A MODEL OF THE STOCK MARKET WITH MANY GOODS

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Introduction

The problem of integrating uncertainty into general equilibrium theory has been the subject of a considerable amount of research in the last twenty years. Most of this research uses as a starting point the pioneering contribution of Arrow [1] and Debreu [3], who recognised that environmental uncertainty can be incorporated in the same way as time or geographical location - by enlarging the commodity space. Arrow and Debreu redefine a commodity so that it is characterised not only by its physical properties and the date and location at which it is available, but also by the state of the world in which it is available. They then show that if markets for all such narrowly defined "contingent" commodities exist, the standard theorems about the existence and optimality of a competitive equilibrium under certainty generalise without modification to the case of uncertainty.

The Arrow-Debreu approach is ingenious, but its applicability as a theory is questionable since so few contingent commodity markets actually exist in the real world. Arrow [1] has shown that the conclusions of the theory continue to hold in the absence of contingent commodity markets, as long as economic agents have correct forecasts of future

prices and there are as many linearly independent securities as states of the world. In most realistic situations, however, the number of states of the world is likely to be very large, so that the assumption that there are as many securities as states is rather a stringent one.

Economists have been led, therefore, to analyse equilibria in situations where contingent commodity markets do not exist and where there may be only a limited number of securities. The main difficulty which must be faced in such an analysis is the specification of the behaviour of economic agents. The consumer side of the market has been studied extensively (see, for example, Lintner [14], Sharpe [21], Mossin [18] and Hart [10]) and poses no real conceptual problems. The analysis of firms' behaviour, however, is considerably more complex since the usual assumption that firms maximise profits is no longer meaningful when profits are a random variable rather than a single number.

Three main approaches have been taken towards specifying firms' behaviour. The first, adopted by Diamond [5], Stiglitz [24], Jensen and Long [11], Fama [9], Merton and Subrahmanyam [16], and Mossin [19], is to assume that firms maximise the market value of their shares. The second, which Dreze [7], Ekern and Wilson [8] and Leland [13] have

followed, is to assume that firms' stockholders get together and make production decisions on a collective basis. Finally, Radner [20] assumes that firms use von Neumann - Morgenstern utility functions to select optimal production plans.

Although quite different specifications of firms' behaviour are assumed in these various approaches, the conclusions derived are relatively uniform. In most cases, equilibria turn out not to be Pareto-optimal. In fact, only Diamond [5], under the assumption that production uncertainty is multiplicative in nature, Leland [13], under a variant of the multiplicative uncertainty assumption, and Ekern and Wilson [8] and Merton and Subrahmanyam [16], under the assumption that consumers have mean-variance utility functions, conclude that equilibria are Pareto-optimal.

The different approaches have another feature in common. With the exception of Radner's work, they all assume that the economy contains a single good and lasts for a single period. The purpose of this paper is to investigate the consequences of relaxing the single good assumption. This will be done using the framework of the Diamond model. That is, it will be assumed that production

uncertainty is multiplicative, and that firms, acting as price takers, maximise their market value. Our main conclusion is that Diamond's result that an equilibrium is Pareto-optimal fails to generalise to the many good case. The reason for this is not the existence of many inputs in the production process, but rather the existence of many outputs and the opportunity to trade in these outputs after the completion of production.[1]

At first sight, it may not seem surprising that an equilibrium fails to be Pareto-optimal in the many good case. After all, if there are several goods in the economy, a new element is introduced: price uncertainty. And, if economic agents have incorrect price expectations, an equilibrium will not be optimal even if there is no uncertainty at all about the environment (see, for example, Dorfman, Samuelson and Solow [6, Chapter 12]).

The results we obtain, however, are much stronger than this. We show that the case of environmental uncertainty differs crucially from the case of environmental certainty in that an equilibrium is not generally optimal even if economic agents have identical self-fulfilling point expectations about prices. In Section 4, examples are given where there are two equilibria, both of which are based on

correct forecasts of future prices, but where one is preferred by everybody in the economy to the other. In other words, everybody can be made better off simply by changing price expectations.

The reason that this sort of situation can arise may be given intuitively as follows. When there are many goods and fewer securities than states of the world, the opportunities for trading depend to some extent on the prices of goods after the completion of production. One equilibrium may be better for everybody than another because, with different goods prices, there is a greater opportunity for all consumers to buy the goods they want to consume, and hence greater gains from trade can be realised. It should be noted that this situation can arise even if each firm produces a single product. What is important is that there are many goods in the economy, not that each firm produces many goods.

The analysis presented in this paper depends crucially on the assumption that firms face multiplicative uncertainty. Our conclusions, however, have wider applicability. If equilibria fail to be Pareto-optimal under the assumption of multiplicative uncertainty and price taking, value maximising behaviour by firms, then it seems highly unlikely

that equilibria will be Pareto-optimal under any other assumptions about firms' behaviour. Hence, our results would appear to imply that, within reason, no stock market equilibrium, however defined, will be Pareto-optimal in the many good case.

The paper is organised as follows. In Section 2, the Diamond model with a single good is reviewed. In Section 3, the model is extended to the case of many goods, and, corresponding to three different restrictions on short-selling behaviour, three self-fulfilling price expectations equilibria of the type investigated by Radner in [20] are defined and discussed. In Section 4, Diamond's notion of constrained Pareto-optimality is extended to the many good case and it is shown that the Section 3 equilibria are not generally optimal in this sense. In Section 5, one of the Section 3 equilibria is shown to be optimal in a weaker sense: it is impossible to make some people better off in some states of the world without making other people worse off in other states of the world. It is shown also that two of the Section 3 equilibria are technologically efficient. Finally, in the appendix, the proofs of some of the more technical results are presented.

2. The Diamond Model with a Single Good

This section consists of a review of the basic Diamond model. Our presentation of the model differs slightly from Diamond's own in [5], but it should be clear that the two are equivalent.

We consider a single period economy with F firms, I consumers and S states of the world; F, I, S are all assumed to be finite. There is a single good in the economy, which appears as an input of the production process at the beginning of the period and as an output of the production process at the end of the period. Consumers have initial endowments of this good and supply these endowments inelastically to firms at the beginning of the period; consumption takes place only at the end of the period.[2] Uncertainty is introduced into the model by assuming that the quantity of output a firm produces depends not only on the quantity of input it uses, but also on which state of the world occurs.[3] The state of the world is assumed to be known by everyone at the end of the period, but to be unknown at the beginning of the period.

Firms

Firm f's production function may be written as

$(h_f^1(L),\ldots,h_f^S(L))$, where $h_f^s(L)$ is the quantity of output produced in state s and L is the quantity of input. We assume that production uncertainty is multiplicative, so that $(h_f^1(L),\ldots,h_f^S(L)) = g_f(L)(a_f^1,\ldots,a_f^S)$ for some $(a_f^1,\ldots,a_f^S) \in R_+^S$ and for some function g_f mapping R_+ into R_+ $(f = 1,\ldots,F)$[4]. In other words, each firm produces a single pattern of returns across states of the world. The assumption that production uncertainty is multiplicative is a crucial part of the Diamond model.

Consumers

Consumer i $(i = 1,\ldots,I)$ has an initial endowment of the good $\bar{L}^i \in R_+$ and initial proportionate shareholdings in firm f given by $\bar{\theta}_f^i \geqq 0$ $(f = 1,\ldots,F)$, where $\sum_i \bar{\theta}_f^i = 1$ for each f.[5] Consumer i's tastes in state s $(s = 1,\ldots,S)$ are represented by a utility function U_s^i defined on R_+ and consumer i's probability beliefs by a vector $\pi^i \in P^S = \{\pi \in R_+^S \mid \sum_s \pi_s = 1\}$ $(i = 1,\ldots,I)$; π_s^i is consumer i's assessment of the probability of state s occurring.

Equilibrium

We define a competitive equilibrium under the assumption that, at the beginning of the period, there are markets

for firms' shares and for the good as an input in production, and that, at the end of the period, consumers consume their shares of firms' outputs. Given the special form of firms' production functions, the existence of a market for firm f's shares is equivalent to the existence of a market for units of the random variable $a_f = (a_f^1,\ldots,a_f^S)$, which yields a_f^s units of output at the end of the period if state s occurs ($s = 1,\ldots,S$). It turns out to be easier to discuss equilibrium in terms of the markets for these random variables rather than in terms of the markets for shares.

Let w be the price of one unit of the good as input, r_f the price of the random variable a_f and $r = (r_1,\ldots,r_F)$. Consider the production decision of firm f. If firm f uses L units of input, its output in state s is given by $g_f(L)a_f^s$ ($s = 1,\ldots,S$). In other words, it is supplying $g_f(L)$ units of the random variable a_f. Its market value is therefore $r_f g_f(L)$, and, under the assumption that the only method of financing the purchase of input is by issuing new shares,[6] the market value of its initial shareholdings is $r_f g_f(L) - wL$. We assume that firms maximise the market value of initial shareholdings, so that firm f chooses L to maximise

$$r_f g_f(L) - wL \ . \qquad (1)$$

A second justification of (1) as firm f's objective function may also be given. Suppose that a complete set of contingent commodity markets existed. Then firm f would maximise

$$\sum_s \sigma_s g_f(L) a_f^s - wL \ , \qquad (2)$$

where σ_s is the price of one unit of the good contingent on state s occurring. (2) cannot be used directly as an objective function in the situation we are considering because the firms do not know the σ_s. However, firm f does know the cost of the random variable a_f and we suppose that it argues that, whatever the σ_s are, the cost of a_f, evaluated using the q_s, must equal r_f. In other words, $\sum_s \sigma_s a_f^s = r_f$. It follows that $\sum_s q_s g_f(L) a_f^s - wL = r_f g_f(L) - wL$ and that maximising (1) is the same as maximising (2).

The argument that firms do not need to know the q_s in order to maximise profits or market value depends, of course, crucially on the assumption that production uncertainty is multiplicative. If this assumption does not hold, there is in general no simple objective function for firms like (1), and the problem of specifying firms' behaviour becomes considerably more difficult.[7]

Consider now the consumers. A consumer's wealth derives from two sources: his initial endowment of the good and his initial shareholdings in firms. Since $\max_{L \in R_+} \{r_f g_f(L) - wL\}$ is the market value of the aggregate initial shareholdings of consumers in firm f, consumer i's wealth is given by $w\bar{L}^i + \sum_f \bar{\theta}_f^i \max_{L \in R_+} \{r_f g_f(L) - wL\}$. Consumers use their wealth to buy shares in firms since this is the only way of obtaining output for consumption purposes at the end of the period. Each consumer is assumed to select a portfolio which maximises expected utility of consumption subject to a budget constraint.

