individual. As long as $\int |r_i| dP_j$ is

finite, the results also hold in the unbounded case provided that $\int |U_j(rx)| dP_j$ is finite for all $x \in R^n$. This is guaranteed if S_j^- is finite and U_j is a non-decreasing function.

Remark 3: The restriction that each feasible set is of the form $\{x \in R^n | A^j x \geqq b^j\}$ appears to be an important one if security returns can be linearly dependent. In the linear independence case, Theorems 2.1, 3.1 and 3.2 hold for more general feasible sets; Theorem 2.2 also holds for more general feasible sets if risk neutrality is ruled out in addition to linear dependence.

4. The Existence of Equilibrium when Individuals' Probability Beliefs are Similar

Let $\mathbb{B}$ be the σ-field of Borel sets of R^n_+, and let $\mathbb{M}$ be the set of all probability measures defined on $\mathbb{B}$. The notion of probability beliefs being similar is made precise by defining a metric on $\mathbb{M}$. If $P_1, P_2 \in \mathbb{M}$, define $d(P_1,P_2) = \inf\{\varepsilon > 0 | P_1(A) \leqq P_2(A^\varepsilon) + \varepsilon$ and $P_2(A) \leqq P_1(A^\varepsilon) + \varepsilon$ for all $A \in \mathbb{B}\}$, where $A^\varepsilon = \{x \in R^n_+ | \; \|x - y\| < \varepsilon$ for some $y \in A\}$. It can be shown (see Billingsley [3, p. 238]) that d is a metric on $\mathbb{M}$.

The metric d is less restrictive than the more obvious metric ρ, defined by $\rho(P_1,P_2) = \inf\{\varepsilon > 0 | \; |P_1(A) - P_2(A)| < \varepsilon$ for all $A \in \mathbb{B}\}$, in the sense that probability measures which are close in terms of ρ are also close in terms of d, whereas the converse is not true.[11] The weak-convergence topology induced on $\mathbb{M}$ by d has been used elsewhere in economics in the study of convergent economies (see Hildenbrand [11]) and in the study of continuity properties of von Neumann-Morgenstern utility functions (see Grandmont [7]).

Let $P_1,\ldots,P_m$ denote the probability measures of individuals $1,\ldots,m$ and $\mathbb{E}(P_1,\ldots,P_m)$ the resulting economy as described in Section 2.[12] Theorem 4.1 says that, under weak assumptions, an equilibrium exists for the economy $\mathbb{E}(P_1,\ldots,P_m)$ if $P_1,\ldots,P_m$ are all sufficiently close to some probability measure P in terms of the metric d.

<u>THEOREM 4.1:</u> Suppose the U_j's are strictly concave and $P \in \mathbb{M}$ satisfies:

(I) $P(\{r \mid rx = 0\}) = 1 \Rightarrow x = 0$;

(II) $P(C) = 1$ for some bounded set C in R^n .

Then there exists $\varepsilon > 0$ such that $\mathbb{E}(P_1,\ldots,P_m)$ has an equilibrium if:

(III) $d(P_j,P) < \varepsilon \quad (j=1,\ldots,m)$;

(IV) $P_j(C) = 1 \quad (j=1,\ldots,m)$;

(V) for each $j (j=1,\ldots,m)$, there exists i such that the i^{th} column of A^j is non-negative and $P_j(\{r \mid r_i > 0\}) > 0$.

<u>PROOF:</u> Suppose not. Then we can find a sequence $\{(P_1^t,\ldots,P_m^t)\}$ such that, for each t , $\mathbb{E}(P_1^t,\ldots,P_m^t)$ has no equilibrium, P_j^t satisfies (IV) and (V), and $P_j^t \xrightarrow{d} P$ as $t \to \infty$ $(j=1,\ldots,m)$. Therefore, by Theorem 3.2, there exist,

for each t, $\hat{x}_t^1, \ldots, \hat{x}_t^m$ such that

$$\sum_j \hat{x}_t^j = 0 , \tag{29}$$

$$\hat{x}_t^j \text{ is a direction of recession of } X^j \quad (j=1,\ldots,m), \tag{30}$$

and

$$s_j^+ \int_{r\hat{x}_t^j \geqq 0} r\hat{x}_t^j \, dP_t^j + s_j^- \int_{r\hat{x}_t^j < 0} r\hat{x}_t^j \, dP_t^j \geqq 0 \quad (j=1,\ldots,m) . \tag{31}$$

