# CONVERGENCE OF THE AGE STRUCTURE:
# APPLICATIONS OF THE PROJECTIVE METRIC*

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In this paper we state necessary and sufficient conditions for the convergence of the age structure (in a discrete time, one sex model of population growth) and we give a new and simple proof of the weak ergodic theorem of stable population theory. The main tool we use to attain these results is Hilbert's notion of the projective metric. This metric provides a way of defining the distance between positive vectors in $\mathbb{R}^n$ which has two important features: first the distance between any two positive vectors depends only on the rays on which the vectors lie; second, positive matrices act as contractions in this metric. These ideas will be made precise in §1.

## §1. The Projective Metric

Let $x$ and $y$ be vectors in $\mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We shall adopt the following conventions for vector inequalities:

i) $x \geq y$ iff (if and only if) $x_i \geq y_i$ for all $i$ ;

ii) $x > y$ iff $x \geq y$ and $x \neq y$ ;

iii) $x >> y$ iff $x_i > y_i$ for all $i$;

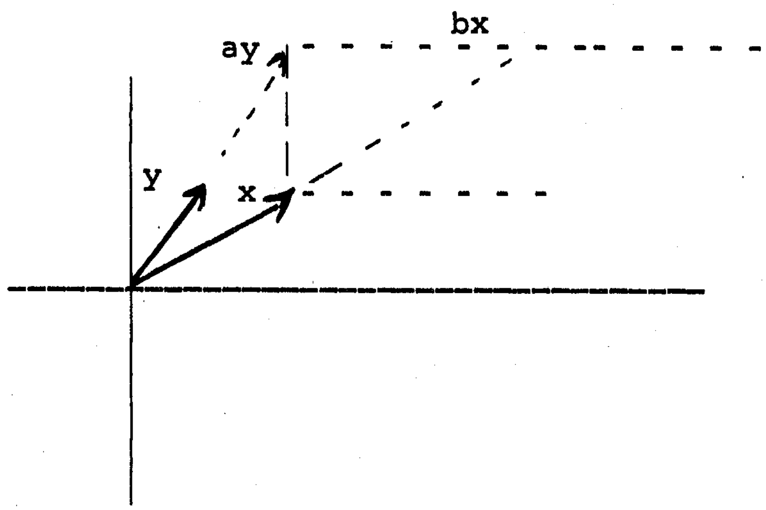
The vector $x$ is <u>positive</u> if $x > 0$ (where $0$ is the vector all of whose components are $0$). The <u>positive orthant</u> is the set in $\mathbb{R}^n$ which consists of all the positive vectors. A vector $x$ is <u>strictly positive</u> if $x >> 0$. The same terminology applies to matrices. In this paper the set of all strictly positive vectors in $\mathbb{R}^n$ will be denoted $\Omega$.

We define a distance between two vectors in the positive orthant of $\mathbb{R}^n$ and then show that this distance depends only on which rays the given vectors lie. (A ray in $R^n$ is a half line starting at the origin.) Let $x$ and $y$ be positive vectors in $\mathbb{R}^n$. Define $p(x,y)$ as follows:

(1) Suppose there exist scalars $a$ and $b$ such that $x \leq a y$ and $a y \leq b x$, then define

$$p(x,y) = \min \ell n(b)$$

where this minimum is taken over all pairs $(a,b)$ satisfying the above inequalities. This figure shows how the choice of $a$ and $b$ can be made.




(2) If no such scalars a and b exist, then define $p(x,y) = \infty$ .

A way of computing $p(x,y)$ when $x$ and $y$ are strictly positive, is as follows: Let $r = \max_{1 \le i \le n} (\frac{x_i}{y_i})$ and $s = \min_{i \le i \le n} (\frac{x_i}{y_i})$ . Then $p(x,y) = \ln(\frac{r}{s})$ . Note this method will not work when $x = (1,0) = y$ .