Let z_f be the number of units of the random variable a_f that consumer i purchases and let $z = (z_1, \ldots, z_F)$. Define

$$B^i(w,r) = \left\{ z \in R_+^F \mid rz \leq w\bar{L}^i + \sum_f \bar{\theta}_f^i \max_{L \in R_+} \{r_f g_f(L) - wL\} \right\}. \quad [8]$$

Then, assuming that short-sales are not permitted,[9] consumer i maximises $\sum_s \pi_s{}^i U_s{}^i (\sum_f z_f a_f{}^s)$ subject to $z \in B^i(w,r)$.

Definition of Equilibrium

$(w,r) \in R_+ \times R_+^F$ is an equilibrium price vector if there exist $L_1, \ldots, L_F$, $z^1, \ldots, z^I$ such that

(I) $L_f \in R_+$ and $r_f g_f(L_f) - wL_f \geq r_f g_f(L) - wL$ for all $L \in R_+$ $(f = 1,\ldots,F)$;

(II) $z^i \in B^i(w,r)$ and $\sum_s \pi_s^i U_s^i(\sum_f z_f^i a_f^s) \geq \sum_s \pi_s^i U_s^i(\sum_f z_f a_f^s)$ for all $z \in B^i(w,r)$ $(i = 1,\ldots,I)$;

(III) $\sum_f L_f = \sum_i \bar{L}^i$;

(IV) $\sum_i z^i = (g_1(L_1),\ldots,g_F(L_F))$.

(I) is the condition that firms maximise profit or market value; (II) is the condition that consumers maximise expected utility; (III) is the condition that there is equilibrium in the input market. Finally, (IV) is the condition that there is equilibrium in the share markets; consumer i's proportionate shareholding in firm f is given by $\frac{z_f^i}{g_f(L_f)}$ if $g_f(L_f) > 0$ and by $\frac{1}{I}$ if $g_f(L_f) = 0$.

We now make some assumptions which guarantee the existence of equilibrium.

A1. g_f is continuous and concave $(f = 1,\ldots,F)$.

A2. g_f is increasing, that is, $g_f(L') > g_f(L)$ if $L \in R_+$ and $L' > L$ $(f = 1,\ldots,F)$.

A3. U_s^i is continuous and concave $(s = 1,\ldots,S;$ $i = 1,\ldots,I)$.

A4. U_s^i is increasing $(s = 1,\ldots,S;\ i = 1,\ldots,I)$.

A5. $\bar{L}^i > 0 \quad (i = 1,\ldots,I)$.

Theorem 2.1. Under assumptions A1-A5, an equilibrium exists.[10]

The proof of Theorem 2.1 will not be given since it consists of a straightforward application of the techniques presented in the proof of Theorem 3.2 in Section 3.

Optimality

Having given sufficient conditions for the existence of equilibrium, we turn next to the definition of optimality and the relationship between equilibria and optima in the Diamond model. We define an allocation[11] to be a vector $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ which satisfies

(V) $L_f \in R_+ \quad (f = 1,\ldots,F)$;

(VI) $x^i \in R_+^S \quad (i = 1,\ldots,I)$;

(VII) $\sum_f L_f = \sum_i \bar{L}^i$;

(VIII) $\sum_i x_s^i = \sum_f g_f(L_f)a_f^s \quad (s = 1,\ldots,S)$.

L_f is firm f's input and x_s^i is consumer i's consumption in state s. (VII) is the condition that the amount of the good used as input equals the total amount

of the good available and (VIII) is the condition that aggregate consumption equals aggregate output in every state of the world.

An allocation is defined to be Pareto-optimal if there is no other allocation which makes some people better off and nobody worse off. That is, the allocation $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is Pareto-optimal if there is no allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$ satisfying

$$\sum_s \pi_s{}^i U_s{}^i(\tilde{x}_s{}^i) \geq \sum_s \pi_s{}^i U_s{}^i(x_s{}^i) \qquad (i = 1,\ldots,I),$$

with inequality for some i.

If an all-knowing central planner took over the economy, he would be able to achieve any allocation. Suppose, however, that the central planner were constrained to allocate consumption at the end of the period by allocating shares in firms to consumers at the beginning at the period. There would then be many allocations which the central planner could not achieve. We call those allocations which could still be achieved SM allocations (short for stock market allocations). More formally, the allocation $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is an SM allocation if there exist $z^1,\ldots,z^I$ such that

$$(IX) \quad z^i \in R_+^F \qquad (i = 1,\ldots,I);$$

$$(X) \quad x_s^i = \sum_f z_f^i a_f^s \qquad (s = 1,\ldots,S;\ i = 1,\ldots,I);$$

$$(XI) \quad \sum_i z^i = (g_1(L_1),\ldots,g_F(L_F)) \ .$$

$z^1,\ldots,z^I$ are the portfolios corresponding to the allocation.

Definition of Constrained Pareto-Optimality

The SM allocation $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is defined to be constrained Pareto-optimal if there is no SM allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$ satisfying

$$\sum_s \pi_s^i U_s^i(\tilde{x}_s^i) \geq \sum_s \pi_s^i U_s^i(x_s^i) \qquad (i = 1,\ldots,I) \ ,$$

with inequality for some i. In other words, an SM allocation is constrained Pareto-optimal if there is no SM allocation making some people better off and nobody worse off. The word constrained is used because SM allocations are being compared with other SM allocations rather than with arbitrary allocations.

It is clear that an equilibrium allocation is an SM allocation. Diamond's main result is that an equilibrium allocation is constrained Pareto-optimal. In order to prove this result, we introduce

another assumption, which says that every consumer believes that some firm is capable of producing a positive amount of output with non-zero probability.

A6. For each i: there exist f and s such that $\pi_s^i > 0$ and $a_f^s > 0$.

Theorem 2.2: An equilibrium allocation is constrained Pareto-optimal if assumptions A4 and A6 are satisfied.

Proof: Let $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ be an equilibrium allocation at prices (w,r) and let the portfolios associated with this allocation be $z^1,\ldots,z^I$. Suppose $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$ is an SM allocation with associated portfolios $\tilde{z}^1,\ldots,\tilde{z}^I$, which satisfies

$$\sum_s \pi_s^i U_s^i(\tilde{x}_s^i) \geq \sum_s \pi_s^i U_s^i(x_s^i) \qquad (i = 1,\ldots,I) ,$$

with inequality for some i. Using A4, A6 and the fact that consumer i chooses z^i instead of the at least as desirable portfolio $\tilde{z}^i$ at prices (w,r), we may deduce that

$$r\tilde{z}^i \geq w\bar{L}^i + \sum_f \bar{\theta}_f^i(r_f g_f(L_f) - wL_f) \qquad (i = 1,\ldots,I) ,$$

with inequality for those i for which $\tilde{z}^i$ is strictly preferred to z^i. Summing over i and using the facts that

$\sum_i \bar{\theta}_f^i = 1$ and $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$ satisfies (VII) and (XI), we obtain

$$\sum_f r_f g_f(\tilde{L}_f) - w \sum_f \tilde{L}_f > \sum_f r_f g_f(L_f) - w \sum_f L_f ,$$

which contradicts $r_f g_f(L_f) - wL_f = \max_{L \in R_+} \{r_f g_f(L) - wL\}$ $(f = 1,\ldots,F)$. Q.E.D.

The following example shows that an equilibrium is not generally an (unconstrained) Pareto-optimum.

Example 1. There are two states of the world and two consumers, each owning $\frac{1}{2}$ unit of the good. There is a single firm, which transforms L units of input into L units of output whichever state of the world occurs. The situation can be represented by means of an Edgeworth box, with the single equilibrium allocation at the centre of the box C.

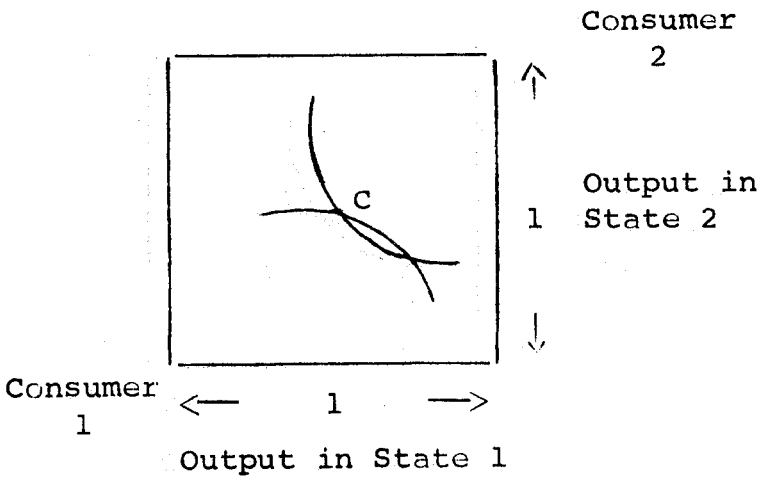

Figure 1

However, there is no reason why the consumers' indifference curves should touch at C, and so this allocation will not generally be Pareto-optimal although it is, of course, constrained Pareto-optimal.

We have shown that any equilibrium allocation is constrained Pareto-optimal. We complete this review of the basic Diamond model by showing that any constrained Pareto-optimum can be achieved as an equilibrium allocation with an appropriate redistribution of initial endowments and shareholdings, thus making complete the formal analogy between the Diamond theory and the Arrow-Debreu theory.

First, we make two more assumptions.

A7. Given i and f, there exists s such that $\pi_s^i > 0$ and $a_f^s > 0$.

A8. $\sum_i \bar{L}^i > 0$.

A7 says that every consumer believes that every firm is capable of producing something with positive probability. A8 says that there is a positive amount of the good available as input at the beginning of the period.

Let $E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$ be the economy in which consumer i's initial endowment of the good is given by L^i

and consumer i's initial proportionate shareholdings in firms by the F-vector θ^i $(i = 1,\ldots,I)$.

Theorem 2.3: Under assumptions A1, A2, A3, A4, A7 and A8, if $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is a constrained Pareto-optimal allocation, there exist $L^1,\ldots,L^I,\theta^1,\ldots,\theta^I$ satisfying

$$L^i \in R_+ \quad (i = 1,\ldots,I),$$

$$\theta^i \in R_+^F \quad (i = 1,\ldots,I),$$

$$\sum_i L^i = \sum_i \bar{L}^i ,$$

$$\sum_i \theta_f^{\,i} = 1 \quad (f = 1,\ldots,F),$$

such that $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is an equilibrium allocation for the economy $E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$.