In view of (III) of Theorem 3.1, we can scale the $\hat{x}_t^j$'s so that

$$\sum_j \|\hat{x}_t^j\| = 1 . \tag{32}$$

Since a subsequence may be chosen if necessary, we may assume that $\hat{x}_t^j$ tends to a limit as $t \to \infty$. Let $\lim_{t\to\infty} \hat{x}_t^j = \hat{x}^j (j=1,\ldots,m)$. (29) and (32) imply that

$$\sum_j \hat{x}^j = 0 \tag{33}$$

and

$$\sum_j \|\hat{x}^j\| = 1 . \tag{34}$$

Using a result of Billingsley [3, p. 17, Exercise 8] and the boundedness of C, we obtain

$$\lim_{t\to\infty} \int_{r\hat{x}_t^j \geqq 0} r\hat{x}_t^j \, dP_j^t = \int_{r\hat{x}^j \geqq 0} r\hat{x}^j \, dP$$

and

$$\lim_{t\to\infty} \int_{r\hat{x}_t^j < 0} r\hat{x}_t^j \, dP_j^t = \int_{r\hat{x}^j < 0} r\hat{x}^j \, dP \quad ,$$

and therefore, by (31),

$$S_j^+ \int_{r\hat{x}^j \geqq 0} r\hat{x}^j \, dP + S_j^- \int_{r\hat{x}^j < 0} r\hat{x}^j \, dP \geqq 0 \ . \qquad (35)$$

We may now argue as in the proof of Proposition 2' to show that (33), (35) and the strict concavity of U_j imply that $P(\{r | r\hat{x}^j = 0\}) = 1$. Hence, by assumption (I) of this theorem, $\hat{x}^j = 0$ $(j=1,\ldots,m)$, which contradicts (34). Q.E.D.

APPENDIX 1

LEMMA 1: Suppose that $\{x_t\}$ is a sequence such that $x_t \in X^j$, $V_j(x_t)$ bounded below and $\lim_{t\to\infty} \frac{x_t}{\alpha_t} = x$, where $\{\alpha_t\}$ is a sequence satisfying $\lim_{t\to\infty} \alpha_t = \infty$. Then, for any $y \in R^n$ and for all $\mu \geqq 0$ $V_j(y + \mu x) \geqq V_j(y)$ and $y + \mu x \in X^j$ if $y \in X^j$.

PROOF: Since V_j is concave it is continuous. Therefore $V_j(y + \mu x) = \lim_{t\to\infty} V_j((1 - \frac{\mu}{\alpha_t})\, y + \frac{\mu}{\alpha_t} x_t)$. For large α_t, since V_j is concave,

$$V_j((1 - \frac{\mu}{\alpha_t})y + \frac{\mu}{\alpha_t} x_t) \geqq (1 - \frac{\mu}{\alpha_t})\, V_j(y) + \frac{\mu}{\alpha_t} V_j(x_t)$$

$$\geqq (1 - \frac{\mu}{\alpha_t})\, V_j(y) + \frac{\mu}{\alpha_t} B,$$

where B is a lower bound for $V_j(x_t)$.

$$\therefore \quad V_j(y + \mu x) \geqq \lim_{t\to\infty} \{(1 - \frac{\mu}{\alpha_t})\, V_j(y) + \frac{\mu}{\alpha_t} B\}$$

$$= V_j(y).$$

If $y \in X^j$, $(1 - \frac{\mu}{\alpha_t})y + \frac{\mu}{\alpha_t} x_t \in X^j$ for large α_t since X^j is convex. Taking limits we obtain $y + \mu x \in X^j$ since X^j is closed. Q.E.D.

LEMMA 2: Assume U_j strictly concave and $0 < \lambda < 1$. Then $V_j(\lambda x + (1-\lambda)x') = \lambda V_j(x) + (1-\lambda) V_j(x')$ implies that $x - x' \equiv_j 0$.