As an example of (1), let $x = (1,1,2)$ and $y = (3,2,1)$; then $p(x,y) = \ln 6$ . As an example of (2) let $x = (1,0)$ and $y = (1,1)$.

<u>Definition:</u> $p$ is called the projective metric and the number $p(x,y)$ is the <u>projective distance</u> from $x$ to $y$.

The following lemma which states some basic facts about $p$ , justifies this terminology.

**LEMMA 1.1:** Let $x$, $y$, and $z$ be positive vectors in $\mathbb{R}^n$. Then

(i) $p(x,y) = p(rx,sy)$ where $r$ and $s$ are positive scalars. Thus $p$ only depends on the rays generated by $x$ and $y$ in the positive orthant.

(ii) $p(x,y) \geq 0$.

(iii) $p(x,y) = 0$ iff $x = ay$ for some positive scalar $a$.

(iv) $p(x,y) = p(y,x)$.

(v) $p(x,y) \leq p(x,z) + p(z,y)$ (the triangle in equality).

**PROOF:** (i) Suppose $x \leq ay \leq bx$, then $rx \leq r(ay) = \frac{ra}{s}(sy)$. Let $a' = \frac{ra}{s}$. Now $a'(sy) = r(ay) \leq r(bx) = b(rx)$. So $rx \leq a'(sy)$ and $a'(sy) \leq b(rx)$. Thus a "b" which works to compute $p(x,y)$ also works to compute $p(rx,sy)$. The process is reversible, so $p(rx,sy) = p(x,y)$.

(ii) If $x \leq ay \leq bx$, then $x \leq bx$, which implies $b \geq 1$. So $\ln b \geq 0$.

(iii) Since $x \leq x$, $p(x,x) = \ln(1) = 0$. By (i) $p(x,ax) = 0$.

Conversely suppose that $p(x,y) = 0$. Then we can take $b = 1$. So that $x \le ay \le x$ and $x = ay$.

(iv) If $x \le ay \le bx$, then $y \le \frac{b}{a} x \le \frac{b}{a}(ay) = by$. So a "b" which works to compute $p(x,y)$ also works to compute $p(y,x)$ and $p(x,y) = p(y,x)$.

(v) If either $p(x,z)$ or $p(x,y) = \infty$, the inequality holds trivially. So let $p(x,z) = \ln b$ and $p(z,y) = \ln b'$. Then $x \le az \le bx$ and $z \le a'y \le b'z$, for constants $a$ and $a'$. So

$$x \le az \le a(a'y) \le a(b'z) = b'(az) \le b'(bx)$$

or

$$x \le (aa')y \le (bb')x .$$

Hence $p(x,y) \le \ln(bb') = \ln b + \ln b' = p(x,z) + p(z,y)$.

Note that the projective distance between any two vectors in $\Omega$, that is, between any two strictly positive vectors, is finite.

LEMMA 1.2: Fix $y$ in $\Omega$. Then the real-valued function $f$ defined on $\Omega$ by $f(x) = p(x,y)$ is continuous.

<u>PROOF:</u> As long as $x$ and $y$ are in $\Omega$, there exist constants $a$ and $b$ such that $x \le ay \le bx$. It should

be clear that the choice of $a$ and $b$ can be made continuously as $x$ varies. (Look at the previous figure).

<u>Proposition 1.3:</u> Let $x$ and $y$ belong to the positive orthant of $\mathbb{R}^n$ and let $S$ be a non-negative $n \times n$ matrix. Then $p(Sx,Sy) \le p(x,y)$. If $S \gg 0$ (all the entries of $S$ are positive) then $S$ is a strict contraction relative to $p$; i.e., $p(Sx, Sy) \le K_S\, p(x,y)$ for all positive vectors $x$ and $y$ where $K_S$ is a constant $< 1$. The contraction constant $K_S$ varies continuously with the entries of $S$.

