GENERALIZATION AND PROOF OF THE HICKS COMPOSITE COMMODITY THEOREM WITHOUT A UTILITY FUNCTION

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1. Introduction

It is a well-known proposition, due to Hicks (1939) and Leontief (1936), but generally associated with the former, that a group of commodities among which the relative prices are constant can, in a natural way, be treated as a single commodity. The phrase "treated as a single commodity" summarizes the assertion that an individual will have preferences over bundles containing the composite commodity and these will have the same general properties - continuity, convexity, etc. - as his more fundamental

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preferences over the underlying goods. The proposition is frequently used, as it was by Hicks, to analyze the demand for a single commodity when all other commodities' prices are held fixed. The standard diagram has units of the commodity in question on one axis, and "dollars" on the other, understood as spendable on the remaining commodities on constant terms.

The usual proof of the composite commodity theorem assumes that consumer preferences are representable by a continuous, twice differentiable utility function, and exploits the second order properties of utility maximization. (See, for example, Green (1971), pp. 308 ff.; Hicks (1946), p. 310 ff.; Samuelson (1947), p. 141 ff.) Versions of the theorem by Gorman (1953) and, most recently, Diewert (1973) are considerably more precise about the properties of preferences which carry over to composite commodities, and relax the (sometimes implicit) restrictions commonly found. Both of these writers assume that preferences are representable by utility functions and analyse preferences in terms of properties of such functions. However, for some purposes it is desirable to deal directly with preference orderings rather than utility functions, particularly as some preferences are not representable by a utility function. As is known (Debreu (1959)) several independent properties of preferences are required to assure the existence of a utility function, and it is of some interest which of these properties separately carry over to preferences over bundles including the composite good "as a single commodity." This question is the principal subject of the present paper.

In the course of answering the question for a representative list of properties of preferences I demonstrate somewhat more general propositions about the conditions under which certain properties of binary relations are inherited by their images in other sets under mapping. I have made a particular effort throughout to deal with properties (such as transitivity and continuity) separately and independently of one another.

The usual composite commodity theorem is a special case of my more general results. The problem of spelling out the composite commodity theorem in terms of properties of preference orderings proved somewhat more difficult and the usual theorem holds under conditions somewhat less general than I had expected. In particular, some care must be exercised to assure the continuity of the derived preference ordering over bundles with the composite commodity as an element.

In section 2 below I describe the properties of preferences which I shall consider and outline the strategy of the remainder of the paper.

2. The Formal Setting

We are given a consumption set X, a subset of n-dimensional Euclidean space, $\mathbb{R}^n$, and a preference relation R on X, with properties to be specified. In the basic composite commodity problem we are interested in a preference relation R^{**} on a set Y of ordered pairs of real numbers $y = (y_1, y_2)$. (Throughout the paper, subscripts on vectors refer to their components.) By choosing a point (y_1, y_2) the individual obtains quantity y_1 of the first good and y_2 "dollars" which he can spend on goods 2 through n at given fixed prices, p_2 through p_n. R^{**} is therefore derived from R, and our inquiry concerns its properties as a function of those of R.

R is a binary relation. The properties I shall assume it may have (separately or in combination) are:

2-1 Definition:

(P1) (Reflexivity) For all x in X, xRx.

(P2) (Completeness) For all x, x' in X, either xRx' or $x'Rx$ or both.

(P3) (Transitivity) For all x, x', x'' in X, if xRx' and $x'Rx''$, then xRx''.

(P4) (Nonsaturation) For all x, x' in X, if $x \geq x'$, then xPx'.

(Here $x \geq x'$ means x no smaller than x' in any component and larger in some. The relation P is derived from R in the usual way: xPx' if and only if xRx' and not $x'Rx$. For general discussions of orderings see, e.g., Debreu (1959), Ch. 1; Fishburn (1972), Ch. 3; Rader (1972), Ch. 3; Sen (1970), Ch. 2.)

(P5) (Convexity) Let $R[x] = \{x' \in X | x'Rx\}$, i.e., the "no worse than x" set. For all x in X, $R[x]$ is convex.

(P6) (Strict Convexity) For all x in X, if x' and x'' are in $R[x]$, and $0 < \tau < 1$, $(\tau x' + (1-\tau)x'')Px$.

(P7) (Continuity) Let $[x]R = \{x' \in X | xRx'\}$, the "no better than x" set. Then for all x in X, $R[x]$ and $[x]R$ are closed.

Properties P1 - P7 are intended to be a representative sample of those customarily ascribed to a preference relation R over the consumption set X. I believe that the methods used here will allow the results to be extended readily to other properties.