Proof: Since the proof is very similar to the usual Arrow-Debreu proof, only a sketch will be given. Let $z^1,\ldots,z^I$ be the portfolios associated with the SM allocation $(L_1,\ldots,L_F,x^1,\ldots,x^I)$. Define

$$V = \Big\{(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{z}^1,\ldots,\tilde{z}^I) \;\Big|\; \tilde{L}_f \in R_+ \quad (f = 1,\ldots,F),$$

$$\tilde{z}^i \in R_+^F \quad (i = 1,\ldots,I),$$

$$\sum_s \pi_s{}^i U_s{}^i\Big(\sum_f \tilde{z}_f{}^i a_f{}^s\Big) \geq \sum_s \pi_s{}^i U_s{}^i\Big(\sum_f z_f{}^i a_f{}^s\Big)$$

$$\text{for each } i\Big\}$$

and

$$W = \Big\{(v,y) \mid v = \sum_f \tilde{L}_f - \sum_i \bar{L}^i,$$

$$y \geqq \sum_i \tilde{z}^i - (g_1(\tilde{L}_1),\ldots,g_F(\tilde{L}_F))$$

$$\text{for some } (\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{z}^1,\ldots,\tilde{z}^I) \in V\Big\} .$$

A1 and A3 imply that W is convex, and, since $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is a constrained Pareto-optimum, $0 \notin \text{int } W$ by A4 and A7. Therefore, we can construct a hyperplane through 0 such that W lies on one side of the hyperplane. That is, there exists $(w,r) \neq 0$ such that

$$wv + ry \geqq 0 \tag{3}$$

for all $(v,y) \in W$.

From (3) it may be deduced that

$$r_f g_f(L) - wL \leqq r_f g_f(L_f) - wL_f \quad \text{for all } L \in R_+ \quad (f = 1,\ldots,F) \tag{4}$$

and

$$z \in R_+^F \text{ and } \sum_s \pi_s{}^i U_s{}^i(\sum_f z_f a_f{}^s) \geqq \sum_s \pi_s{}^i U_s{}^i(\sum_f z_f{}^i a_f{}^s)$$

$$\Longrightarrow rz \geqq rz^i \quad (i = 1,\ldots,I) . \tag{5}$$

(4), (5), A2, A4, A7 and A8 now establish that $(w,r) > 0$ and that

$$z \in R_+^F \quad \text{and} \quad \sum_s \pi_s{}^i U_s{}^i\Big(\sum_f z_f a_f{}^s\Big) > \sum_s \pi_s{}^i U_s{}^i\Big(\sum_f z_f{}^i a_f{}^s\Big)$$

$$\Longrightarrow \quad rz > rz^i \qquad (i = 1,\ldots,I) \ .$$

It follows that $(L_1,\ldots,L_F,z^1,\ldots,z^I)$ is an equilibrium allocation at prices (w,r) with initial shareholdings and endowments given by

$$\theta_f{}^i = \begin{cases} \dfrac{z_f{}^i}{g_f(L_f)} & \text{if} \quad g_f(L_f) > 0 \\ \dfrac{1}{I} & \text{if} \quad g_f(L_f) = 0 \end{cases} \qquad \begin{matrix} (f = 1,\ldots,F; \\ i = 1,\ldots,I) \end{matrix}$$

$$L^i = \sum_f L_f \theta_f{}^i \qquad (i = 1,\ldots,I).$$

Q.E.D.

3. The Extension of the Diamond Model to the Many Good Case

We now generalise the Diamond model to the case where there are M inputs at the beginning of the period and N outputs at the end of the period.

Firms

The assumption that production uncertainty is multiplicative is retained, so that firm f's production function may be written as $g_f(L)(a_f^1,\ldots,a_f^S)$, where $g_f(L)a_f^s$ is now the vector of outputs firm f produces in state s and L is its vector of inputs ($f = 1,\ldots,F$). It is assumed that g_f is a function mapping R_+^M into R_+ and that, for each s, $a_f^s \in R_+^N$ ($f = 1,\ldots,F$).

Consumers

Consumer i ($i = 1,\ldots,I$) has an initial endowment of inputs $\bar{L}^i \in R_+^M$ and initial, proportionate shareholdings in firm f given by $\bar{\theta}_f^i \geq 0$ ($f = 1,\ldots,F$), where $\sum_i \bar{\theta}_f^i = 1$ for each f. Consumer i's tastes in state s ($s = 1,\ldots,S$) are represented by a utility function U_s^i defined on R_+^N and consumer i's probability beliefs by a vector $\pi^i \in P^S = \{\pi \in R_+^S \mid \sum_s \pi_s = 1\}$ ($i = 1,\ldots,I$).

Equilibrium

We define an equilibrium under the assumption that there are markets for firms' shares and for inputs at the beginning of the period and markets for outputs at the end of the period when production is completed. The existence of end of period markets is a new feature of the model. It should be clear that without such markets the many good case differs in no significant way from the single good case of Section 2. In the many good case, however, unlike the single good case, there will in general be incentives to trade at the end of the period, and it therefore seems desirable to consider the case where such trading actually takes place.

As in Section 2, we may regard firm f as a producer of units of the (vector-valued) random variable $a_f = (a_f^1, \ldots, a_f^S)$, which yields the vector a_f^s of outputs at the end of the period if state s occurs ($s = 1, \ldots, S$). It will again be easier to discuss equilibrium in terms of the markets for these random variables rather than in terms of the markets for firms' shares.

Let w be the M-vector of input prices, r_f the price of the random variable a_f and $r = (r_1, \ldots, r_F)$. It is assumed, as in the basic Diamond model, that firm f chooses

L to maximise $r_f g_f(L) - wL$. Exactly the same arguments as before may be used to justify this particular objective function. It should be noted that firms do not require any knowledge of output prices at the end of the period to make their production decisions.

Consumer behaviour is a little more complicated than in the basic Diamond model. As before, consumers are assumed to supply their endowments of inputs inelastically at the beginning of the period and to use their wealth to buy shares in firms. However, since consumers are assumed to be aware of the fact that trading takes place at the end of the period, consumer investment behaviour depends not only on consumer tastes and probability beliefs about states of the world, but also on expectations about end of period prices.

The equilibria we will be considering in this paper are of a very special type. We confine ourselves to cases where, first, each consumer has a point forecast of end of period output prices; secondly, the forecasts of different consumers are the same (this does not mean that a consumer has the same price forecasts in different states of the world); and, thirdly, these forecasts are correct. Equilibria of this kind have been investigated by Radner in [20].

The assumption that consumers' forecasts are correct is, of course, very unrealistic. However, if consumers' forecasts are allowed to be incorrect, it is well-known that an equilibrium will be sub-optimal even if there is no uncertainty at all about the state of the world (see Dorfman, Samuelson and Solow [6, Chapter 12]). Our purpose is to show that in the case of environmental uncertainty, there is a more fundamental source of sub-optimality; an equilibrium will be sub-optimal even if consumers' forecasts are correct.

Let the N-vector p_s represent the price forecasts of consumers in state s $(s = 1,\ldots,S)$. If consumer i holds z_f units of the random variable a_f $(f = 1,\ldots,F)$, consumer i will have claim to the vector of outputs $\sum_f z_f a_f^s$ in state s before trading re-opens, and consumer i's income in this state at prices p_s will be given by $p_s(\sum_f z_f a_f^s)$. If price forecasts are correct, consumer i's utility in state s will therefore be

$$V_s^i(p_s, p_s(\sum_f z_f a_f^s))$$

$$= \max \{U_s^i(x) \mid x \in R_+^N \text{ and } p_s x \leq p_s(\sum_f z_f a_f^s)\},$$

where V_s^i, which is an indirect utility function, is

well-defined only if $p_s(\sum_f z_f a_f^s) \geqq 0$. Hence, consumer i's expected utility at the beginning of the period is given by $\sum_s \pi_s{}^i V_s{}^i(p_s, p_s(\sum_f z_f a_f^s))$. Consumer i is assumed to choose z so as to maximise $\sum_s \pi_s{}^i V_s{}^i(p_s, p_s(\sum_f z_f a_f^s))$ subject to a budget constraint.

We consider three different regimes for consumers. In the first regime, short-selling is prohibited as in the basic Diamond model. In the second regime, short-selling is permitted, but the vector of outputs which each consumer owns at the end of the period before trading re-opens is constrained to be non-negative in every state of the world. In the third regime, the only requirement is that each consumer's income be non-negative in each state of the world at the forecasted prices. The first regime is more restrictive than the second regime, which in turn is more restrictive than the third regime. In all three regimes, bankruptcy in any state of the world, even in those states which are believed to occur with zero probability, is ruled out.[12]

The reason for considering these three regimes is that, as we will see in Section 5, their equilibria have somewhat different optimality properties. Equilibria in

the second and third regimes, for example, are technologically efficient, whereas equilibria in the first regime may not be. Another reason for distinguishing between the three regimes is that whereas equilibria exist under standard assumptions in the first and second regimes, quite stringent assumptions are required to insure the existence of equilibria in the third regime.

Let

$$B_1^i(w,r,p_1,\ldots,p_S)$$

$$= \left\{z \in R_+^F \mid rz \leqq w\bar{L}^i + \sum_f \bar{\theta}_f^i \max_{L\in R_+^M} \{r_f g_f(L) - wL\}\right\},$$

$$B_2^i(w,r,p_1,\ldots,p_S)$$

$$= \left\{z \in R^F \mid \sum_f z_f a_f^s \geqq 0 \quad (s = 1,\ldots,S) \quad \text{and} \right.$$

$$\left. rz \leqq w\bar{L}^i + \sum_f \bar{\theta}_f^i \max_{L\in R_+^M} \{r_f g_f(L) - wL\}\right\},$$

$$B_3^i(w,r,p_1,\ldots,p_S)$$

$$= \left\{z \in R^F \mid p_s\left(\sum_f z_f a_f^s\right) \geqq 0 \quad (s = 1,\ldots,S) \quad \text{and} \right.$$

$$\left. rz \leqq w\bar{L}^i + \sum_f \bar{\theta}_f^i \max_{L\in R_+^M} \{r_f g_f(L) - wL\}\right\},$$

for each i. B_k^i is consumer i's budget set at the beginning of the period in regime k $(k = 1,2,3)$. Let

$$G_s^i(p_s,z) = \{x \in R_+^N \mid p_s x \leq p_s(\sum_f z_f a_f^s)\}$$

for each i and s. G_s^i is consumer i's budget set at the end of the period in state s if consumer i's portfolio is z.

We are now ready to define equilibria corresponding to each of the three different regimes.