<u>PROOF:</u> The complete proof is too complicated to present here. It is given in Chapter 16 of Birkhoff [1]. The first part however, is easy. If $p(x,y) = \infty$, then $p(Sx,Sy) \le p(x,y)$. If $p(x,y) < \infty$ then for some $a$ and $b$,

$$x \le ay \le bx .$$

Since $S$ is a positive matrix,

$$Sx \le aSy \le bSx$$

so that if $p(x,y) = \ln b$, $p(Sx,Sy) \le \ln b$.[1/]

[1] See [1, Chap. 16] and [4] for other discussions and applications of the projective metric.

## §2. The Weak Ergodic Theorem

The discrete, one sex model of population growth may be sketched as follows: break the population into $n$ equally spaced age groups and let $v_o$ be the vector whose $i^{th}$ component is the number of people in the $i^{th}$ age group; $v_o$ is a vector in the positive orthant of $\mathbb{R}^n$. If we let $\|v\| = |v_1|+\ldots+|v_n|$ where $v = (v_1,\ldots,v_n)$, then $\tilde{v}_o = v_o/\|v_o\|$ gives the vector of percentages of people in each age group; $\tilde{v}_o$ is the age structure vector at time $0$. Suppose that each age group has the birth rates $(b_1, \ldots, b_n)$ and survival rates $(s_1,\ldots,s_{n-1})$. The survival rate for the oldest age group, $s_n$, is necessarily equal to $0$. In the discrete time model, birth rates necessarily include some component of survival, only in continuous models is fertility completely separated from mortality. This is an inconvenient, although not unimportant fact, which we shall henceforth ignore.

Let

$$T = \begin{pmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & 0 \end{pmatrix}.$$

Then the population and age structure vectors next period are given by $v_1 = T v_o$ and $\tilde{v}_1 = v_1/\|v_1\|$. Of course, we must measure time so that it takes exactly one period to move from one age category to the next. Thus, if the data in T represent birth and survival rates for 5 year intervals, a single time period is 5 years long. T is called a <u>population matrix</u> and is a positive $n \times n$ matrix. The Perron-Frobenius Theorem[1] states that a positive matrix like T has a unique positive eigenvalue $\lambda$ whose modulus is exceeded by no other eigenvalue and a unique positive eigenvector e with $Te = \lambda e$ and $\|e\| = 1$. Such a matrix T is <u>primitive</u> if T has no other eigenvalue whose modulus is $\lambda$. A sufficient condition for a population matrix T to be primitive is that the survival rates $s_i$, the last birthrate, $b_n$, and birth-rates in the middle age groups are non-zero (i.e., two successive age groups -- not including the first -- have positive birth rates).[2] With these conditions $T^{\ell} \gg 0$ for some integer $\ell$.

If the birth and survival rates are constant, then after k time periods the age structure is $\tilde{v}_k = T^k v_o/\|T^k v_o\|$.

These conditions are sufficient to ensure the convergence of the age structure, $v_k$, to the vector e independent of the initial population. This result is the Strong Ergodic Theorem of stable population theory.

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[1] See [8] for a proof. We assume that T is indecomposable.

[2] If $b_n = 0$ and $b_k$ is the last non-zero birth rate, it is common to consider only the $k \times k$ matrix composed of the first k rows and columns of T. This essentially determines all facts of demographic interest. See [7], [8] and [9].

THEOREM 2.1: (The Strong Ergodic Theorem). Under the conditions stated $\lim_{k\to\infty} \tilde{v}_k = e$ .

PROOF: This is a simple consequence of the Perron-Frobenius Theorem. We present a different proof here -- one based on Proposition 1.3 -- which is almost identical to the proof of the weak ergodic theorem which we give below. Since both $e$ and $\tilde{v}_k$ are of unit length, it will suffice to prove that $\lim_{k\to\infty} p(\tilde{v}_k, e) = \lim_{k\to\infty} p(T^k v_o, e) = 0$. Let $S = T^{\ell} >> 0$

then $T^k = T^{k-\ell[k/\ell]} S^{[k/\ell]} = U_k \; S^{[k/\ell]}$

where $[k/\ell]$ is the greatest integer in $k/\ell$ and $U_k = T^{k-\ell[k/\ell]}$ . If $k/\ell$ is an integer $U_k$ is the identity matrix; in any event, $U_k$ is positive and, by Proposition 1.3, $p(U_k x, U_k y) \leq p(x,y)$ for all positive $x$ and $y$ .