The general outline of the way in which R induces a preference relation R** on the set Y of bundles with the composite commodity is simple enough. If $y = (y_1, y_2)$ and $y' = (y_1', y_2')$ are elements of Y we say that yR**y' if the bundle in X consisting of the best combination of y_1 units of the first commodity and such amounts of the remaining commodities as can be purchased at prices $p_2, \ldots, p_n$ for y_2 dollars is as good as (according to R) the best combination of y_1' units of the first commodity and such amounts of the remaining commodities as can be purchased at prices $p_2, \ldots, p_n$ for y_2' dollars. Given that R has one of the properties P1 - P7, we wish to know whether R** has that property.

The choice of one bundle with composite commodity over another is seen to be a choice of one subset of X over another, it being understood that, having selected a subset of X, a further choice of a particular element of that subset can be made according to preference relation R. The relation R** on Y is thus based on a relation R* on 2^X, the set of subsets of X, where R^* is as yet only implicitly defined. We can divide the analysis of the ordering R^{**} of bundles with composite goods into the questions (a) when does an appropriately defined ordering R^* on 2^X preserve the properties of R and (b) when does the ordering R^{**} on Y induced by R^* preserve the properties of R^*?

I shall take up the latter question first. The ordering R^{**} of two elements of Y is defined by reference to the ordering of their images in 2^X under the simple mapping described verbally above. This situation has a simple general form. Given sets V and W and a relation R on W, a mapping f from V into W induces a relation on V which may be devoted R_f. We say of two elements v and v' of V that $vR_f v'$ if and only if $f(v)Rf(v')$. In section 3 sufficient conditions are established for R_f to preserve any of properties P1 - P7 displayed by R.

In section 4 I show that the "composite good mapping" satisfies the sufficient conditions described in section 3, provided that the consumption set X (and hence its power set 2^X) satisfies certain restrictions. Counterexamples are shown for some cases when X does not satisfy these restrictions. As a preliminary to the results in section 4 it is necessary to spell out the definitions of scalar multiplication and addition, of "vector inequality" and of closedness applying to $2^{\mathbb{R}^n}$ (and thus to 2^X) required to make sense of properties P1 - P7.

There remains the problem of relating preferences on 2^X in the appropriate way to preferences on X. This is taken up in section 5. A natural definition is adopted (a set is preferred to another if for every element of the second there is a better element of the first), and it is shown that properties P1 - P6 of R carry over to R* quite generally. Property P7, continuity, requires special treatment, basically because 2^X has both open and closed sets as elements.

Finally, in section 6, I state two composite good theorems which follow directly from the earlier results.

3. Mappings Which Preserve Properties P1 - P7

Suppose that V is a subset of $\mathbb{R}^m$ and W is a subset of $\mathbb{R}^n$, that R is a relation on W with properties P1 - P7, and that f maps V into W. It is convenient to regard R as a subset of $W \times W$, such that if w and w' are elements of W, then wRw' if and only if (w,w') is an element of R. The mapping f induces a relation R_f on V. Again, thinking of R_f as a subset of $V \times V$, we say that if

v and v' are in V, (v,v') is an element of R_f if and only if $(f(v),f(v'))$ is an element of R.

In the language of R as a set of ordered pairs our seven properties become

3-1 <u>Definition</u>:

(P1') For all w in W, $(w,w) \in R$.

(P2') For all w, w' in W, either $(w,w') \in R$, $(w',w) \in R$ or both.

(P3') For all w, w', w'' in W, $(w,w') \in R$ and $(w',w'') \in R$ implies $(w,w'') \in R$.

(P4') For all w, w' in W, if $w \geq w'$ then $(w,w') \in R$ and $(w',w) \notin R$.

(P5') For all w in W the set $R[w] = \{w' \in W \mid (w',w) \in R\}$ is convex.

(P6') For all w in W, w' and $w'' \in R[w]$ implies for $0 < \tau < 1$, $((\tau w' + (1-\tau)w''),x) \in R$ and $(w,(\tau w' + (1-\tau)w'')) \notin R$.

(P7') If $[w]R = \{w' \in W \mid (w,w') \in R\}$, then for all w in W, $R[w]$ and $[w]R$ are both closed.

If W' is a subset of W, $f^{-1}(W')$ is defined to be the subset of elements v' of V such that $f(v')$ is in W'. We require the following lemmas.

3-2 Lemma: If V is convex and f is linear then for all $W' \subset W$, W' convex implies $f^{-1}(W')$ convex.

Proof: Suppose v and v' in $f^{-1}(W')$. Since V is convex, $\tau v + (1-\tau)v'$ is in V. Because f is linear $f(\tau v + (1-\tau)v') = \tau f(v) + (1-\tau)f(v')$. Because W' is convex, this is in W'. Hence $f^{-1}(W')$ is convex. Q.E.D.

3.3 Lemma: If V is closed and f is continuous then for all $W' \subset W$, W' closed implies $f^{-1}(W')$ closed.

Proof: Suppose a sequence of elements $v^1, v^2, \ldots$ in $f^{-1}(W')$ converges to a limit $\bar{v}$. Since V is closed, $\bar{v}$ is in V. Because f is continuous, the sequence $f(v^1), f(v^2), \ldots$ converges in W' to $f(\bar{v})$. If W' is closed $f(\bar{v})$ must be in W'. Hence $\bar{v}$ is in $f^{-1}(W')$; $f^{-1}(W')$ is closed.