PROOF: $\int \{U_j(\lambda r x + (1-\lambda) r x') - \lambda U_j(rx) - (1-\lambda) U_j(rx')\} dP_j = 0$ since $V_j(\lambda x + (1-\lambda)x') - \lambda V_j(x) - (1-\lambda) V_j(x') = 0$. However, $U_j(\lambda r x + (1-\lambda)rx') - \lambda U_j(rx) - (1-\lambda) U_j(rx') \geqq 0$ since U_j is concave, and so $U_j(\lambda r x + (1-\lambda)rx') - \lambda U_j(rx) - (1-\lambda) U_j(rx') = 0$ with probability 1. By the strict concavity of U_j it follows that $rx = rx'$ with probability 1 and hence $x - x' \equiv_j 0$.

Q.E.D.

LEMMA 3: Suppose that $\{x_t\}$ is a sequence such that $x_t \geqq -\underset{\sim}{t}$, $\lim_{t\to\infty} \frac{x_t}{\alpha_t} = x$, $\alpha_t \leqq \gamma t$, where $\gamma > 0$ is independent of t, and $\lim_{t\to\infty} \alpha_t = \infty$. Then there exists T such that $(x_t - x) \in X^j$ and $x_t - x > -\underset{\sim}{t}$ for all $t \geqq T$.

PROOF: $X^j = \{x \in R^n | A^j x \geqq b^j\}$, where $A^j = \{a_{hi}^j\}$. Therefore, $A^j x_t \geqq b^j$, and, since $\lim_{t\to\infty} \alpha_t = \infty$, $A^j x \geqq 0$.

Suppose $\sum_{i=1}^{n} a_{hi}^j x_i > 0$. Then $\lim_{t\to\infty} \sum_{i=1}^{n} a_{hi}^j x_{it} = \infty$ since $\lim_{t\to\infty} \alpha_t = \infty$, and so we can choose T_h such that

$$\sum_{i=1}^{n} a_{hi}^j (x_{it} - x_i) \geqq b_h^j \qquad (36)$$

for all $t \geqq T_h$. If $\sum_{i=1}^{n} a^j_{hi} x_i = 0$, on the other hand, (36) holds for all t , so that $T_h = 1$. Let $T' = \max_h T_h$. Then, for all $t \geqq T'$, $A^j(x_t - x) \geqq b^j$, and consequently $(x_t - x) \in X^j$.

Suppose it is not the case that $x_t - x > -\underset{\sim}{t}$ eventually. Then, choosing a subsequence if necessary, we may assume that $x_{it} - x_i \leqq -t$ for some i and all t . Since $x_{it} \geqq -t$, $x_i \geqq 0$. On the other hand, $x_{it} - x_i \leqq -t$ implies that $x_{it} < 0$ eventually, and so $x_i = \lim_{t\to\infty} \frac{x_{it}}{\alpha_t} \leqq 0$. Hence $x_i = 0$ and $x_{it} = -t$. Therefore $\frac{x_{it}}{\alpha_t} = -\frac{t}{\alpha_t} \leqq -\frac{1}{\gamma}$ and so $x_i \leqq -\frac{1}{\gamma}$, which contradicts $x_i = 0$. Q.E.D.

APPENDIX 2

Throughout the paper we have assumed that security returns are non-negative. Since some important cases, including the case of normally distributed returns, are thereby excluded, we discuss briefly in this appendix the consequences of dropping the non-negativity assumption. It turns out that, with small changes in other assumptions, all our theorems hold in the negative return case, as long as equilibrium security prices are permitted to be negative.

In Theorems 2.1, 2.2, 3.2 and 4.1, A7 must be replaced by the stronger assumption that $\bar{x}^j \in \text{int}\, X^j$ $(j=1,\ldots,m)$ in order to insure that demand correspondences are upper semi-continuous when prices are negative. In addition, the assumption that no individual has a bliss point, that is for each $x \in X^j$ there exists $x' \in X^j$ such that $V_j(x') > V_j(x)$ $(j=1,\ldots,m)$, is required in Theorems 2.1, 2.2 and 3.2, and the assumption that individual j has no bliss point when his probability beliefs are given by P_j $(j=1,\ldots,m)$ replaces (V) of Theorem 4.1. A5 and A6 are no longer required.