Since $e$ is an eigenvector of $T$ , $e$ is an eigenvector of $T^k$ . Thus,

$$p(T^k v_o, e) = p(T^k v_o, T^k e) =$$

$$p(U_k \; S^{[k/\ell]} v_o \; , \; U_k \; S^{[k/\ell]} e) \leq p(S^{[k/\ell]} v_o \; , \; S^{[k/\ell]} e)$$

$$< K_S^{[k/\ell]-1} \; p(Sv_o, Se) \; ,$$

where $K_S < 1$ is the contraction constant whose existence is guaranteed by Proposition 1.3. Since $S >> 0$ , $Sv_o$ and $Se$ are in $\Omega$ so that $p(Sv_o, Se)$ is finite. Clearly $\lim_{k\to\infty} p(T^k v_o, e) \to 0$.

It is unrealistic to assume that the birth and survival rates do not change with time. However, if these rates are constant over a single time period, then ergodic analysis is still possible. Let $T_k$ be the matrix of birth and survival rates during the $k^{th}$ time period. Then after $k$ periods population and age structure vectors are $v_k = T_k \cdot T_{k-1} \cdots T_1 v_o$ and $\tilde{v}_k = v_k / \|v_k\|$. It is no longer true that the $\tilde{v}_k$'s converge but under rather modest assumptions on the $T_k$'s, it is still true that for large $k$'s, the vectors $\tilde{v}_k$ are independent of $v_o$. This is the Weak Ergodic Theorem which we now prove.

<u>THEOREM 2.2:</u> (The Weak Ergodic Theorem). Let $T_1, T_2, \ldots$ be a sequence of primitive population matrices satisfying $M \leq T_k \leq N$ for all $k$, where M and N are fixed positive matrices and $M^{\ell} >> 0$. Let $S_k = T_k \cdot T_{k-1} \cdots T_1$. Then if $v_o$ and $w_o$ are any strictly positive vectors $\lim_{k\to\infty} p(S_k v_o, S_k w_o) = 0$.

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<u>Remark:</u> Since the projective distance depends only on rays, this theorem states that $p(\tilde{v}_k, \tilde{w}_k) \to 0$. It is possible to use these techniques to prove a slightly stronger result -- that

$$p(\tilde{v}_k, \tilde{w}_k) < C\, K^{[k/\ell]-1}, \qquad K < 1$$

where the constants C and K and $\ell$ depend on M and not on $v_o, w_o$ or the $T_k$'s. Thus the speed of convergence can be bounded independent of $v_o$ and $w_o$. See [5] for details.

As in the proof of the strong ergodic theorem, the key is to apply proposition 1.3 to products of the $T_k$'s taken $\ell$ at a time. The following Lemma states that this can be done.

LEMMA 2.3: There exists a constant $K < 1$ such that is $S$ is any $\ell$-fold product of the $T_k$'s ; i.e., $S = T_{k_1}, \ldots, T_{k_\ell}$, then $S$ contracts projective distance by at least $K$.

PROOF OF LEMMA 2.3: The boundedness assumption in the $T_k$'s implies that $M^\ell \leq S \leq N^\ell$. By Proposition 1.3, $S$ contracts distance in the projective metric by a factor $K_S$. Recall that a set in $\mathbb{R}^{n^2}$ is compact if it is closed and bounded and that any continuous function on a compact set achieves its minimum and maximum. Now the set of matrices $S$ satisfying $M^\ell \leq S \leq N^\ell$ is closed and bounded in $\mathbb{R}^{n^2}$ and therefore compact. Since $K_S$ varies continuously with $S$ (Proposition 1.3 again) there is a $K > 0$ such that $K_S \leq K < 1$ for all such $S$.