Q.E.D.

Now we can state sufficient conditions on V and f to assure that R_f on V has the same properties as R on W:

3-4 Theorem: If V is a convex and closed subset of $\mathbb{R}^n$ and if f is a continuous linear mapping into a subset W of $\mathbb{R}^m$ which preserves natural partial ordering ($\geq$) of vectors then any of properties P1 - P7 (equivalently P1' - P7') displayed by R over W are displayed also by R_f over V.

Proof:

(P1') If R is reflexive, for all v in V, $(f(v),f(v)) \in R$. Hence $(v,v) \in R_f$.

(P2') If R is complete, for all $v,v' \in V$, either $(f(v),f(v')) \in R$ or $(f(v'),f(v)) \in R$ or both. Hence R_f is complete.

(P3') For all $v,v',v'' \in V$, if $(v,v') \in R_f$ and $(v',v'') \in R_f$ then $(f(v),f(v')) \in R$ and $(f(v'),f(v'')) \in R$. If R is transitive $(f(v),f(v'')) \in R$, implying $(v,v'') \in R_f$. Thus R_f is transitive.

(P4') If f preserves vector inequality, $v,v' \in V$, $v \geq v'$ implies $f(v) \geq f(v')$. If R displays nonsaturation $(f(v),f(v')) \in R$ and $(f(v'),f(x)) \notin R$. Hence $(v,v') \in R_f$ and $(v',v) \notin R_f$; R_f also displays nonsaturation.

(P5') If R is convex, for all v in V, the set $R[f(v)]$ is convex. By lemma 3-2, if f is linear and V convex, the set $f^{-1}(R[f(v)])$ is convex. It is readily seen that $f^{-1}(R[f(v)]) = R_f[v]$. For $v' \in f^{-1}(R[f(v)])$ if and only if $(f(v'),f(v)) \in R$ if and only if $(v',v) \in R_f$. Thus for all $v \in V$, $R_f[v]$ is convex.

(P6') Suppose v' and v'' in $R_f[v]$. Since V is convex, $\tau v' + (1-\tau)v''$ is in V for $0 < \tau < 1$. If R displays strict convexity $(\tau f(v') + (1-\tau)f(v''),v) \in R$ and $(v, \tau f(v') + (1-\tau)f(v'')) \notin R$. f being linear this says equivalently, $(f(\tau v' + (1-\tau)v''),v) \in R$ and $(v,f(\tau v' + (1-\tau)v'')) \notin R$. Hence $(\tau v' + (1-\tau)v'', v) \in R_f$ and $(v, \tau v' + (1-\tau)v'') \notin R_f$. Thus R_f is strictly convex.

(P7') If R is continuous $[f(v)]R$ and $R[f(v)]$ are closed for all v in V. If V is closed and f continuous, by lemma 3-3, $f^{-1}([f(v)]R)$ and $f^{-1}(R[f(v)])$ are closed; equivalently $R_f[v]$ and $[v]R_f$ are closed. Hence R_f is continuous

Q.E.D.

4. The Composite Commodity Mapping Preserves Properties of Orderings on 2^X

The composite commodity theorem concerns the connection between a preference relation on the set Y of bundles with composite goods and the preference relation R on X. However, as we have discussed we need to establish along the way that the composite good mapping preserves the properties of a relation on 2^X. In particular we want to show that if a relation R* on 2^X has any of properties P1 - P7, then its image on A via the composite commodity mapping has the same properties. This is basically a matter of showing that the situation satisfies the hypotheses of Theorem 3-4.

Before proceding, however, we must adopt definitions of scalar multiplication, of vector inequality, and of closedness, on $2^{\mathbb{R}^n}$.

4-1 Definition: If X^1 and X^2 are in $2^{\mathbb{R}^n}$ and τ a scalar then

$$\tau X^1 = \{x \in \mathbb{R}^n \mid x = \tau x^1 \text{ for some } x^1 \in X^1\}$$

$$X^1 + X^2 = \{x \in \mathbb{R}^n \mid x = x^1 + x^2 \text{ for some } x^1 \in X^1 \text{ and } x^2 \in X^2\} .$$

These are standard definitions, by which $2^{\mathbb{R}^n}$ becomes a vector space, where the set consisting of the zero vector is the origin.

4-2 Definition: If X^1 and X^2 are in $2^{\mathbb{R}^n}$, $X^1 \geq X^2$ if and only if for all x^2 in X^2 there exists x^1 in X^1 such that $x^1 \geq x^2$.