Definition of Equilibrium

$(w,r,p_1,\ldots,p_S)$, where $w \in R_+^M$, $r \in R_+^F$ and $p_s \in R_+^N$ $(s = 1,\ldots,S)$, is a type k equilibrium $(k = 1,2,3)$ if there exist $L_1,\ldots,L_F,z^1,\ldots,z^I,x^1,\ldots,x^I$ such that

(XII) $L_f \in R_+^M$ and $r_f g_f(L_f) - wL_f \geq r_f g_f(L) - wL$ for all $L \in R_+^M$ $(f = 1,\ldots,F)$;

(XIII) $z^i \in B_k^i(w,r,p_1,\ldots,p_S)$ and
$\sum_s \pi_s^i V_s^i(p_s, p_s(\sum_f z_f^i a_f^s)) \geq \sum_s \pi_s^i V_s^i(p_s,p_s(\sum_f z_f a_f^s))$
for all $z \in B_k^i(w,r,p_1,\ldots,p_S)$ $(i = 1,\ldots,I)$;

(XIV) $x_s^i \in G_s^i(p_s,z^i)$ and $U_s^i(x_s^i) \geq U_s^i(x_s)$ for all $x_s \in G_s^i(p_s,z^i)$ $(s = 1,\ldots,S;\ i = 1,\ldots,I)$;

$$(XV)\quad \sum_f L_f = \sum_i \bar{L}^i \ ;$$

$$(XVI)\quad \sum_i z^i = (g_1(L_1),\ldots,g_F(L_F)) ;$$

$$(XVII)\quad \sum_i x_s^{\,i} = \sum_f g_f(L_f)a_f^{\,s} \quad (s = 1,\ldots,S) \ .$$

(XII) is the condition that L_f is a profit-maximising input for firm f; (XIII) is the condition that z^i is an optimal portfolio for consumer i; (XIV) is the condition that $x_s^{\,i}$ is an optimal consumption vector for consumer i in state s at prices p_s, given the portfolio z^i; (XV) is the condition that the demand for inputs equals the supply of inputs; (XVI) is the condition that the demand for shares equals the supply of shares. Finally, (XVII) is the condition that p_s is an equilibrium price vector at the end of the period if state s occurs, thus guaranteeing that consumers' price forecasts are actually fulfilled.

It should be noted that, as long as $\pi_s^{\,i} > 0$ for all s, conditions (XIII) and (XIV) for consumer i are equivalent to the following condition:

(XVIII) $(z^i, x_1^{\,i},\ldots,x_S^{\,i})$ is a solution of the problem:

$$\text{maximise} \quad \sum_s \pi_s^{\,i} U_s^{\,i}(x_s)$$

subject to

$$p_s x_s \leq p_s(\sum_f z_f a_f^s) \quad (s = 1,\ldots,S) ,$$

$$z \in B_k^i(w,r,p_1,\ldots,p_S) .$$

However, if $\pi_s^i = 0$ for some s, (XIII) and (XIV) are stronger than (XVIII). For suppose that $\pi_1^i = 0$. Then any x_1^i such that $p_1 x_1^i \leq p_1(\sum_f z_f^i a_f^1)$, in particular $x_1^i = 0$, satisfies (XVIII). Yet, $x_1^i = 0$ may well not satisfy (XIV). The point is that, at the beginning of the period, consumer i will be content with zero consumption in state 1 if he believes that the probability of state 1 occurring is zero, but, at the end of the period, if state 1 does occur, consumer i will desire positive consumption if $p_1(\sum_f z_f^i a_f^1) > 0$.

Having defined an equilibrium, we now make some assumptions which guarantee its existence.

B1. g_f is continuous and concave ($f = 1,\ldots,F$).

B2. For each f: given $L \in R_+^M$, there exists $L' \in R_+^M$ such that $g_f(L') > g_f(L)$.

B3. Given i and f, there exists s such that $\pi_s^i > 0$ and $a_f^s \geq 0$.

B4. Given $m = 1,\ldots,M$, there exists f such that g_f is increasing in L_m.

B5. U_s^i is continuous and concave ($s = 1,\ldots,S$; $i = 1,\ldots,I$).

B6. U_s^i is increasing, that is, $U_s^i(x') > U_s^i(x)$ if $x \in R_+^N$ and $x' \geq x$ ($s = 1,\ldots,S$; $i = 1,\ldots,I$).

B7. $\bar{L}^i \geq 0$ ($i = 1,\ldots,I$).

B8. $\sum_i \bar{L}^i > 0$.

B9. Given $n = 1,\ldots,N$ and s, there exists f such that $a_f{}^s{}_n > 0$.

B10. Given s, there exists i such that $\pi_s^i > 0$.

B1, B5, B6, B7 and B8 are standard assumptions. B2 says that there is no maximum to the amount of the random variable a_f that firm f can produce. B3 says that each consumer believes that each firm is capable of producing something with positive probability. B4 says that each input can be used by some firm to produce more output. B9 says that every good can be produced in every state of the world. Finally, B10 says that no state of the world is believed by everybody to occur with zero probability.

Theorem 3.1: Under assumptions B1 - B10, a type 1 equilibrium and a type 2 equilibrium exist.

The proof of this theorem is given in the appendix.

Unfortunately, assumptions B1 - B10 are not sufficient to insure the existence of a type 3 equilibrium. We require a further assumption.

Let

$$Y = \Big\{ p \mid p = (p_1,\ldots,p_S), \text{ where } p_s \in R^N,\ p_s > 0 \text{ and } \sum_{n=1}^{N} p_{sn} = 1 \text{ for each } s \Big\},$$

and define $R_f(p) = (p_1 a_f^1,\ldots,p_S a_f^S)$ for $p \in Y$. $R_f(p)$ is the vector of monetary returns in different states of the world yielded by the random variable a_f at prices p. Fix p and consider the subset of Euclidean S-space, $\{R_1(p),\ldots,R_F(p)\}$. We may choose $f_1,..,f_\varphi \in \{1,..,F\}$ such that the vectors $R_{f_1}(p),..,R_{f_\varphi}(p)$ are a basis of this subset. In general, the numbers $f_1,\ldots,f_\varphi$ will vary with p. Assumption B11 restricts us to cases where $f_1,\ldots,f_\varphi$ may be chosen independently of p.

B11. We may choose $f_1,\ldots,f_\varphi \in \{1,\ldots,F\}$ such that, for each p in Y, $R_{f_1}(p),\ldots,R_{f_\varphi}(p)$ are linearly independent vectors which span $\{R_1(p),\ldots,R_F(p)\}$.

It should be emphasised that assuming that $R_{f_1}(p),\ldots,R_{f_\varphi}(p)$ are linearly independent for all p is far stronger than assuming that $a_{f_1},\ldots,a_{f_\varphi}$ are linearly independent vectors. The linear independence of $a_{f_1},\ldots,a_{f_\varphi}$ is a necessary condition for the linear independence of $R_{f_1}(p),\ldots,R_{f_\varphi}(p)$, but by no means a sufficient condition.

Theorem 3.2: Under assumptions B1 - B11, a type 3 equilibrium exists.

The proof of this theorem is given in the appendix.

The need for assumption B11 may not be obvious to the reader, particularly since Radner does not require it in [20]. Radner, however, assumes that the vector of outputs which each consumer holds at the end of the period before trading re-opens must be greater than some fixed vector, thus making a Radner equilibrium closer to our type 2 equilibrium than to our type 3 equilibrium.

The following example, in which assumptions B1 - B10 hold, B11 does not, and a type 3 equilibrium fails to exist, should give some indication of the importance of B11.

Example 2

There are two consumers, two firms, two inputs, two outputs and two states of the world. The first firm transforms the first input into the first output and the second firm transforms the second input into the second output. There is no production uncertainty. We assume $g_1(L) = L_1$, $g_2(L) = L_2$, $a_1^1 = a_1^2 = (1,0)$ and $a_2^1 = a_2^2 = (0,1)$.

Each consumer owns $\frac{1}{2}$ unit of each input. Consumer 1 believes that state 1 occurs with certainty; consumer 2 believes that state 2 occurs with certainty. U_1^1, U_2^1, U_1^2, U_2^2 are assumed to satisfy B5 and B6. We assume also that U_1^1, U_2^1, U_1^2, U_2^2 are differentiable and that

$$\frac{\frac{\partial U_1^1}{\partial x_1}(1,1)}{\frac{\partial U_1^1}{\partial x_2}(1,1)} = \frac{\frac{\partial U_2^2}{\partial x_1}(1,1)}{\frac{\partial U_2^2}{\partial x_2}(1,1)} . \qquad (6)$$

$\left(\frac{\partial U^1}{\partial x_1}(1,1)\right.$ is $\frac{\partial U^1}{\partial x_1}(x_1,x_2)$ evaluated at $(x_1,x_2) = (1,1).\Big)$

Assumptions B1 - B10 are clearly satisfied in this example, but we will now show that a type 3 equilibrium does not generally exist.

Suppose $(\hat{w},\hat{r},\hat{p}_1,\hat{p}_2)$ is a type 3 equilibrium price vector. It is clear from our assumptions that $(\hat{w},\hat{r},\hat{p}_1,\hat{p}_2) > 0$ and, because of the constant returns to scale technology and the fact that both firms produce in equilibrium, $\hat{w} = \hat{r}$.

We may describe the situation by means of an Edgeworth box. In equilibrium, the first firm uses all of the first input and the second firm uses all of the second input, so that one unit of each output is produced in each state of the world.

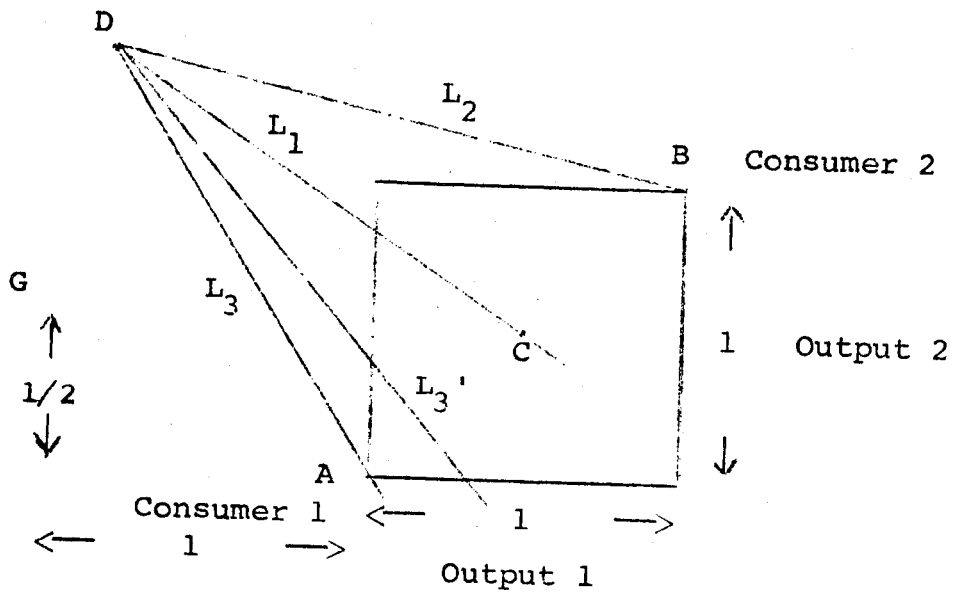

Figure 2

Consumption is represented in this diagram in the usual way; two points are required since there are two states of the world. Investment can also be represented

in the diagram. Suppose, for example, consumer 1 holds -1 unit of random variable a_1 and $\frac{1}{2}$ unit of random variable a_2. Then consumer 1 has contracted to supply 1 unit of output 1 and to receive $\frac{1}{2}$ unit of output 2 in each state of the world. This portfolio is therefore represented by the point G.