PROOF OF THEOREM 2.1: Again, let $[k/\ell]$ be the greatest integer in $k/\ell$. Then

$$S_k = U_k V_{[k/\ell]} \cdots V_1 ,$$

where $V_i = T_{i\ell}, T_{i\ell-1} \cdots T_{(i-1)\ell+1}$ and $U_k$ is the identity matrix if $k/\ell$ is an integer and otherwise $U_k = T_k \cdots T_{[k/\ell]+1}$. In either case $U_k$ is a positive (but not necessarily strictly positive) matrix so that by Proposition 1.3, $p(U_k x, U_k y) \le p(x,y)$ for all positive $x$ and $y$. Then

$$\begin{aligned} p(S_k v_o, S_k w_o) &= p(U_k V_{[k/\ell]} \cdots U_1 v_o , U_k V_{[k/\ell]} \cdots V_1 w_o) \\ &\le p(V_{[k/\ell]} \cdots V_1 v_o, V_{[k/\ell]} \cdots V_1 w_o) \\ &\le K^{[k/\ell]-1} p(V_1 v_o, V_1 w_o) . \end{aligned}$$

where $K < 1$ is the contraction constant whose existence is guaranteed by Lemma 2.3. Since $V_1 >> 0$, $p(V_1 v_o, V_1 w_o)$ is finite and $K^{[k/\ell]-1} p(V_1 v_o, V_1 w_o)$ converges to $0$.[1]

## §3. Necessary and Sufficient Conditions for Convergence of Age Structure

The strong ergodic theorem states that if the population matrices $T_k$ are constant in time ($T_k = T$ for all $k$) and primitive then the age structure converges. The weak ergodic theorem allows $T_k$ to vary (with some boundedness assumptions) and concludes that the age structure -- in the long run -- does not depend on the initial population distribution. In general, it is *not true that* the age structure must converge. We now present conditions which are both necessary and sufficient *for* the convergence of the age structure.

We make the boundedness assumption of section §2 on the sequence $T_k$, namely, $M \le T_k \le N$ for all $k$ where

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[1] For other discussions of the basic results of stable population theory see, e.g. [2], [6], [7] and [9].

$M^{\ell} > 0$ for some $\ell$. This implies that the survival rates and the middle and last birth rates of each $T_k$ are bounded away from zero. Thus each $T_k$ is primitive, has a unique positive eigenvalue $\lambda_k$ which dominates all other eigenvalues of $T_k$ (in modulus) and has a unique positive engenvector $e_k$ with $\|e_k\| = 1$.

<u>THEOREM 3.1:</u> With the assumptions and notation just given, the age structure vector converges to a vector $e$ iff $\lim_{k\to\infty} e_k = e$.

<u>Remark:</u> Recall that in a compact set any sequence has a convergent subsequence. Moreover a sequence in a compact set converges if every convergent subsequence converges to the same point.

<u>PROOF:</u> Given an initial age structure vector $\tilde{v}_o = v_o$, define, as usual, $\tilde{v}_k = T_k \tilde{v}_{k-1}/\|T_k \tilde{v}_{k-1}\|$. The problem is not changed if we multiply each $T_k$ by some positive scalar. So we may assume that $\lambda_k = 1$ for all $k$.