4-3 Definition: A sequence $X^1, X^2, \ldots,$ of elements of $2^{\mathbb{R}^n}$ converges to a limit $\bar{X}$ if and only if

(a) every sequence of elements of $\mathbb{R}^n$, one from each set X^i, has a limit point, and

(b) any limit point of such a sequence is in $\bar{X}$, and

(c) any element of $\bar{X}$ can be written as the limit of such a sequence of elements of X^i.

4-4 Definition: A subset $\mathcal{S}$ of $2^{\mathbb{R}^n}$ is closed if and only if the limit of any convergent sequence of elements of $\mathcal{S}$ is itself in $\mathcal{S}$.

With these definitions all of properties P1 - P7 become meaningful as applied to relations defined on $2^{\mathbb{R}^n}$. We turn then to the composite commodity mapping from the set Y, a subset of $\mathbb{R}^2$, into 2^X. Although I shall continue thus to deal with the case of a single "natural" good and a single composite of the remaining goods, it should be obvious that the same reasoning applies to any number of composite goods.

4-5 Definition: For $y \in \mathbb{R}^2$, $\rho(y) = \{x \in \mathbb{R}^n | \, x_1 \leq y_1$ and $p_2x_2 + \cdots + p_nx_n \leq y_2\}$ where $p_2,\ldots,p_n$ are non-negative real numbers, not all zero.

4-6 Definition: The composite commodity consumption set, $Y \subset \mathbb{R}^2$, is defined by

$$Y \equiv \rho^{-1}(X)$$

The composite commodity mapping $\rho_X: Y \to 2^X$ is defined for $y \in Y$:

$$\rho_X(y) = \rho(y) \cap X \, .$$

4-7 Lemma: The mapping ρ is linear into $2^{\mathbb{R}^n}$, and if X is (a) convex and closed and (b) has the property that for any x in X there is x' in X, $x' \geq x$, then ρ_X is a continuous, order-preserving mapping from Y into 2^X.

Before proving the lemma it will be helpful to show the function of the restrictions on X. The convexity of X is needed to assure the continuity of ρ_X, as the diagrammed counterexample to the lemma in figure 1 suggests.

In the diagram the area to the northeast of the curved boundary represents the available second and third components of X in 3-space when x_1 is held constant at $\bar{y}_1$. The "budget lines" labelled k_1, k_2 and $\bar{k}$ constrain the quantities of x_2 and x_3 purchasable with k_1 dollars, k_2 dollars and $\bar{k}$ dollars, $k_1 < k_2 < \bar{k}$, when their prices are set at p_2 and p_3. Suppose the sequence of points in Y, $(\bar{y}_1,k_1),(\bar{y}_1,k_2),\ldots$ converges to $(\bar{y}_1,\bar{k})$. Then the corresponding sequence of subsets of X converges to a set with x_2 and x_3 components constrained by the line segment x'x" in figure 1. However the point $(\bar{y}_1,\bar{k})$ in A is mapped into a set with x_2 and x_3 possibilities including the point labeled x"' in figure 1.

To show the order-preserving property of ρ we must use the fact that for any x in X there exists x' in X, $x' \geq x$. Otherwise the counterexample of figure 2 might arise. In the diagram, there is no point in X to the northeast of x', a point in $\rho(y')$, so even though $y' \leq y''$ we do not have $\rho(y'') \geq \rho(y')$. For some plausible preferences the consumer would be indifferent between $\rho(y')$ and $\rho(y'')$, because the extra points in the latter are no better than points already available in $\rho(y')$.

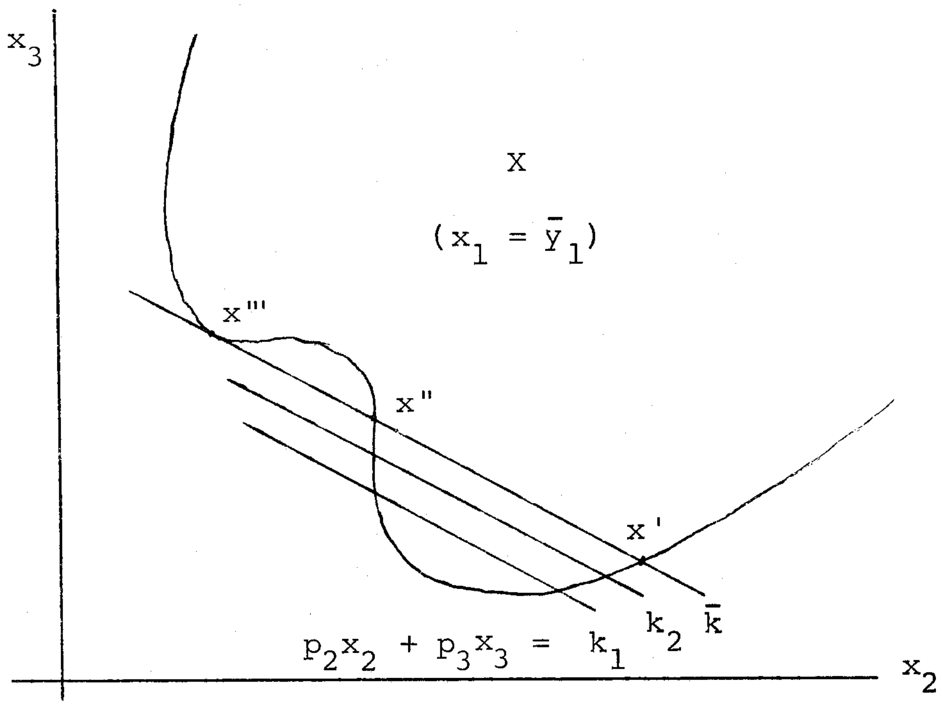

Figure 1. If X is not convex, ρ_X may not be continuous.