Since $\hat{w} = \hat{r}$, each consumer can afford the portfolio represented by C, the centre of the box. In fact, the portfolio budget line for consumers is given by the line L_1 through C with slope $-\frac{\hat{r}_1}{\hat{r}_2}$. We establish first that $\frac{\hat{p}_{11}}{\hat{p}_{12}} = \frac{\hat{p}_{21}}{\hat{p}_{22}} = \frac{\hat{r}_1}{\hat{r}_2}$ is a necessary condition for equilibrium, where $\hat{p}_{12}$ is the price of the second output in the first state and $\hat{p}_{21}$ is the price of the first output in the second state.

Suppose that $\frac{\hat{p}_{11}}{\hat{p}_{12}} \neq \frac{\hat{r}_1}{\hat{r}_2}$. Let L_2 be the line through B with slope $-\frac{\hat{p}_{11}}{\hat{p}_{12}}$. Since $-\frac{\hat{p}_{11}}{\hat{p}_{12}} \neq -\frac{\hat{r}_1}{\hat{r}_2}$, L_2 and L_1 intersect at D, say. Let L_3 be the line through D with slope $-\frac{\hat{p}_{21}}{\hat{p}_{22}}$. Clearly consumer 2's equilibrium consumption in state 2 must lie on L_3 and hence L_3 must have at least one point in common with the Edgeworth box.

However, if L_3 has more than one point in common with the Edgeworth box (see L_3' in Figure 2), consumer 1 will choose a portfolio to the North-west of D on L_1, and consumer 1's consumption in state 1 will be given by a point outside the Edgeworth box. This is impossible in equilibrium and so L_3 must pass through A.

It follows that the equilibrium portfolios of consumers 1 and 2 are represented by D and that equilibrium consumption in state 1 is given by B and equilibrium consumption in state 2 by A. Hence the slope of consumer 1's state 1 indifference curve at B must equal $-\frac{\hat{p}_{11}}{\hat{p}_{12}}$, and the slope of consumer 2's state 2 indifference curve at A must equal $-\frac{\hat{p}_{21}}{\hat{p}_{22}}$. By (6), however, the slopes of these indifference curves are equal and so $-\frac{\hat{p}_{11}}{\hat{p}_{12}} = -\frac{\hat{p}_{21}}{\hat{p}_{22}}$, which is impossible since L_2 and L_3 intersect.

This proves that $\frac{\hat{p}_{11}}{\hat{p}_{12}} = \frac{\hat{r}_1}{\hat{r}_2}$. A similar argument shows that $\frac{\hat{p}_{21}}{\hat{p}_{22}} = \frac{\hat{r}_1}{\hat{r}_2}$ and therefore $\frac{\hat{p}_{11}}{\hat{p}_{12}} = \frac{\hat{p}_{21}}{\hat{p}_{22}} = \frac{\hat{r}_1}{\hat{r}_2}$. If this is the case, however, it is clear that there is no need for markets to re-open at the end of the period. This means that there are only two prices r_1, r_2 to bring four markets, the market for each output in each state of

the world, into equilibrium. In other words, as is shown in Figure 3, there must be a line through C such that the U_1^1 and U_1^2 indifference curves touch at some point on the line and the U_2^1 and U_2^2 indifference curves touch at another point on the line (unless there is a corner solution). The utility functions can obviously be chosen so that this is impossible. For example, let $U_1^1(x_1,x_2) = U_1^2(x_1,x_2) = U_2^2(x_1,x_2) = \log(x_1+1) + \log(x_2+1)$ and let $U_2^1(x_1,x_2) = \log(x_1+1) + 2\log(x_2+1)$. Hence a type 3 equilibrium does not necessarily exist.

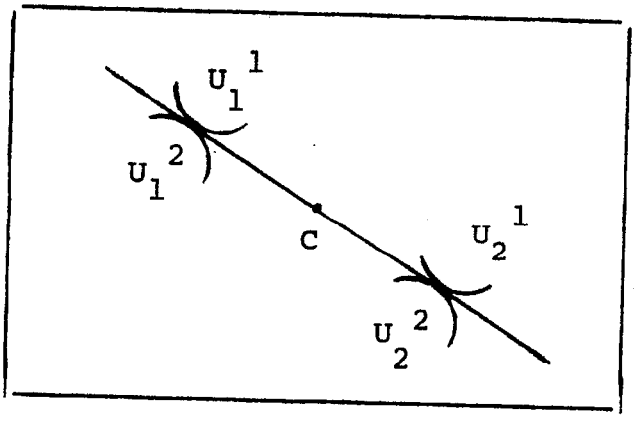

Figure 3

The basic reason for the failure of a type 3 equilibrium to exist in this example is that there is a discontinuity in consumer demand. Consider again Figure 2 and

let $-\frac{p_{11}}{p_{12}}$ be given by the slope of L_2, $-\frac{p_{21}}{p_{22}}$ by the slope of L_3, and $-\frac{r_1}{r_2}$ by the slope of L_1, where

$$\frac{r_1}{r_2} = \frac{\frac{\partial U_1^1}{\partial x_1}(1,1)}{\frac{\partial U_1^1}{\partial x_2}(1,1)} . \qquad (7)$$

Consumer 1's optimal portfolio is given by D. Now let D move along L_1 away from C with the p's adjusting accordingly. As $CD \to \infty$, $\frac{p_{11}}{p_{12}}$ and $\frac{p_{21}}{p_{22}} \to \frac{r_1}{r_2}$, and consumer 1's consumption demand in state 1 tends to B by (7). However, when $\frac{p_{11}}{p_{12}} = \frac{p_{21}}{p_{22}} = \frac{r_1}{r_2}$, B is not even an attainable consumption for consumer 1 in state 1.

This discontinuity results from the fact that B11 is not satisfied: $R_1(p) = (p_{11},p_{21})$ and $R_2(p) = (p_{12},p_{22})$ are linearly independent when $\frac{p_{11}}{p_{12}} \neq \frac{p_{21}}{p_{22}}$, but are linearly dependent when $\frac{p_{11}}{p_{12}} = \frac{p_{21}}{p_{22}}$.

In this example, consumers' utility functions are allowed to be different in different states. If tastes are the same in both states, a type 3 equilibrium obviously exists with $\frac{\hat{p}_{11}}{\hat{p}_{12}} = \frac{\hat{p}_{21}}{\hat{p}_{22}} = \frac{\hat{r}_1}{\hat{r}_2}$. It should be clear that

more complicated examples can be constructed where assumptions B1 - B10 are insufficient to insure the existence of a type 3 equilibrium even if tastes are the same in all states.

B11, then, does seem to have an important role to play. Unfortunately, it is very easy to construct examples, like Example 2, where B11 is violated. Moreover, there is no reason to believe that these examples are in any sense pathological. In fact, if the number of firms, the number of outputs and the number of states of the world are all large, it seems probable that, for any $f_1,\ldots,f_\varphi, R_{f_1}(p),\ldots,R_{f_\varphi}(p)$ are linearly dependent for some $p \in Y$, thus leading us to believe that B11 will fail to hold.

One important special case in which B11 is satisfied is when there is a single output. Then, Y contains the single element $(1,1,\ldots,1)$. This, of course, is the basic Diamond model of Section 2 (except that we now allow many inputs) in which type 2 and type 3 equilibria coincide.

4. Constrained Pareto-Optimality.

In the basic Diamond model of Section 2, the concept of constrained Pareto-optimality was defined. The reason for our interest in such a concept is that it seems unreasonable to judge the efficiency of a competitive system where complete markets do not exist in terms of what can be achieved by a central planner who is subject to no restrictions at all. A better approach is to compare competitive outcomes with outcomes which the central planner can achieve under the existing market structure.

In the single good case and also in the many input-single output case, it is easy to characterise in purely physical terms those allocations which can be achieved through the stock market. In the many output case, however, this is much more difficult because of the existence of trading at the end of the period. The problem can best be understood as follows. A central planner, if he is to be subject to the same constraints as the market mechanism, must be allowed to make certain exchanges of outputs at the end of the period. If he is allowed to make arbitrary exchanges, however, he can achieve any allocation at all and therefore is subject to no constraints. It follows that the central planner's exchanges at the end

of the period must be restricted in some way. The difficulty is finding a suitable restriction.

Because of this problem, we proceed somewhat differently in the many good case. Instead of trying to characterise allocations which can be achieved through the existing market structure in physical terms, we characterise them in terms of competitive equilibria.

Specifically, we say that an allocation can be achieved through the existing market structure if and only if it is an equilibrium allocation with respect to some initial distribution of endowments and shareholdings. We then define an allocation to be a constrained Pareto-optimum if it can be achieved through the existing market structure and if it is Pareto-optimal relative to all other allocations which can be achieved through the existing market structure. That is, an allocation is a constrained Pareto-optimum if it is an equilibrium allocation with respect to some initial distribution of endowments and shareholdings, and if there is no other equilibrium allocation, with respect to a possibly different initial distribution of endowments and shareholdings, which makes some people better off and nobody worse off.

This definition of constrained Pareto-optimality can be justified in two ways. First, Theorems 2.2 and 2.3 tell

us that, under weak assumptions, the definition is equivalent to the Section 2 definition in the single good case. Secondly, the definition seems to be the weakest possible one which makes any sense. For however we define a constrained Pareto-optimum, we surely do not want to call an equilibrium allocation constrained Pareto-optimal if there exists another equilibrium allocation which makes some people better off and nobody worse off. Since we will see that even under the above definition an equilibrium is not necessarily constrained Pareto-optimal, our approach seems to be a reasonable one.

In order to formalise what we have been saying, we first extend the definition of an allocation to the many good case. We define an allocation to be a vector $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ which satisfies

$$L_f \in R_+^M \qquad (f = 1,\ldots,F) \ ;$$

$$x^i = (x_1^i,\ldots,x_S^i) \in \prod_{s=1}^{S} R_+^N \qquad (i = 1,\ldots,I) \ ;$$

$$\sum_f L_f = \sum_i \bar{L}^i \ ;$$

$$\sum_i x_s^i = \sum_f g_f(L_f) a_f^s \qquad (s = 1,\ldots,S) \ ;$$

L_f is firm f's vector of inputs and x_s^i is consumer i's consumption of outputs in state s.

We define next $E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$ to be the economy in which consumer i's initial endowment of inputs is given by L^i and consumer i's initial proportionate shareholdings in firms by the F-vector θ^i $(i = 1,\ldots,I)$.

Definition of Constrained Pareto-Optimality

A vector $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is defined to be a constrained Pareto-optimal allocation in regime k $(k = 1,2,3)$ if

(a) it is a type k equilibrium allocation for the economy $E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$, where

$$L^i \in R_+^M (i=1,\ldots,I), \qquad \sum_i L^i = \sum_i \tilde{L}^i , \tag{8}$$

$$\theta^i \in R_+^F \quad (i = 1,\ldots,I) , \tag{9}$$

$$\sum_i \theta_f^i = 1 \quad (f = 1,\ldots,F) ; \tag{10}$$

(b) there do not exist $\tilde{L}^1,\ldots,\tilde{L}^I,\tilde{\theta}^1,\ldots,\tilde{\theta}^I$ satisfying (8), (9) and (10) such that $E(\tilde{L}^1,\ldots,\tilde{L}^I,\tilde{\theta}^1,\ldots,\tilde{\theta}^I)$ has a type k equilibrium allocation

$(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$, where

$$\sum_s \pi_s{}^i U_s{}^i(\tilde{x}_s{}^i) \geq \sum_s \pi_s{}^i U_s{}^i(x_s{}^i)$$

for all i, with inequality for some i.

