

A PROOF OF THE IMPOSSIBILITY OF OBTAINING  
GENERAL WEALTH EFFECT COMPARATIVE STATICS  
PROPERTIES IN PORTFOLIO THEORY

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1. Introduction

Consider an investor who has a certain amount of wealth to invest in a riskless security and several risky securities. The investor's optimal portfolio will depend on his attitudes towards risk, his wealth and the probability distribution of the security returns. An interesting question to ask is how the investor's optimal portfolio is affected by changes in his wealth, given that all other things remain constant. For example, does the total amount invested in risky securities increase as wealth increases? Does the proportion of wealth invested in risky securities decrease as wealth increases?

Questions such as these have been investigated by Arrow [1, Chapter 3], who showed that in the case of one riskless security and one risky security, the amount invested in the risky security is an increasing function of wealth, and the proportion of wealth invested in the risky security is a decreasing function of wealth, if the investor's von Neumann-Morgenstern utility function exhibits decreasing absolute risk aversion and increasing relative risk aversion. More recently,

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Cass and Stiglitz [3] have shown that Arrow's results do not generalize to the case of many risky securities. They give an example where an investor who can purchase one riskless security and two risky securities invests a greater proportion of his wealth in the two risky securities when his wealth increases, even though his utility function exhibits increasing relative risk aversion. Cass and Stiglitz note, however, that Arrow's results do generalize for an important, if highly restrictive, class of utility functions -- those for which the mix of risky securities in the investor's optimal portfolio depends only on the probability distribution of security returns and is independent of the investor's wealth. Such utility functions are said to possess the separation property.

The purpose of this paper is to prove that the separation property is not only a sufficient condition for the generalization of Arrow's results to the many risky security case, but is also a necessary condition. That is, given more than one risky security and a utility function which does not possess the separation property, it is always possible to pick probability distributions for the returns of the risky securities so that the total amount (proportion of wealth) invested in the risky securities decreases (increases) as wealth increases, even though the utility function exhibits decreasing absolute risk aversion (increasing relative risk aversion). In fact,

it is always possible to find cases where the amount (proportion of wealth) invested in every risky security decreases (increases) as wealth increases.

The separation property, then, is essential for the generalization of Arrow's comparative statics results. But we will prove a much stronger result. Exactly the same argument which establishes the importance of the separation property for Arrow's results may also be used to establish the importance of the separation property for every wealth effect comparative statics property. That is, given a utility function which does not possess the separation property, every wealth effect comparative statics property will be violated for some probability distribution of security returns in the many risky security case.

One consequence of this result is that we cannot deduce from portfolio-theoretic considerations alone that the demand for money (regarded as a riskless security) is an increasing function of wealth. A second consequence is that the only hope of obtaining wealth effect comparative statics properties for utility functions which do not possess the separation property is to restrict the probability distributions of security returns in some way. The proof of our main theorem will give us some useful information about precisely the sorts of restrictions that will have to be placed on security returns if progress is to be made in this direction.

## 2. The Relationship Between Comparative Statics Properties and the Separation Property

Consider an investor who has a real-valued von Neumann-Morgenstern utility function  $U$ , which is defined on some interval of the real line. The investor has initial positive wealth  $W$  to invest in one riskless and  $n$  risky securities. We will assume for simplicity that  $n=2$ , but all our results generalize a fortiori to the case  $n > 2$ . The riskless security pays a certain gross rate of return  $r_0$  and the two risky securities pay uncertain gross rates of return  $r_1, r_2$ . We will assume that  $r_0$  is fixed, but that the probability distribution of  $r_1, r_2$  may be chosen arbitrarily.

The investor is assumed to choose real numbers  $x_0, x_1$  and  $x_2$  so as to maximize the expected utility of his final wealth

$$EU(x_0 r_0 + x_1 r_1 + x_2 r_2) \quad (1)$$

subject to his budget constraint  $x_0 + x_1 + x_2 = W$ .  $x_i$  is the monetary amount invested in security  $i$ . If  $x_i > 0$ , the investor is holding the  $i^{\text{th}}$  security long, and if  $x_i < 0$ , the investor is holding the  $i^{\text{th}}$  security short (if  $i=0$ ,  $x_i < 0$  means that the investor is borrowing).<sup>1</sup>  $E$  is the expectations operator.