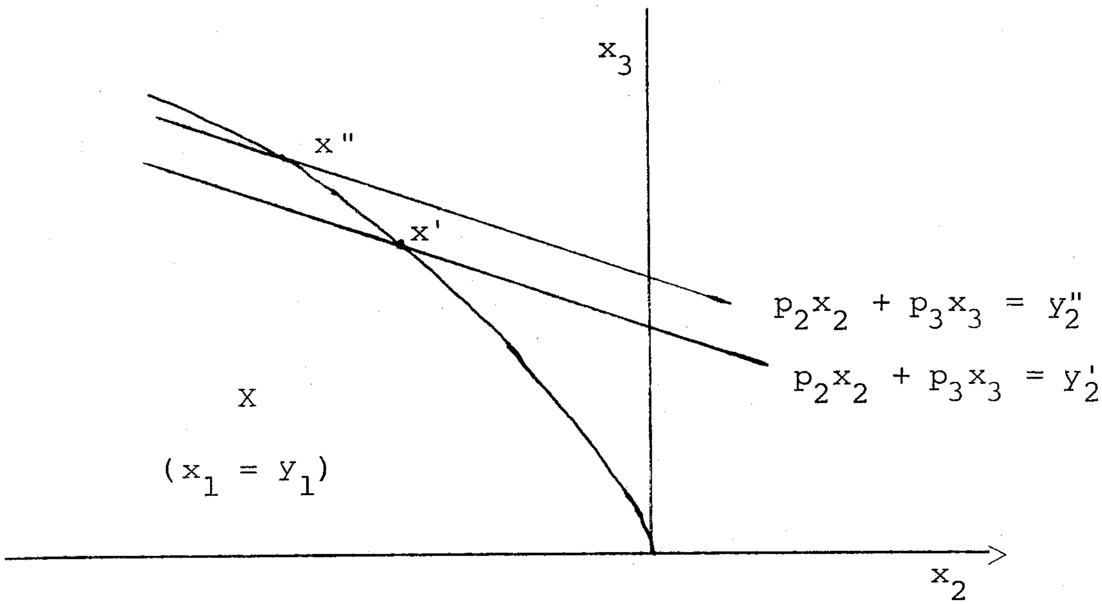

Figure 2. Even though $y' = (y_1, y_2') \leq y'' = (y_1, y_2'')$, there is no element of $\rho(y'')$ northeast of x' in $\rho(y')$.

We turn now to the proof of Lemma 4-7.

Proof of Lemma 4-7: It is apparent that ρ is linear into $\mathbb{R}^n$. In view of the preceding discussion it is further readily shown that ρ_X is order-preserving, and I omit the details. To show the continuity of ρ_X into 2^X, we must show that if a sequence $y^1, y^2, \ldots,$ of elements of Y converges to $\bar{y}$ in Y then the sequence, $\rho_X(y^1), \rho_X(y^2), \ldots,$ converges to $\rho_X(\bar{y})$.

Let $\bar{X}$ be the element of 2^X consisting of all the limit points obtainable by picking sequences of elements of X, one each from $\rho_X(y^1), \rho_X(y^2), \ldots$. It is straightforward to show that $\bar{X}$ is contained in $\rho_X(\bar{y})$. For suppose $\bar{x} \in \bar{X}$. By definition of $\bar{X}$ there is a sequence of elements x^i, with x^i in $\rho_X(y^i)$ with a subsequence converging to $\bar{x}$. By definition of ρ, for each x^i, $x_1^i = y_1^i$ and $p_2 x_2^i + p_3 x_3^i + \ldots + p_n x_n^i \leq y_2^i$. Since the sequence y^i converges to $\bar{y}$, any limit point $\hat{x}$ of the sequence x^i must satisfy $\hat{x}_1 = \bar{y}_1$, $p_2 \hat{x}_2 + \ldots + p_n \hat{x}_n \leq \bar{y}_2$. In particular this must hold for $\bar{x}$. Since X is closed, $\bar{x} \in X$ and hence, by definition of ρ_X, $\bar{x} \in \rho_X(\bar{y})$.