The next two theorems establish the existence of a constrained Pareto-optimum.

Theorem 4.1: Under assumptions B1 - B10, a constrained Pareto-optimal allocation exists in regimes 1 and 2.

Theorem 4.2: Under assumptions B1 - B11, a constrained Pareto-optimal allocation exists in regime 3.

The proofs of these theorems are given in the appendix.

We turn now to the relationship between constrained Pareto-optima and equilibria. It is evident from the definition of a constrained Pareto-optimum that any constrained Pareto-optimum can be achieved as an equilibrium (with an appropriate redistribution of initial wealth). The following examples show that the converse of this proposition is false: an equilibrium is not necessarily constrained Pareto-optimal.

Example 3

There are two consumers, one firm, one input, two outputs and two states of the world. We assume that $g_1(L) = L$, $a_1^1 = a_1^2 = (1,1)$, so that there is no production uncertainty. Each consumer owns $\frac{1}{2}$ unit of the input and consumers' tastes are the same in both states of the world. Assumptions B1, B2, B4, B7, B8, B9 and B11 clearly hold and consumers' utility functions and probability beliefs may be chosen so that B3, B5, B6 and B10 also hold.

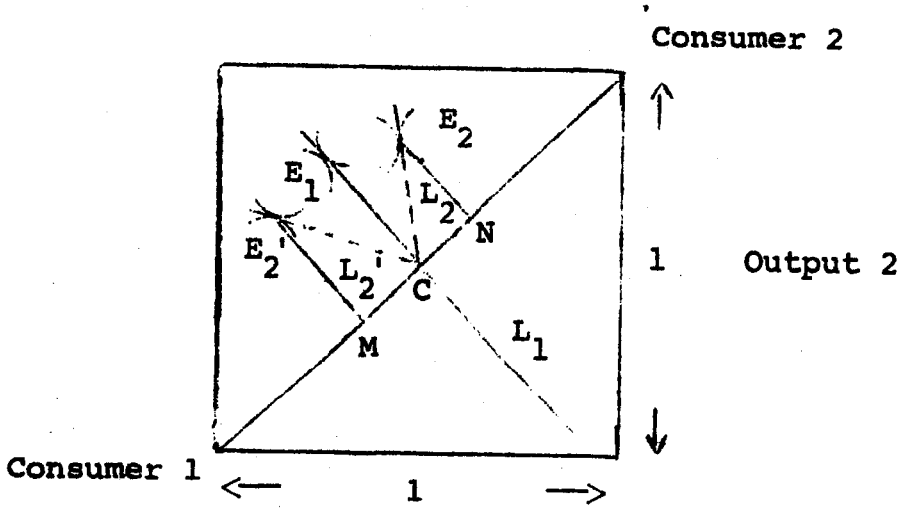

Figure 3

The situation can be represented by means of an Edgeworth box, as in Example 2. Since there is only one firm, type 1, 2 and 3 equilibria coincide. In equilibrium,

each consumer owns $\frac{1}{2}$ of the firm and, hence, at the end of the period, each consumer has claim to $\frac{1}{2}$ unit of each output in both states of the world. Consumers' equilibrium portfolios are therefore represented by the centre of the box, C.

Since there is no production uncertainty and tastes are the same in both states, there is an equilibrium with output prices at the end of the period the same in both states. The price line corresponding to this equilibrium is represented by L_1 in the diagram. However, we can evidently choose the utility functions so that there is another equilibrium with output prices represented by the line L_2 in state 1 and the line L_2' in state 2. In this second equilibrium, consumer 1 is better off in state 1 and worse off in state 2 and the opposite is true for consumer 2. It is clear, therefore, that if consumer 1 believes that state 1 occurs with a high enough probability and consumer 2 believes that state 1 occurs with a low enough probability, both consumers will be better off ex-ante in the second equilibrium than in the first equilibrium. Hence, the first equilibrium is not constrained Pareto-optimal.

This example can be modified very simply to show that the same problem can arise if consumers' probability beliefs

are the same. Assume that consumers agree that the probability of each state occurring is $\frac{1}{2}$. In the first equilibrium (E_1), each consumer has the same utility in each state. In the second equilibrium (E_2, E_2'), each consumer has a higher utility in one state and a lower utility in the other state. Clearly, if consumers are sufficiently risk-averse, they will prefer the first equilibrium in which there is no risk to the second equilibrium in which there is risk, and so now the second equilibrium is not constrained Pareto-optimal.

The reader may wonder where the usual proof of the optimality of equilibrium breaks down. The usual proof may be summarised as follows. If feasible situation A is preferred to the equilibrium situation by everybody, each consumer's bundle in situation A must be more expensive at the equilibrium prices than his equilibrium bundle. If we sum over consumers, it follows that profits are higher in situation A than in equilibrium, which contradicts the assumption that firms maximise profits.

Let us apply this proof to Example 3. Consider the equilibrium represented by L_1 and E_1 in Figure 4. In order to obtain the consumption given by E_2 in state 1 and E_2' in state 2 when output prices are given by L_1

in both states, consumer 1 must hold the portfolio represented by the point N. In other words he must own more than 50% of the firm not 50%. Of course, he will then be able to obtain a better consumption that E_2' in state 2, but this is beside the point. Similarly, consumer 2 must hold more than 50% of the firm in order to obtain the consumption given by E_2 in state 1 and E_2' in state 2.

The first part of the usual proof is therefore still correct. The more desirable consumption bundle (E_2, E_2') does cost more at the equilibrium prices than the less desirable bundle (E_1, E_1) since more than 50% of the firm is more expensive than 50% of the firm. If we now sum over consumers, however, all we may deduce is that more than 100% of the firm is more expensive than 100%. This evidently contradicts nothing!

In this example, starting off from the less desirable equilibrium, we can make both consumers better off simply by changing their price expectations. No change in the initial distribution of wealth is required. The example relies, of course, on the existence of multiple equilibria. If utility functions obey the gross substitutes assumption (see Arrow and Hahn [2]), equilibria of the (E_2, E_2') type, in which relative output prices differ in the two states

of the world, are ruled out, and any equilibrium of the E_1 type will therefore be constrained Pareto-optimal. The following example also depends on the existence of multiple equilibria, but these multiple equilibria are not ruled out by the gross substitutes condition or by any other simple condition. This example also shows that the assumption in Example 3 that firms produce multiple products is unimportant; an equilibrium may be sub-optimal even if each firm produces a single product.

Example 4

The example is almost identical to Example 2. There are two consumers, two firms, two inputs, two outputs and two states of the world. $g_1(L) = L_1$, $g_2(L) = L_2$, $a_1^1 = a_1^2 = (1,0)$, $a_2^1 = a_2^2 = (0,1)$. Each consumer owns $\frac{1}{2}$ unit of each input. Consumer 1 believes that state 1 occurs with certainty; consumer 2 believes that state 2 occurs with certainty. Consumers' utility functions satisfy B5 and B6 and are differentiable. The only change from Example 2 is that we assume that tastes are the same in both states of the world and that

$$\frac{\frac{\partial U^1}{\partial x_1}(1,1)}{\frac{\partial U^1}{\partial x_2}(1,1)} \neq \frac{\frac{\partial U^2}{\partial x_1}(1,1)}{\frac{\partial U^2}{\partial x_2}(1,1)} . \qquad (11)$$

(Assumptions B1 - B10 are clearly satisfied; assumption B11 is not.)

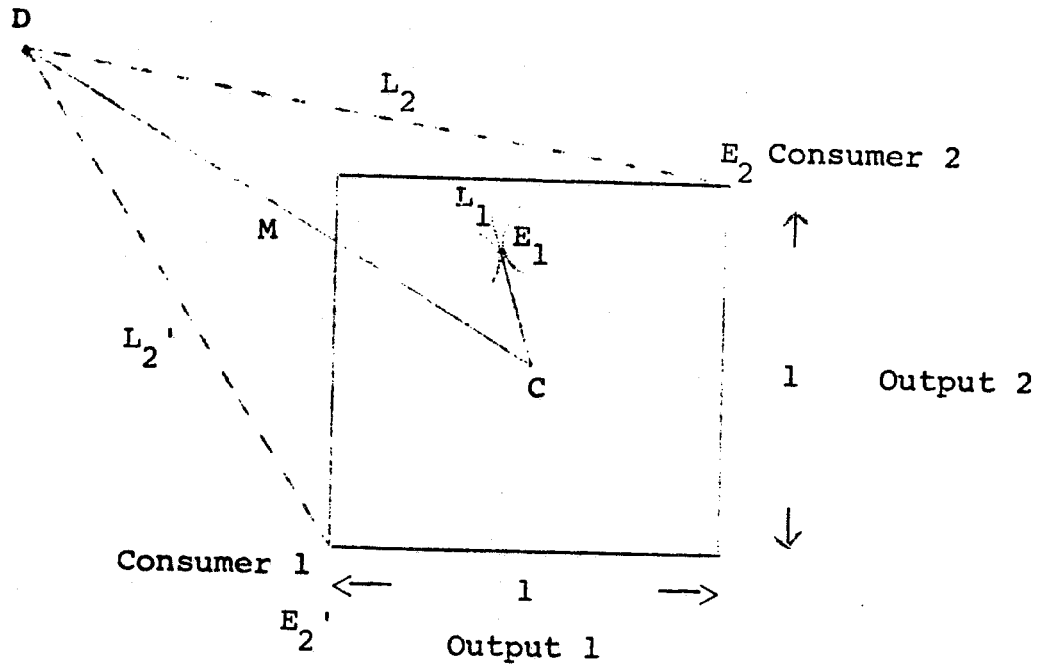

Figure 5

Type 1, 2 and 3 equilibria do not coincide in this example. We will concentrate on type 3 equilibria, but the same problems may arise with type 1 and 2 equilibria.

Because there is no production uncertainty and tastes are the same in both states, there exists an equilibrium in which $w = r = p_1 = p_2$ (see Example 2). This is represented by L_1. Equilibrium portfolios are given by C and equilibrium consumption by E_1 in both states of the world. As a result of (11), however, there is another equilibrium in which

$$\frac{w_1}{w_2} = \frac{r_1}{r_2} = - \text{ slope of } M ,$$

$$\frac{p_{11}}{p_{12}} = -\text{ slope of } L_2 = \frac{\frac{\partial U^1}{\partial x_1}(1,1)}{\frac{\partial U^1}{\partial x_2}(1,1)},$$

and

$$\frac{p_{21}}{p_{22}} = -\text{ slope of } L_2' = \frac{\frac{\partial U^2}{\partial x_1}(1,1)}{\frac{\partial U^2}{\partial x_2}(1,1)}.$$

Equilibrium portfolios are given by D, equilibrium consumption in state 1 by E_2 and equilibrium consumption in state 2 by E_2'.