The proofs of Theorems 2.1 and 2.2 are modified in the following way. $\mathbb{E}_t$ is now defined to be the economy in which individuals can hold only portfolios satisfying $\|x\| \leqq t$.

The following theorem from Hart and Kuhn [10, Theorem 2.4] is applied to the excess demand correspondence of $\mathbb{E}_t$.

> Let Z be a compact subset of R^n and let E be an upper semi-continuous correspondence mapping points in $S^{n-1} = \{p \in R^n \mid \sum_{i=1}^{n} p_i^2 = 1\}$ to non-empty, convex subsets of Z . Suppose that, for each $p \in S^{n-1}$, $z \in E(p) \Rightarrow pz \leqq 0$. Then either
>
> (a) there exists $p \in S^{n-1}$ with $0 \in E(p)$, or
>
> (b) there exists $p \in S^{n-1}$ with $z \in E(p)$, $z' \in E(-p)$, such that $\lambda z + (1-\lambda)z' = 0$ for some $\lambda \in (0,1)$.

If (a) holds for infinitely many t , the same argument as in the non-negativity proof shows that equilibria of $\mathbb{E}_t$ are also equilibria of $\mathbb{E}$ for large t . A modified version of Lemma 3 is used in which the assumption that $\|x_t\| \leqq t$ for all t and the conclusion that $\|x_t - x\| < t$ eventually replace the assumption that $x_t - x > -\underset{\sim}{t}$ for all t and the conclusion that $x_t - x > -\underset{\sim}{t}$ eventually. The case where, for infinitely many t , (b) holds and (a) does not can be shown to be impossible.

No changes are required in Theorem 3.1. In the proof, $\hat{F}_t$ is defined to be $\{(x^1,\ldots,x^m) \in \hat{F} \mid \|x^j\| \leqq t$ for all j $\}$ and the modified version of Lemma 3 is used.

FOOTNOTES

1. Even if risky securities cannot be held in negative amounts, no lower bound is placed on the amount of the riskless security held.

2. If $x \in R^n$, $x \geqq 0$ means $x_i \geqq 0$ $(i=1,\ldots,n)$; $x \geq 0$ means $x \geqq 0$ and $x \neq 0$; $x > 0$ means $x_i > 0$ $(i=1,\ldots,n)$.

3. This assumption is made only to simplify the analysis. A discussion of the negative return case may be found in Appendix 2.

4. The concavity of U_j implies that U_j is unbounded unless it is a constant function. The unboundedness of U_j , as Arrow has pointed out in [1, Chapter 2], is inconsistent with the usual assumptions about preferences over risky alternatives which are used to justify the existence of a von Neumann-Morgenstern utility function. There seems no alternative to assuming concavity, however, if we want to prove the existence of equilibrium when there are a finite number of individuals.

5. This assumption is relaxed in Remark 2 at the end of Section 3.

6. $P_j(\{r \in R^n_+ | r_i > 0\})$ is the probability that $r_i > 0$ according to individual j .

7. For $a, b \in R^n$, ab denotes the inner product $\sum_{i=1}^{n} a_i b_i$.

8. It might seem more sensible to assume that budget constraints hold with equality in a securities model. It turns out that, under our assumptions, budget constraints do indeed hold with equality in equilibrium.

9. $\sum_j$ is used as a short-hand for $\sum_{j=1}^{m}$. Similarly, for all j is used to mean for $j=1,\ldots,m$ and for all i to mean for $i=1,\ldots,n$.

10. Rockafellar defines directions of recession only for $x \neq 0$ whereas, in our definition, 0 is automatically a direction of recession.

11. It is shown in Billingsley that $P^t \xrightarrow{d} P \iff \lim_{t\to\infty} P^t(A) = P(A)$ for all $A \in \mathbb{B}$ with P (boundary A) = 0, whereas $P^t \xrightarrow{\rho} P \iff \lim_{t\to\infty} P^t(A) = P(A)$ for all $A \in \mathbb{B}$.

12. Throughout this section, the U_j's, $\bar{x}^j$'s and X^j's are assumed fixed, and hence this notation makes sense. The U_j's, $\bar{x}^j$'s and X^j's are assumed to satisfy assumptions A1, A2, A4, A5, A7.

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