Using the budget constraint to solve for  $x_0$  and defining  $e_1 = (r_1 - r_0)$  and  $e_2 = (r_2 - r_0)$ , we may rewrite the investor's maximization problem as:

$$\text{Maximize } EU(Wr_0 + x_1e_1 + x_2e_2) \quad . \quad (2)$$

Since, by assumption, the random variables  $r_1, r_2$  may be chosen arbitrarily, it follows that the random variables  $e_1, e_2$  are also arbitrary. We will find it easier to work with  $e_1, e_2$  rather than with  $r_1, r_2$ .

Without further restrictions, (2) may not have a solution, and even if it does, the solution may not be unique.<sup>2</sup> If (2) does have a unique solution, we will write it as  $x_1 = x_1(W, F)$ ,  $x_2 = x_2(W, F)$ , or, in vector form,  $x = x(W, F)$ . We will also say that  $x(W, F)$  is well-defined in this case.  $F$  refers to the probability distribution function of  $e_1, e_2$ , that is,  $F(b_1, b_2) = \text{Prob}\{e_1 \leq b_1, e_2 \leq b_2\}$ . The solution of (2) also depends on  $r_0$ , but this dependence may be ignored since  $r_0$  is assumed to be fixed.

Consider what happens to the solution of (2), assuming that it exists, as  $W$  varies and  $F$  remains constant. In general, the investor will choose to purchase the risky securities in different proportions at different levels of

wealth, that is,  $\frac{x_1(W,F)}{x_2(W,F)}$  will vary with  $W$ . In the special case where  $\frac{x_1(W,F)}{x_2(W,F)}$  depends only on  $F$  and is independent of  $W$ , we will say that  $U$  possesses the separation property.

Definition:  $U$  possesses the separation property if  $x(W_0, F)$  and  $x(W_1, F)$  are linearly dependent vectors whenever they are well-defined.

Let us turn now to the types of comparative statics properties that we might be interested in. Arrow [1, Chapter 3] has shown that in the case of one risky security, the absolute value of the amount invested in the risky security is non-decreasing in wealth if  $U$  exhibits non-increasing absolute risk aversion.<sup>3,4</sup> An appropriate generalization of this result to the case of two risky securities is:

P1)  $|x_1(W,F)| + |x_2(W,F)|$  is non-decreasing in  $W$ ;<sup>5</sup>

or

P2)  $x_1^2(W,F) + x_2^2(W,F)$  is non-decreasing in  $W$ .

Arrow has also shown that in the case of one risky security, the absolute value of the proportion of wealth invested in the risky security is non-increasing in wealth if  $U$  exhibits non-decreasing relative risk aversion.<sup>6</sup> An appropriate generalization of this result to the case of two risky securities is:

$$P3) \quad \frac{|x_1(W,F)| + |x_2(W,F)|}{W} \quad \text{is non-increasing in } W;$$

or

$$P4) \quad \frac{x_1^2(W,F) + x_2^2(W,F)}{W^2} \quad \text{is non-increasing in } W .$$

In each of P1 - P4 , some function of the optimal  $x_1, x_2$  is non-decreasing in  $W$  . In some cases, we might want wealth also to be an argument of this function. For example, we might be interested in conditions under which the total amount invested in the riskless security is non-decreasing in wealth:

$$P5) \quad (W - x_1(W,F) - x_2(W,F)) \quad \text{is non-decreasing in } W .$$

This leads us to consider comparative statics properties of the form

$$P6) \quad f(x_1(W,F), x_2(W,F), W) \quad \text{is non-decreasing in } W ,$$

where  $f$  is a function from  $R^2 \times \Omega$  to  $R$  .<sup>7</sup> This formulation will not be general enough for our purposes, however, since it rules out "either - or" comparative statics properties, such as the following generalizations of Arrow's results which are much weaker than P1 - P4 :

$$P7) \quad \text{If } W_1 > W_0 \quad \text{and } x(W_0, F), x(W_1, F) \quad \text{are well-defined,} \\ \text{then either } |x_1(W_1, F)| \geq |x_1(W_0, F)| \quad \text{or } |x_2(W_1, F)| \geq |x_2(W_0, F)|;$$