Clearly the second equilibrium is preferred ex-ante by both consumers to the first since both consumers get more in the state they believe will occur. Hence, the first equilibrium is not constrained Pareto-optimal. Furthermore, both equilibria exist whether or not the gross substitutes condition holds. The gross substitutes condition fails to rule out multiple equilibria in this sort of situation basically because of the existence of budget constraints at the end of the period as well as a budget constraint at the beginning of the period.

An interesting feature of both Examples 3 and 4 is that, at the prices which rule in the less desirable

equilibrium, there does not exist a portfolio which provides exactly the income in each state required for the purchase of the consumption corresponding to the more desirable equilibrium. That is, at the prices represented by L_1, if consumer 1 wants enough income in state 1 to buy consumption E_2, he must hold a portfolio which provides him with more than enough income to buy E_2' in state 2.

Theorem 4.3 tells us that it is this feature of the many output case which explains why an equilibrium fails to be constrained Pareto-optimal. Before stating Theorem 4.3, we make another assumption.

B12. For each i: there exist f and s such that $\pi_s^i > 0$ and $a_f^s \geq 0$.

B12 says that each consumer believes that some firm is capable of producing something with non-zero probability.

Theorem 4.3: Let $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ be a type 3 equilibrium allocation for the economy $E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$ at equilibrium prices $(w,r,p_1,\ldots,p_S)$, where $L^1,\ldots,L^I,\theta^1,\ldots,\theta^I$ satisfy (8), (9) and (10). Under assumptions B6 and B12, there does not exist an allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$ such that

(a) $\sum_s \pi_s^i U_s^i(\tilde{x}_s^i) \geq \sum_s \pi_s^i U_s^i(x_s^i)$ for all i, with inequality for some i;

(b) for each i: the equations $p_s \tilde{x}_s^i = \sum_f y_f^i (p_s a_f^s)$ $(s = 1,\ldots,S)$ have a solution $y^i \in R^F$.

In other words, the allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$ cannot be Pareto-superior to the equilibrium allocation $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ if, at the equilibrium prices, there exist portfolios which yield in each state exactly the income required to purchase the $\tilde{x}^i$.

Proof of Theorem 4.3: Let

$$z^i = y^i \qquad (i = 1,\ldots,I-1)$$

and

$$z^I = (g_1(\tilde{L}_1),\ldots,g_F(\tilde{L}_F)) - \sum_{i=1}^{I-1} z^i .$$

Since the allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^I)$ is feasible, $\sum_{i=1}^{I} \tilde{x}_s^i = \sum_f g_f(\tilde{L}_f) a_f^s$, and hence by (b),

$$p_s \tilde{x}_s^i = \sum_f z_f^i (p_s a_f^s) \qquad (s = 1,\ldots,S)$$

for each i. In other words, the consumption bundle $\tilde{x}^i$ can be purchased with the portfolio z^i at the equilibrium prices. By B6, B12 and the fact that $\tilde{x}^i$ is at least as desirable as x^i for consumer i,

$$rz^i \geq wL^i + \sum_f \theta_f{}^i (r_f g_f(L_f) - wL_f)$$

for each i, with inequality if consumer i prefers $\tilde{x}^i$ to x^i. If we sum over i, the assumption that firms maximise market value is contradicted. Q.E.D.

Theorem 4.3 tells us that any allocation which does not involve exchanges at the end of the period (in which case $x_s{}^i = \sum_f z_f{}^i a_f{}^s$ and (b) is satisfied with $y^i = z^i$) cannot be Pareto-superior to an equilibrium allocation. It is clear, therefore, that a type 3 equilibrium is constrained Pareto-optimal in the many input - single output case since no equilibrium allocations involve exchanges at the end of the period.

Theorem 4.3 also tells us that if there are at least as many firms as states of the world and the monetary returns vectors of S of the firms are linearly independent for all $p_1, \ldots, p_S$, a type 3 equilibrium allocation is unconstrained Pareto-optimal since (b) is satisfied for

all allocations.

Similar results to Theorem 4.3 can be proved for type 1 and 2 equilibria. In particular, these equilibria are constrained Pareto-optimal in the many input - single output case. A more stringent condition than

$$\text{rank} \begin{bmatrix} p_1 a_1^1 & \cdots & p_S a_1^S \\ \vdots & & \vdots \\ p_1 a_F^1 & \cdots & p_S a_F^S \end{bmatrix} = S,$$

however, is required to insure the unconstrained Pareto-optimality of type 1 and 2 equilibria.

5. Weak Pareto-Optimality and Technological Efficiency.

In Section 4, we saw that an equilibrium is not generally constrained Pareto-optimal in the many output case. In this section, we show that a type 3 equilibrium is optimal in a somewhat weaker sense and that both a type 2 and a type 3 equilibrium are technologically efficient.

Definition of Weak Pareto-Optimality

An allocation $(L_1,\ldots,L_F,x^1,\ldots,x^S)$ is said to be weak Pareto-optimal if there is no allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^S)$ such that

$$U_s^i(\tilde{x}_s^i) \geq U_s^i(x_s^i) \qquad (s = 1,\ldots,S;\ i = 1,\ldots,I)\ ,$$

with inequality for some s and i.

That is, an allocation is weak Pareto-optimal if there is no allocation which makes some people better off in some states of the world without making other people worse off in other states of the world. This type of optimality has been discussed by Malinvaud in [15].

Weak Pareto-optimality differs from constrained Pareto-optimality in that our concern is with consumers' utilities at the end of the period when the state of the world has been determined rather than with their ex-ante

utility at the beginning of the period. In particular, no probability beliefs enter into the definition of weak Pareto-optimality. It should also be noted that weak Pareto-optimality is an unconstrained concept: an allocation is being compared with all other allocations, not just with those which can be achieved under the existing market structure.

Theorem 5.1: A type 3 equilibrium is weak Pareto-optimal under assumptions B6, B10 and B12.

Proof: Let $(w,r,p_1,\ldots,p_S)$ be an equilibrium price vector and $(L_1,\ldots,L_F,x^1,\ldots,x^S)$ a corresponding equilibrium allocation. Suppose the allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^S)$ satisfies $U_s^i(\tilde{x}_s^i) \geq U_s^i(x_s^i)$ for each s and i, with inequality for some s and i. Then, clearly, by B6 and equilibrium condition (XIV) of Section 3,

$$p_s\tilde{x}_s^i \geq p_s x_s^i \quad \text{for each } s \text{ and } i,$$

with inequality for those s and i for which $U_s^i(\tilde{x}_s^i) > U_s^i(x_s^i)$. Summing over i and using the facts that $\sum_i x_s^i = \sum_f g_f(L_f)a_f^s$ and $\sum_i \tilde{x}_s^i = \sum_f g_f(\tilde{L}_f)a_f^s$, we obtain

$$\sum_f g_f(\tilde{L}_f)p_s a_f^s \geq \sum_f g_f(L_f)p_s a_f^s \qquad (12)$$

for each s, with inequality for some s.

We now apply a result which follows from a theorem proved in Kuhn [12, Theorem II, Section 3].

Proposition: Let B be a $k \times \ell$ matrix and c a k-vector. Then, either

(a) $Bq = c$ has a solution $q > 0$,

or

(b) there exists a k-vector y such that $yc \leqq 0$ and $yB \geqq 0$, with $yB = 0 \Rightarrow yc < 0$.

Putting $B = \begin{bmatrix} p_1 a_1^1 & \cdots & p_S a_1^S \\ \vdots & & \vdots \\ p_1 a_F^1 & \cdots & p_S a_F^S \end{bmatrix}$ and $c = r$, we may deduce that

(a) $\begin{bmatrix} p_1 a_1^1 & \cdots & p_S a_1^S \\ \vdots & & \vdots \\ p_1 a_F^1 & \cdots & p_S a_F^S \end{bmatrix} q = r$ has a solution $q > 0$,

or

(b) there exists y such that $yr \leqq 0$ and

$$y \begin{bmatrix} p_1 a_1^1 & \cdots & p_S a_1^S \\ \vdots & & \vdots \\ p_1 a_F^1 & \cdots & p_S a_F^S \end{bmatrix} \geqq 0, \text{ with}$$

$$y\begin{bmatrix} p_1 a_1^{\,1} & \cdots & p_S a_1^{\,S} \\ \vdots & & \vdots \\ p_1 a_F^{\,1} & \cdots & p_S a_F^{\,S} \end{bmatrix} = 0 \Rightarrow yr < 0.$$

In equilibrium (b) is impossible by assumptions B6, B10 and B12. For, (b) says that either there exists a portfolio with non-positive cost which yields non-negative income in each state of the world and positive income in some states or there exists a portfolio with negative cost which yields zero income in each state of the world.

Hence (a) holds. Multiplying (12) by q_s and summing over s, we obtain

$$\sum_f g_f(\tilde{L}_f) \sum_s q_s p_s a_f^{\,s} > \sum_f g_f(L_f) \sum_s q_s p_s a_f^{\,s} ,$$

and, hence, by (a),

$$\sum_f g_f(\tilde{L}_f) r_f > \sum_f g_f(L_f) r_f .$$

Since $\sum_f \tilde{L}_f = \sum_f L_f$, it follows that

$$\sum_f g_f(\tilde{L}_f) r_f - w \sum_f \tilde{L}_f > \sum_f g_f(L_f) r_f - w \sum_f L_f ,$$

which contradicts $r_f g_f(L_f) - wL_f = \max_{L \in R_+^M} \{r_f g_f(L) - wL\}$ $(f = 1,\ldots,F)$. Q.E.D.

A type 1 or type 2 equilibrium may not be weak Pareto-optimal since we cannot use the same argument to rule out alternative (b). The above argument does however show that a type 1 equilibrium is weak Pareto-optimal if there exists i such that

$$z^i > 0 \qquad (13)$$

and consumer i believes that each state of the world occurs with positive probability. In the case of a type 2 equilibrium, the condition

$$\sum_f z_f^i a_f^s > 0 \qquad (s = 1,\ldots,S)$$

may be substituted for (13).

We turn now to a consideration of technological efficiency.

Definition of Technological Efficiency

An allocation $(L_1,\ldots,L_F,x^1,\ldots,x^S)$ is said to be technologically efficient if there is no allocation $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^S)$ such that

$$\sum_f g_f(\tilde{L}_f) a_f^s \geqq \sum_f g_f(L_f) a_f^s \qquad (s = 1,\ldots,S)\ ,$$

with $\geq$ replacing $\geqq$ for some s.

That is, an allocation is efficient if there is no

allocation such that more of some outputs are produced in some states of the world without less of other outputs being produced in other states of the world.