To complete the proof we must show that $\rho_X(\bar{y})$ is contained in $\bar{X}$. Suppose $\bar{x} \in \rho_X(\bar{y})$. Can we find a sequence x^i, one each from $\rho_X(y^i)$, such that x^i converges to $\bar{x}$?

Figure 1 tells us that the convexity of X will play a role in the answer. Suppose the answer were no. That would mean that there is some positive distance ϵ such that, given a number m, arbitrarily large, I can find a larger number m' such that for all $x^{m'}$ in $\rho_X(y^{m'})$, $|\bar{x} - x^{m'}| > \epsilon$. Without loss of generality we may assume that we can pick m large enough so that this is true for all $m' > m$ (simply drop from the original sequence of y's any for which it is not true). The situation described is illustrated in Figure 1, with x''' in Figure 1 playing the role of $\bar{x}$ here. However, since X is assumed to be convex, for any given j if we connect $\bar{x}$ to a point x^j in $\rho_X(y^j)$ by a straight line, all of the points on the line are in X. Such a point is of the form

$$\hat{x}^j = (\tau\bar{x}_1 + (1-\tau)x_1^j,\ \tau\bar{x}_2 + (1-\tau)x_2^j,\ \ldots,\ \tau\bar{x}_n + (1-\tau)x_n^j)\ ,$$

where x^j is in $\rho_X(y^j)$ and where $0 \leq \tau \leq 1$. Such points satisfy

$$\hat{x}_1^j \leq \tau\bar{y}_1 + (1-\tau)y_1^j$$

$$(p_2\hat{x}_2^j + \ldots + p_n\hat{x}_n^j) \leq \tau\bar{y}_2 + (1-\tau)y_2^j$$

By the convergence of y^i to $\bar{y}$, for any fixed $0 < \tau < 1$, and for any fixed j, there is a number m^* large enough so that for $m' > m^*$,

$$\tau\bar{y}_1 + (1-\tau)y_1^j \leq y_1^{m'}$$

$$\tau\bar{y}_2 + (1-\tau)y_2^j \leq y_2^{m'}$$

This means the point in question (the convex combination of $\bar{x}$ and a point in $\rho_X(y^j)$) is in $\rho_X(y^{m'})$ for all $m' > m^*$. By making τ small enough, we can make the point as close to $\bar{x}$ as we wish, in particular closer than ε. Hence there must be no element $\bar{x}$ in $\rho_X(\bar{y})$ and not in $\bar{X}$. Q.E.D.

Lemma 4-7 and Theorem 3-4 together establish that if X is closed and convex the composite commodity mapping generates preferences on Y which have any of properties P1 - P7 holding for preferences in 2^X. We turn then to the origins of preferences on 2^X.

5. Preferences on 2^X Derived from R

The connection between preferences R on X and preferences on 2^X which seems to accord with intuition is the direct analogue of Definition 4-2 of partial ordering of vectors in $2^{\mathbb{R}^n}$. If X^1 and X^2 are subsets of X we say that X^1 is weakly preferred to X^2 if for every element of X^2 there is an element of X^1 weakly preferred according to R. Formally,

5-1 <u>Definition</u>: If X^1 and X^2 are in 2^X, then $X^1R^*X^2$ (equivalently $(X^1,X^2) \in R^*$, viewed as a subset of $2^X \times 2^X$) if and only if for all $x^2 \in X^2$ there exists $x^1 \in X^1$ such that x^1Rx^2 (equivalently, $(x^1,x^2) \in R$).

Now it is a simple matter to prove

5-2 <u>Theorem</u>: If R on X has any of properties P1 - P6, then R* on 2^X has the same properties.

<u>Proof</u>:

(P1) (Reflexivity) To show for X' in 2^X that X'R*X' we need only observe that for all x' in X', x'Rx', by R reflexive.

(P2) (Completeness) Suppose R* were not complete. Then for some X^1 and X^2 in 2^X, neither $X^1R^*X^2$ nor $X^2R^*X^1$. The first means for some $\bar{x}^2$ in X^2 there is no x^1 in X^1 for which $x^1R\bar{x}^2$. Hence, because R is complete, $\bar{x}^2Px^1$ for all x^1 in X^1. Similarly, from the second, there is $\bar{x}^1$ in X^1 such that $\bar{x}^1Px^2$ for all x^2 in X^2. But this would imply $\bar{x}^1 P \bar{x}^2$ and $\bar{x}^2 P \bar{x}^1$, which is impossible. Hence R* must be complete.

(P3) (Transitivity)
(P4) (Nonsaturation)
(P5) (Convexity)
(P6) (Strict Convexity)

} left as exercises.

Q.E.D.

The remaining property, continuity, presents some problems. Figure 3 shows a case in which R is continuous but R* not continuous. The curved line is an ordinary indifference curve, which is asymptotic to the vertical line $(x_1 = 1, x_2 \geq 0)$. The set B consists of all the points in the first quadrant on the vertical line and to the left of it. Define B^i to be the set consisting of all the points obtained by adding to B the vector $\left(\left(\frac{1}{2}\right)^i, 0\right)$. Clearly the sequence of sets B^i converges to B. It is also true that, by our definition each set B^i is better than the set consisting of the single element, x': $B^i P^* \{x'\}$. Yet we see that $\{x'\} P^* B$.