P8) If  $W_1 > W_0$  and  $x(W_0, F)$ ,  $x(W_1, F)$  are well-defined,

then either  $\frac{|x_1(W_1, F)|}{W_1} \leq \frac{|x_1(W_0, F)|}{W_0}$  or

$$\frac{|x_2(W_1, F)|}{W_1} \leq \frac{|x_2(W_0, F)|}{W_0} .$$

P7 says that when wealth increases, the holding of some risky security rises in absolute terms; P8 says that when wealth increases, the proportion of wealth invested in some risky security falls in absolute terms. In order to consider cases like P7 and P8, we will consider the following general comparative statics property:

#### General Comparative Statics Property

P\*) If  $W_1 > W_0$  and  $x(W_0, F)$ ,  $x(W_1, F)$  are well-defined, then  $f_j(x(W_1, F), W_1) \geq f_j(x(W_0, F), W_0)$  for some  $j=1, \dots, m$ , where  $f_1, \dots, f_m$  are given functions mapping  $R^2 \times \Omega$  to  $R$ .

In the case of P7,  $f_1 = |x_1(W, F)|$ ,  $f_2 = |x_2(W, F)|$   
and in the case of P8,  $f_1 = -\frac{|x_1(W, F)|}{W}$  and  $f_2 = -\frac{|x_2(W, F)|}{W}$ .

It is clear that for some functions  $f_1, \dots, f_m$ , P\* is a tautology. For example, if  $f_2 = -f_1$ , then either

$f_1$  or  $f_2$  is non-decreasing between any two points. In fact, for  $P^*$  not to be a tautology, there must exist  $\hat{x}, \bar{x} \in \mathbb{R}^2$  and  $\hat{W} > \bar{W} > 0$  such that  $f_j(\hat{x}, \hat{W}) < f_j(\bar{x}, \bar{W})$  for all  $j=1, \dots, m$ . We will make a slightly stronger assumption about the functions  $f_1, \dots, f_m$ :

A1) Given  $\hat{W}, \bar{W}$  satisfying  $\hat{W} > \bar{W} > 0$ , we may find linearly independent vectors  $\hat{x}, \bar{x} \in \mathbb{R}^2$  such that  $f_j(\hat{x}, \hat{W}) < f_j(\bar{x}, \bar{W})$  for all  $j=1, \dots, m$ .

A1 is a very weak assumption which is satisfied in all the cases considered above. All it really says is that the functions  $f_j$  do not depend too strongly on  $W$ . A1 would not be satisfied if  $f_1(x, W) = \sin x_1 + \sin x_2 + W$ .

THEOREM 1: Suppose that A1 holds and there exist functions  $f_1, \dots, f_m$  such that property  $P^*$  is satisfied for all distribution functions  $F$ . Then  $U$  possesses the separation property.

Proof: The main result required to prove the theorem is contained in the following lemma.

Lemma: Suppose  $e_1, e_2$  have distribution function  $F$  and  $x(W, F)$  is the unique solution of the problem:

maximize  $EU(Wr_0 + x_1 e_1 + x_2 e_2)$ . Let  $\alpha, \beta, \gamma, \delta$  be real numbers such that  $\alpha\delta - \beta\gamma \neq 0$  and define the new random variables  $e_1^* = \frac{\alpha e_1 + \beta e_2}{\alpha\delta - \beta\gamma}$  and  $e_2^* = \frac{\gamma e_1 + \delta e_2}{\alpha\delta - \beta\gamma}$ . Then,

the unique solution of the problem: maximize

$EU(Wr_0 + x_1 e_1^* + x_2 e_2^*)$  is given by

$$\begin{aligned} x_1^* &= \delta x_1(W, F) - \gamma x_2(W, F), \\ x_2^* &= -\beta x_1(W, F) + \alpha x_2(W, F). \end{aligned} \quad (3)$$