Necessity: Assume $\lim_{k\to\infty} \tilde{v}_k = e$. We must show that

$\lim_{k\to\infty} e_k = e$ . Since the $e_k$'s are all unit length vectors, they vary within a compact set. Let $e_{k_1}, e_{k_2}, \ldots$ be a subsequence converging to $f$ . By the remark above it is sufficient to show that $e = f$. Since the $T_{k_i}$'s vary within a compact subset, there is a convergent subsequence. So, by passing to a subsequence if necessary, we may assume that $\lim_{i\to\infty} T_{k_i} = T$ , with $M \leq T$ . The assumptions on $M$ guarantee that $T$ has a unique positive eigenvector with eigenvalue $1$ . (Note the modulus of the largest eigenvalue depends continuously on the matrix. Since that eigenvalue is $1$ for each $T_{k_i}$ it must be $1$ for $T$.) In fact $Tf = \lim_{i\to\infty} T_{k_i}(e_{k_i}) = \lim_{i\to\infty} e_{k_i} = f$ , so $f$ must be that positive eigenvector. On the other hand $\lim_{i\to\infty} \tilde{v}_{k_i} = e = \lim_{i\to\infty} T_{k_i}(\tilde{v}_{k_i-1})/\|T_{k_i}(\tilde{v}_{k_i-1})\| = Te/\|Te\|$ . Since $T^{\ell} >> 0$ , $Te \neq 0$ for any positive vector $e$ . So $e$ is also an eigenvector for $T$ with positive entries and unit length. By the uniqueness of such a vector $e = f$ .

<u>Sufficiency:</u> We assume that $\lim_{k\to\infty} e_k = e$ and show that $\lim_{k\to\infty} \tilde{v}_k = e$ .

Recall that $\ell$ is the integer for which $M^{\ell} >> 0$ .

Part I: It is clearly sufficient to show that $\lim_{k\to\infty} \tilde{v}_{a+k\ell} = e$ for $a = 1,2,\ldots,\ell-1$ . Moreover, since the $\tilde{v}_k$'s all have unit length, all we need do is show that any convergent subsequence of the sequence $\tilde{v}_{a+\ell}, \tilde{v}_{a+2\ell}, \ldots$ converges to $e$ . (Recall the remark above.)

Since $T_k^{\ell} >> 0$ for all $k$ , each $e_k$ is strictly positive. Furthermore $T_k^{\ell} \geq M^{\ell}$ for all $k$ implies that all the entries of $e_k$ are bounded away from zero so all the $e_k$'s and any limit point of the $e_k$'s are in $\Omega$ .

Part II: For any $\epsilon > 0$, there is an integer $K > 0$ such that $p(e_k, e) < \epsilon$ for all $k \geq K$ . (Use Lemma 1.2 and the fact that $e_k \to e$ in $\Omega$ .) For such a $k$ we claim that

$$(*) \qquad p(\tilde{v}_{k+\ell}, e) < C\, p(\tilde{v}_k, e) + 2\ell\epsilon$$

where $C$ is a constant $< 1$ independent of $k$ .

Let $S = T_{k+\ell} \cdot T_{k+\ell-1} \cdots T_{k+1}$ . Then

$$p(\tilde{v}_{k+\ell}, e) = p(S\,\tilde{v}_k / \|S\,\tilde{v}_k\|, e) = p(S\,\tilde{v}_k, e)$$

by Lemma 1.1 (i). Now $p(S\,\tilde{v}_k, e) \leq p(S\,\tilde{v}_k, Se) + p(S\,e, S\,e_{k+1}) + p(S\,e_{k+1}, e)$ by Lemma 1.1 (v). By Proposition 1.3, there is a $C < 1$ so that $p(Sx, Sy) \leq C\, p(x,y)$ for all $x$, $y$, and $S$. So

$$p(\tilde{v}_{k+\ell}, e) \leq C\, p(v_k, e) + \epsilon + p(S\,e_{k+1}, e) \quad .$$

So we need only show that $p(S\, e_{k+1}, e) \leq (2\ell-1)\,\epsilon$ . Let $S' = T_{k+\ell} \cdots T_{k+2}$ so that $S = S' \cdot T_{k+1}$ . Then since $T_{k+1}\, e_{k+1} = e_{k+1}$, $p(S\, e_{k+1}, e) = p(S' e_{k+1}, e)$

$$\leq p(S'\, e_{k+1}, S'e) + p(S'e, S'e_{k+2}) + p(S'e_{k+2}, e)$$

$$\leq p(e_{k+1}, e) + p(e, e_{k+2}) + p(S'e_{k+2}, e)$$

since any non-negative matrix is a contraction in the projective metric. (See the proof of Proposition 1.3). So