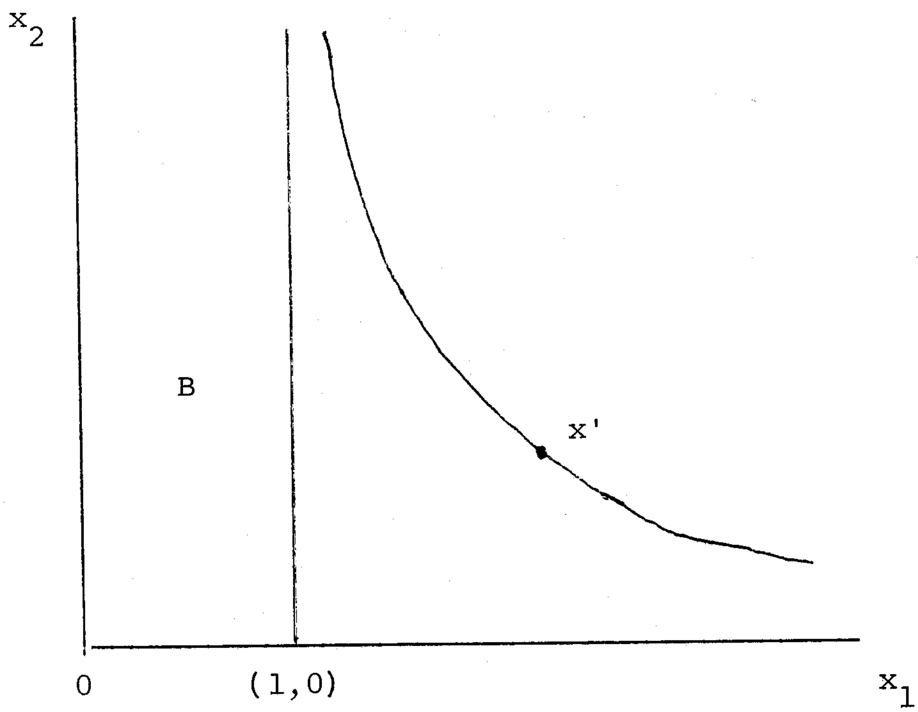

Figure 3

Figure 4 shows a somewhat different case of R* not continuous.

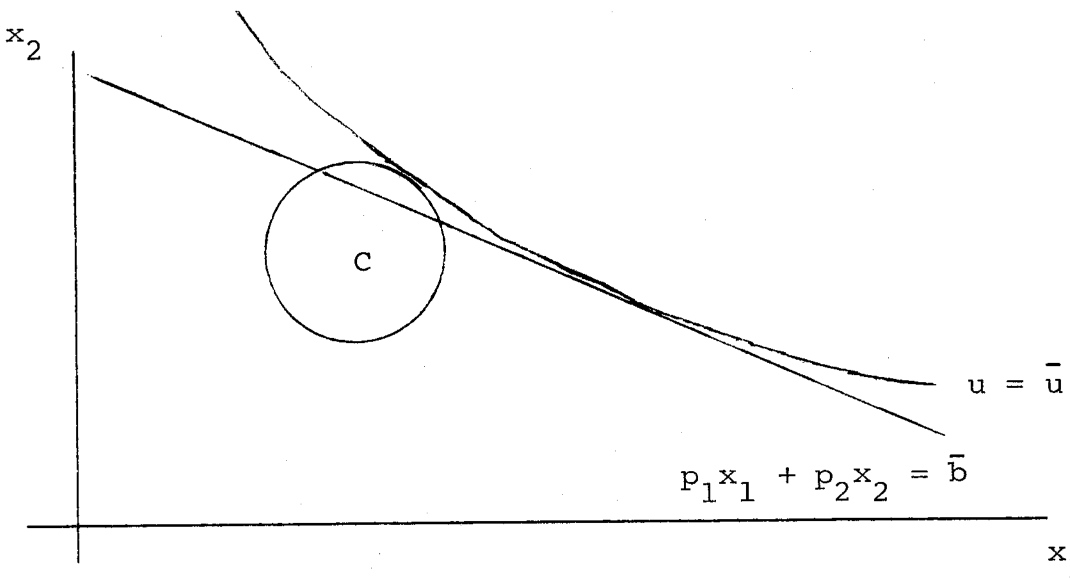

Figure 4

The set C consists of the interior of the circle, but not its boundaries. The curve $u = \bar{u}$ is an ordinary indifference curve tangent to C, and tangent as well to the line $p_1x_1 + p_2x_2 = \bar{b}$. If we take the sequence of ordinary budget sets described by $p_1x_1 + p_2x_2 \leq \bar{b} - \left(\frac{1}{2}\right)^i$, we see that it converges to $p_1x_1 + p_2x_2 \leq \bar{b}$. The latter set is preferred to C, but every set in the sequence is strictly worse than C.