Proof: Suppose that for some  $x_1', x_2'$ ,

$$EU(Wr_0 + x_1' e_1^* + x_2' e_2^*) \geq EU(Wr_0 + x_1^* e_1^* + x_2^* e_2^*). \quad (4)$$

Define

$$\begin{aligned} \tilde{x}_1 &= \frac{\alpha x_1' + \gamma x_2'}{\alpha\delta - \beta\gamma}, \\ \tilde{x}_2 &= \frac{\beta x_1' + \delta x_2'}{\alpha\delta - \beta\gamma}. \end{aligned} \quad (5)$$

Then, by (3), (4), (5) and the definition of  $e_1^*, e_2^*$ ,

$$\begin{aligned} EU(Wr_0 + \tilde{x}_1 e_1 + \tilde{x}_2 e_2) &= EU(Wr_0 + \left(\frac{\alpha x_1' + \gamma x_2'}{\alpha\delta - \beta\gamma}\right) e_1 + \left(\frac{\beta x_1' + \delta x_2'}{\alpha\delta - \beta\gamma}\right) e_2) \\ &= EU(Wr_0 + x_1' \left(\frac{\alpha e_1 + \beta e_2}{\alpha\delta - \beta\gamma}\right) + x_2' \left(\frac{\gamma e_1 + \delta e_2}{\alpha\delta - \beta\gamma}\right)) \end{aligned}$$

$$\begin{aligned}
&\geq EU(Wr_0 + x_1^* \left( \frac{\alpha e_1 + \beta e_2}{\alpha\delta - \beta\gamma} \right) + x_2^* \left( \frac{\gamma e_1 + \delta e_2}{\alpha\delta - \beta\gamma} \right)) \\
&= EU(Wr_0 + (\delta x_1(W,F) - \gamma x_2(W,F)) \left( \frac{\alpha e_1 + \beta e_2}{\alpha\delta - \beta\gamma} \right) \\
&\quad + (\alpha x_2(W,F) - \beta x_1(W,F)) \left( \frac{\gamma e_1 + \delta e_2}{\alpha\delta - \beta\gamma} \right)) \\
&= EU(Wr_0 + x_1(W,F) e_1 + x_2(W,F) e_2) \quad . \quad (6)
\end{aligned}$$

Since  $x(W,F)$  is the unique solution of (2), (6) implies that

$$\begin{aligned}
\tilde{x}_1 &= x_1(W,F) , \\
\tilde{x}_2 &= x_2(W,F) .
\end{aligned} \tag{7}$$

If we now solve (5) for  $x'_1, x'_2$  and use (7), we obtain

$$\begin{aligned}
x'_1 &= \delta \tilde{x}_1 - \gamma \tilde{x}_2 = \delta x_1(W,F) - \gamma x_2(W,F) , \\
x'_2 &= -\beta \tilde{x}_1 + \alpha \tilde{x}_2 = -\beta x_1(W,F) + \alpha x_2(W,F) .
\end{aligned}$$

Hence, by (3),

$$x_1' = x_1^* ,$$

and

$$x_2' = x_2^* ,$$

which proves that  $(x_1^*, x_2^*)$  is the unique solution of the problem: maximize  $EU(Wr_0 + x_1 e_1^* + x_2 e_2^*)$  . Q.E.D.

The lemma tells us that if short-sales are permitted, a linear transformation of the random variables  $e_1, e_2$  can always be "undone" by a linear transformation of  $x_1, x_2$  .