$$p(S\, e_{k+1}, e) \leq 2\,\epsilon + p(S'e_{k+2}, e) \quad .$$

A simple induction shows that

$$p(S\, e_{k+1}, e) \leq 2(\ell-1)\epsilon + p(T_{k+\ell} e_{k+\ell}, e)$$

$$\leq (2\ell-1)\,\epsilon \quad .$$

<u>Part III:</u> From (*) we see that when $k > K$ ,

$$p(\tilde{v}_{a+(k+1)\ell}, e) \leq C\, p(\tilde{v}_{a+k\ell}, e) + 2\,\ell\,\epsilon \; .$$

Repeated applications of this formula show that

$$p(\tilde{v}_{a+(k+t)\ell}, e) \leq C^t\, p(\tilde{v}_{a+k}, e) + 2\,\ell\,\epsilon\, (1 + C^e + \ldots + C^{t-1})$$

$$= C^t p(\tilde{v}_{a+k}, e) + 2\,\ell\,\epsilon\, \frac{(1-C^t)}{(1-C)} \; .$$

Since $c < 1$ and $\epsilon$ may be arbitrarily small, it follows that $\lim_{k\to\infty} p(\tilde{v}_{a+k\ell}, e) = 0$. Thus if $\tilde{v}_{a+k_1\ell}$, $\tilde{v}_{a+k_2\ell}, \ldots$ is a convergent subsequence of the $\tilde{v}_{a+k\ell}$'s we have that $p(\lim_{i\to\infty} \tilde{v}_{a+k_i\ell}, e) = 0$. By Lemma 1.1 (iii), $\lim_{i\to\infty} \tilde{v}_{a+k_i\ell}$ is a scalar multiple of $e$. Since both are positive and of unit length, they must be equal.

Q.E.D.

<u>Corollary 3.2:</u> Suppose the sequence $T_k$ converges to $T$ where $T^{\ell} >> 0$. Then the age structure vectors always converge to the unique unit length positive eigenvector of $T$.

Finally we make some remarks on the demographic interest of these results.

<u>Corollary 3.3:</u> Suppose that the sequence of age structure vectors $\tilde{v}_k$ converge to $e = (e_1, \ldots, e_n)$ and that $T$ and $T'$ are limit points of the sequence of population matrices $T_k$. Let $(b_1, \ldots, b_n, s_1, \ldots, s_{n-1})$ and $(b_1', \ldots, b_n', s_1', \ldots, s_{n-1}')$ be the birth and survival rates of $T$ and $T'$ respectively. Then there exist positive constants $a$ and $a'$ such that

(1) $$a \sum_{i=1}^{n} b_i e_i = a' \sum_{i=1}^{n} b_i' e_i \ ,$$

and

(2) $$a\, s_j = a' s_j' \qquad \text{for} \qquad 1 \leq j \leq n-1 \ .$$

PROOF: Let $a = \dfrac{1}{\|Te\|}$ and $a' = \dfrac{1}{\|T'e\|}$. In proving the necessity part of Proposition 3.1, we showed that

$$e = \frac{Te}{\|Te\|} = a\, Te$$

$$= \frac{T'e}{\|T'e\|} = a'\, T'e \ .$$

Writing the equation $a\,T\,e = a'\,T'\,e$ component by component yields the desired conclusion.

The demographic meaning of these results is clear: An age structure will approach a constant only if the crude birth rates $(\sum_{i=1}^{n} b_i e_i)$ and each age specific survival rate approach constants or if, in the limit, these rates vary proportionally and simultaneously. The fact that the age structure converges imposes no other restrictions on the asymptotic behavior of the entries of $T_k$. The "sufficiency" part of Theorem 3.1 guarantees that this asymptotic behavior for the crude birth rates and the age specific survival rates is sufficient to guarantee convergence of the age structure.

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