These examples indicate it will be necessary to restrict our attention to the relation induced by R^* on some subset of 2^X in order to obtain a continuity condition. The most obvious candidate is the set 2^X_C of compact subsets of X. In doing so we must be cautious about the meaning of convergence.

5-3 __Definition__: A sequence of elements, X^i, of 2^X_C will be said to __converge in__ 2^X_C if and only if there is an element $\bar{X}$ in 2^X_C such that $\{X^i\}$ converges to $\bar{X}$ in 2^X. Similarly, we say that the relation induced by R^* on 2^X_C is __continuous on__ 2^X_C if, for $X \in 2^X_C$, the limit of any sequence in $R^*[X] \cap 2^X_C$ which converges in 2^X_C is in $R^*[X]$ and the limit of any sequence in $[X]R^* \cap 2^X_C$ which converges in 2^X_C is in $[X]R^*$.

5-4 __Theorem__: If R is continuous on X, R^* is continuous on 2^X_C.

Proof: Suppose a sequence X^i of compact subsets of X converges to the compact subset $\overline{X}$, and that for all i, $X^i \in R^*[X']$, where $X' \in 2^X_C$. We wish to show that $\overline{X} \in R^*[X']$.

We know that for all x' in X' there is at least one element x^i in X^i such that $x^i R x'$. Consider a sequence of such elements x^i. Since the sequence of sets X^i converges the sequence of points x^i has a limit point; call it x^*. Since R is continuous, $x^* R x'$. By definition of $\overline{X}$, $x^* \in \overline{X}$. Hence, for all x' in X' there is an element x^* in $\overline{X}$ such that $x^* R x'$. By definition 4-1, $\overline{X} R^* X'$, i.e. $\overline{X} \in R^*[X']$.

The analogous argument applies to $[X']R^*$. Q.E.D.

6. Composite Good Theorems

It is now a simple matter to state two composite commodity "theorems," which are direct corollaries of Theorems 4-2 and 4-5. These theorems concern the ordering R^{**} on the set Y of bundles with composite commodity, where, in terms of our previous notation, $R^{**} = R^*_{\rho_X}$, where ρ_X, the composite commodity mapping, is defined by 4-6, and R^* is defined by 5-1. For reference we note

6-1 Definition: If y' and y'' are elements of Y, $y' R^{**} y''$ if and only if $\rho_X(y') R^* \rho_X(y'')$.

Then we can state

6-2 Corollary: If X satisfies the conditions of Lemma 4-7, and R has any of the properties P1 - P6, then R** has the same properties on Y. If, furthermore, for all $y \in Y$, $\rho_X(y)$ is compact, then R** is continuous if R is continuous.

6-3 Corollary: (Hicks-Leontief) If X is a closed convex subset of the non-negative orthant, such that $x \in X$ implies $x' \in X$ for all $x' \geq x$, and the prices of the component goods are all positive, then R** on Y has any of the properties P1 - P7 characterizing R on X.

We see that as far as properties P1 - P6 are concerned, the conditions for the composite good theorem in the Corollary 6-1 version are reasonably mild. The condition to assure continuity, however, is less agreeable (although we should keep in mind it is sufficient, perhaps not necessary). It is fulfilled in the Hicks-Leontief version, basically by the non-negativity of X. For many purposes their restriction of X to the non-negative orthant is not acceptable. It is by now fairly standard, for example, to treat an individual's sales of labor as negative consumption of leisure. In externality problems the individual may be subject to arbitrarily large negative effects. It may be necessary in such models to adopt explicit bounding assumptions to assure validity of the composite commodity device.

Bibliography

Debreu, G., Theory of Value, New York: John Wiley & Sons, 1959.

Diewert, W. E., "Hicks' Aggregation Theorem and the Existence of a Real Value Added Function," Technical Report No. 84, The Economics Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, January 1973.

Fishburn, P. C., Mathematics of Decision Theory, The Hague: Mouton, 1972.

Gorman, M. W., "Community Preference Fields," Econometrica, XXI (1953), pp. 63-80.

Green, H. A., Consumer Theory, Baltimore: Penguin Books (1971).

Hicks, J. R., Value and Capital, Oxford: Clarendon Press, 1939, 2nd. ed., 1946.

Leontief, W. W., "Composite Commodities and the Problem of Index Numbers," Econometrica, IV (1936), pp. 39-59.

Rader, T., Theory of Microeconomics, New York: Academic Press, 1972.

Samuelson, P. A., Foundations of Economic Analysis, Cambridge, Mass.: Harvard University Press, 1947.

Sen, A. K., Collective Choice and Social Welfare, San Francisco: Holden-Day, 1970.