Let us see now how the lemma may be used to prove the theorem. Suppose that  $U$  does not possess the separation property. Then we may find a distribution function  $F$  and  $\hat{W} > \bar{W}$  such that  $x(\hat{W}, F)$  and  $x(\bar{W}, F)$  are linearly independent. Also, by assumption A1, we may choose linearly independent vectors  $\hat{x}$  and  $\bar{x}$  such that

$$f_j(\hat{x}, \hat{W}) < f_j(\bar{x}, \bar{W}) \quad \text{for all } j=1, \dots, m . \quad (8)$$

The idea of the proof is to construct a distribution function  $F^*$  such that  $x(\bar{W}, F^*) = \bar{x}$  and  $x(\hat{W}, F^*) = \hat{x}$  . Since by (8),  $f_j$  is decreasing between  $\bar{W}$  and  $\hat{W}$  for all  $j$  , this shows that property  $P^*$  does not hold for wealth levels  $\bar{W}$  and  $\hat{W}$  and distribution function  $F^*$  , and hence the theorem is proved by contradiction.

Let  $\alpha, \beta, \gamma, \delta$  be given by

$$\alpha = \frac{\bar{x}_2 x_1(\hat{W}, F) - \hat{x}_2 x_1(\bar{W}, F)}{x_2(\bar{W}, F) x_1(\hat{W}, F) - x_2(\hat{W}, F) x_1(\bar{W}, F)},$$

$$\beta = \frac{\bar{x}_2 x_2(\hat{W}, F) - \hat{x}_2 x_2(\bar{W}, F)}{x_2(\bar{W}, F) x_1(\hat{W}, F) - x_2(\hat{W}, F) x_1(\bar{W}, F)},$$

$$\gamma = \frac{\hat{x}_1 x_1(\bar{W}, F) - \bar{x}_1 x_1(\hat{W}, F)}{x_2(\bar{W}, F) x_1(\hat{W}, F) - x_2(\hat{W}, F) x_1(\bar{W}, F)},$$

$$\delta = \frac{\hat{x}_1 x_2(\bar{W}, F) - \bar{x}_1 x_2(\hat{W}, F)}{x_2(\bar{W}, F) x_1(\hat{W}, F) - x_2(\hat{W}, F) x_1(\bar{W}, F)}.$$

$\alpha, \beta, \gamma, \delta$  are well-defined since the linear independence of  $x(\bar{W}, F)$  and  $x(\hat{W}, F)$  implies that  $x_2(\bar{W}, F) x_1(\hat{W}, F) - x_2(\hat{W}, F) x_1(\bar{W}, F) \neq 0$ . It may be verified that, in addition, the linear independence of  $\bar{x}$  and  $\hat{x}$  (see assumption A1) implies that  $\alpha\delta - \beta\gamma \neq 0$ . We may therefore apply the lemma to deduce that

$$\begin{aligned} x_1(W, F^*) &= \delta x_1(W, F) - \gamma x_2(W, F), \\ x_2(W, F^*) &= -\beta x_1(W, F) + \alpha x_2(W, F), \end{aligned} \tag{9}$$

where  $F^*$  is the distribution function of  $\frac{\alpha e_1 + \beta e_2}{\alpha\delta - \beta\gamma}$ ,

$\frac{\gamma e_1 + \delta e_2}{\alpha\delta - \beta\gamma}$ . Substituting the values of  $\alpha, \beta, \gamma, \delta$  in (9), we obtain

$$x(\bar{W}, F^*) = \bar{x},$$

$$x(\hat{W}, F^*) = \hat{x}.$$

But, since  $\hat{W} > \bar{W}$ , it follows by property  $P^*$  that

$$f_j(\hat{x}, \hat{W}) \geq f_j(\bar{x}, \bar{W})$$

for some  $j=1, \dots, m$ . This contradicts (8) and establishes that  $U$  must have the separation property. Q.E.D.

We may use this theorem to derive necessary and sufficient conditions for  $P1 - P4$  and  $P7$  and  $P8$  to hold. We have shown that a necessary condition is that  $U$  possess the separation property. But, if  $U$  possesses the separation property, the risky securities may be regarded as a composite risky security (the composition of this security depending on the probability distribution of security returns), and so we may use Arrow's results for the single risky security case to prove the following: (1) If  $U$  is twice differentiable and  $U' > 0$ ,  $U'' < 0$ , a necessary and sufficient condition for

P1, P2 and P7 to hold is that  $U$  possess the separation property and  $U$  exhibit non-increasing absolute risk aversion; (2) if  $U$  is twice differentiable and  $U' > 0$ ,  $U'' < 0$ , a necessary and sufficient condition for P3, P4 and P8 to hold is that  $U$  possess the separation property and  $U$  exhibit non-decreasing relative risk aversion.

Cass and Stiglitz [2] have shown that the class of utility functions which possess the separation property for all probability distributions is very small. Theorem 1 tells us, therefore, that the range of comparative statics properties in the many security case is very limited. One approach to finding new properties is to restrict the admissible probability distributions of returns in some way. It should be noted, however, that in the proof of Theorem 1, we needed only the assumption that the class  $C$  of admissible two dimensional random variables satisfies:

$$(r_1, r_2) \in C \Rightarrow (\lambda r_1 + \mu r_2 + (1-\lambda-\mu)r_0, \pi r_1 + \rho r_2 + (1-\pi-\rho)r_0) \in C \quad (10)$$

for all real numbers  $\lambda, \mu, \pi, \rho$ . This condition holds, for example, for the class of bivariate normally distributed random variables. For this class, of course, as is well known from mean-variance analysis, all utility functions possess the separation property, Arrow's results generalize to the many security case, and Theorem 1 is trivially true.



This suggests two possible directions which could be taken in restricting the probability distributions of security returns. The first is to assume that the class  $C$  of two dimensional random variables satisfies (10) and to investigate under what conditions a large number of utility functions possess the separation property for the class  $C$ . The second possibility is to assume that  $C$  does not satisfy (10), in which case the separation property may not be necessary for the existence of comparative statics properties. Some progress has been made in this second direction by Cass and Stiglitz [3], who are able to obtain comparative statics properties for the case where each risky security yields a return in only one state of nature and there are as many securities as states of nature.

### Conclusion

We have shown in this paper that in the many security case a large class of comparative statics properties can hold only for utility functions possessing the separation property. Throughout, we have confined our attention to wealth effect comparative statics properties. It would be interesting to know whether the separation property is necessary also for the existence of comparative statics properties involving changes in the riskless interest rate  $r_0$  or in the distribution function  $F$ . The arguments used in the proof of Theorem 1

do not appear to carry over to these cases. However, it seems likely that the separation property is important also for these types of comparative statics properties, since the Slutsky equations of portfolio theory (see Fischer [4]) indicate that there is a close relationship between portfolio adjustments resulting from changes in  $r_0$  and  $F$  and portfolio adjustments resulting from changes in wealth.

## FOOTNOTES

1. Many of our results can be generalized to the case where short-selling is prohibited.
2. We will say that  $(x'_1, x'_2)$  is the unique solution of (2) if the following three conditions are met:
  - (a)  $(x'_1, x'_2) \in D = \{(x_1, x_2) | (Wr_0 + x_1e_1 + x_2e_2) \text{ assumes values for which } U \text{ is well-defined with probability 1}\}$ ;
  - (b)  $EU(Wr_0 + x_1e_1 + x_2e_2)$  is well-defined for all  $(x_1, x_2) \in D$ ;
  - (c)  $EU(Wr_0 + x'_1e_1 + x'_2e_2) > EU(Wr_0 + x_1e_1 + x_2e_2)$  for all  $(x_1, x_2)$  such that  $(x_1, x_2) \in D$  and  $(x_1, x_2) \neq (x'_1, x'_2)$ .

We will assume that  $U$  is measurable, so that

$EU(Wr_0 + x_1e_1 + x_2e_2)$  is well-defined for all  $(x_1, x_2) \in D$  as long as  $e_1, e_2$  are bounded with probability 1. If  $e_1, e_2$  are unbounded, however,  $EU(Wr_0 + x_1e_1 + x_2e_2)$  may not be well-defined.

3. Arrow rules out short-sales, in which case there is no need to worry about absolute values. If the risky security is held short, however, the amount invested in the risky security is non-increasing, rather than non-decreasing, in wealth if the utility function exhibits non-increasing absolute risk

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