In view of Theorem 5.1, it should be no surprise that, under assumptions B6, B10 and B12, a type 3 equilibrium is technologically efficient. Theorem 5.2 says that in addition a type 2 equilibrium is efficient.

Theorem 5.2: Under assumptions B6, B10 and B12, a type 2 equilibrium and a type 3 equilibrium are technologically efficient.

Proof: Since the proof is almost the same as that of Theorem 5.1, only a sketch will be given. Let $(L_1,\ldots,L_F,x^1,\ldots,x^S)$ be an equilibrium allocation and $(\tilde{L}_1,\ldots,\tilde{L}_F,\tilde{x}^1,\ldots,\tilde{x}^S)$ an allocation satisfying

$$\sum_f g_f(\tilde{L}_f)a_f^s \geqq \sum_f g_f(L_f)a_f^s \qquad (s = 1,\ldots,S)$$

with $\geq$ for some s.

We apply Kuhn's result with $B = \begin{bmatrix} a_1^1 & \cdots & a_1^S \\ \vdots & & \vdots \\ a_F^1 & \cdots & a_F^S \end{bmatrix}$.

(b) can be shown to

be impossible, so that

$$\begin{bmatrix} a_1^1 & \cdots & a_1^S \\ \vdots & & \vdots \\ a_F^1 & \cdots & a_F^S \end{bmatrix} q = r$$

has a solution $q > 0$. A contradiction of the fact that firms are market value maximisers is obtained as before.

Q.E.D.

A type 1 equilibrium may not be technologically efficient. Such an equilibrium is efficient, however, if some consumer who believes that all states occur with positive probability holds positive shareholdings of all firms.

The proof of Theorem 5.2 is interesting because it tells us that in a stock market model, in contrast to the usual Arrow-Debreu model, technological efficiency depends in a crucial way on utility maximising behaviour by consumers as well as on market value maximising behaviour by firms. It should also be noted that, since it is assumed nowhere in the proof that price expectations are correct, Theorem 5.2 generalises to situations where there is price uncertainty.

Conclusion

We have shown in this paper that Diamond's result that a stock market equilibrium is constrained Pareto-optimal does not generalise to the many good case even if consumers have correct forecasts of future prices. The reason for this is not the existence of many inputs at the beginning of the period, but rather the existence of many outputs at the end of the period. When there are many outputs, the opportunities that exist for trading are not fixed, but depend to some extent on prices at the end of the period. One equilibrium may be better for everybody than another simply because it enables consumers to realise greater gains from trade.

Although an equilibrium is not necessarily constrained Pareto-optimal in the many good case, we have shown that, if consumers are subject only to a solvency constraint, an equilibrium is optimal in the weaker sense that some people cannot be made better off in some states of the world without other people being made worse off in other states of the world. This result cannot be regarded as all that comforting, however, since a rather restrictive assumption appears to be required to insure the existence of this sort of equilibrium.

FOOTNOTES

1. Diamond's optimality result may also be shown to break down in the many period-single good case. We will not consider this case here.

2. All our results can be generalised to the case where consumption takes place at the beginning of the period as well as at the end of the period.

3. States of the world are assumed to be determined independently of economic activity.

4. The following notation will be used. If $x \in R^K$, $x \geqq 0$ means $x_k \geqq 0$ $(k = 1,\ldots,K)$; $x \geq 0$ means $x \geqq 0$ and $x \neq 0$; $x > 0$ means $x_k > 0$ $(k = 1,\ldots,K)$. R^K_+ is defined to be $\{x \in R^K \mid x \geqq 0\}$.

5. Σ_i, Σ_f, Σ_s are used as a short-hand for $\sum_{i=1}^{I}$, $\sum_{f=1}^{F}$ and $\sum_{s=1}^{S}$ respectively. Similarly for all i, for all f, and for all s are used to mean for $i = 1,\ldots,I$, for $f = 1,\ldots,F$, and for $s = 1,\ldots,S$.

6. The case where firms can raise funds by issuing bonds as well as by issuing shares creates no additional complications as long as firms do not go bankrupt. For a proof of the irrelevance of firms' financial

policy in this situation, see Modigliani and Miller [17] and Stiglitz [23]. For a discussion of the problems caused by bankruptcy, see Smith [22].

7. For a slight weakening of the multiplicative uncertainty assumption, see Ekern and Wilson [8] and Leland [13].

8. rz denotes the inner product $\sum_f r_f z_f$.

9. This assumption is relaxed in Section 3.

10. Assumptions A2 and A4 are required in the absence of the free disposal assumption. The free disposal assumption is inconsistent with the assumption that firms produce a single pattern of returns across states of the world.

11. No distinction is made between a feasible allocation and an allocation.

12. Once we allow short-selling, a slightly strange situation can arise in which some consumers hold non-zero amounts of the random variable a_f, but firm f does not produce because the negative holdings cancel out the positive holdings. What we are implicitly assuming in regimes 2 and 3 is that markets for the a_f exist even when firms do not produce.

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APPENDIX

In this appendix, we prove Theorems 3.1, 3.2, 4.1 and 4.2. Theorem 3.2 is proved first.

Theorem 3.2: Under assumptions B1 - B11, a type 3 equilibrium exists.

Proof: The proof is basically the same as the proof of existence of an Arrow-Debreu equilibrium (see Debreu [3]). Modifications are required because of the existence of several budget constraints, the unboundedness of portfolios in regime 3, and the fact that demand correspondences are not always upper-semicontinuous.

The existence of several budget constraints creates no problems if we use a different normalisation of prices from the usual one. Let P^K be the (K-1) dimensional unit simplex, that is,

$$P^K = \{x \in R_+^K \mid \sum_{k=1}^{K} x_k = 1\} ,$$

and let

$$T = P^{M+F} \times \prod_{s=1}^{S} P^N .$$

We will consider prices in T. This normalisation has been used by Radner in [20].

We prove the theorem in three steps. The first step is to establish the existence of equilibrium when an artificial lower bound on portfolios is imposed and under the assumption that all the π_s^i are positive. The second step is to show that the assumption that the π_s^i are positive can be dropped. Finally, the third step is to show that the bound on portfolios can be removed.

Let E be the regime 3 economy defined in Section 3 and let ^bE denote the economy in which there is an additional constraint that consumer i can purchase only portfolios $z \geqq -\underset{\sim}{b}$ $(i = 1,\ldots,I)$; $\underset{\sim}{b}$ is defined to be the F-vector which has each component equal to b, and b is chosen to be non-negative.

Step 1: We prove that ^bE has an equilibrium if all the π_s^i are positive.

Define

$$H = \Big\{(L_1,\ldots,L_F,z^1,\ldots,z^I,x^1,\ldots,x^I) \;\Big|\; L_f \in R_+^M \ (f=1,\ldots,F),$$
$$z^i \in R^F \ (i=1,\ldots,I), \quad z^i \geqq -\underset{\sim}{b} \ (i=1,\ldots,I),$$
$$x^i = (x_1^i,\ldots,x_S^i) \in \prod_{s=1}^{S} R_+^N \ (i=1,\ldots,I),$$
$$\sum_f L_f \leqq \sum_i \bar{L}^i, \quad \sum_i z^i \leqq (g_1(L_1),\ldots,g_F(L_F)),$$
$$\sum_i x_s^i \leqq \sum_f g_f(L_f)a_f^s \ (s=1,\ldots,S)\Big\}.$$

H may be regarded as the feasible set for b_E (if free disposal is allowed). L_f is firm f's input vector, z^i is consumer i's portfolio and x_s^i is consumer i's consumption of outputs in state s.

Clearly H is bounded, so that we may choose c (depending on b) such that

$$(L_1,\ldots,L_F,z^1,\ldots,z^I,x^1,\ldots,x^I) \in H$$

$$\Longrightarrow \quad \begin{aligned} &\|L_f\| < c \quad (f = 1,\ldots,F) \\ &\|z^i\| < c \quad (i = 1,\ldots,I) \\ &\|x^i\| < c \quad (i = 1,\ldots,I) \end{aligned} \tag{14}$$

where $\|\ \|$ is the Euclidean norm. We use the bound c to define restricted demand and supply correspondence for b_E.

For $y = ((w,r),p_1,\ldots,p_S) \in T$, define

$$\alpha_f(y) = \Big\{L_f \in R_+^M \mid \|L_f\| \leqq c \text{ and } r_f g_f(L_f) - wL_f \geqq r_f g_f(L) - wL \text{ for all } L \in R_+^M \text{ satisfying } \|L\| \leqq c\Big\}$$

and

$$W^i(y) = w\bar{L}^i + \sum_f \bar{\theta}_f^i (r_f g_f(L_f) - wL_f) \ ,$$

where $L_f \in \alpha_f(y)$. α_f is firm f's (restricted) demand correspondence and W^i is consumer i's (restricted) wealth at prices y. Define also

$$\beta^i = \Big\{(z,x_1,\ldots,x_S) \mid z \in R^F,\ z \geqq -\underset{\sim}{b},\ \|z\| \leqq c,\ rz \leqq W^i(y),$$
$$x_s \in R^N_+ \ (s = 1,\ldots,S),\ \|x_s\| \leqq c \ (s = 1,\ldots,S),$$
$$p_s x_s \leqq p_s\Big(\sum_f z_f a_f^s\Big) \ (s = 1,\ldots,S)\Big\}.$$

β^i is consumer i's budget set at the beginning of the period. We may regard α_f as a correspondence from T to R^M_+ and β^i as a correspondence from T to $R^F \times \prod_{s=1}^{S} R^N_+$.

<u>Proposition 1</u>: α_f, β^i are upper-semicontinuous and $\alpha_f(y)$, $\beta^i(y)$ are convex.

For a definition of upper semi-continuity (and lower semi-continuity), the reader is referred to Debreu [3]. The proof of Proposition 1 is straightforward and will not be given.

<u>Proposition 2</u>: β^i is lower-semicontinuous at y if $W^i(y) > 0$.

<u>Proof</u>: Suppose $(z,x_1,\ldots,x_S) \in \beta^i(y)$ and $y^t \to y$. We construct $(z^t,x_1^t,\ldots,x_S^t) \in \beta^i(y^t)$ such that

$(z^t, x_1^t, \ldots, x_S^t) \longrightarrow (z, x_1, \ldots, x_S)$. Define

$$^tz_f = z_f + \varepsilon^t \qquad (f = 1, \ldots, F),$$

where ε^t is a non-negative real number chosen so that $p_s{}^t x_s \leq \sum_f {}^tz_f (p_s{}^t a_f{}^s)$ $(s = 1, \ldots, S)$ and $\varepsilon^t \to 0$ as $t \to \infty$. There exists such an ε^t since $p_s x_s \leq \sum_f z_f (p_s a_f{}^s)$ and B9 implies that $\sum_f p_s{}^t a_f{}^s > 0$. Now let

$$z_f{}^t = \frac{^tz_f}{\max_f \left[1, \frac{\|^tz\|}{c}\right] \times \max \left[1, \frac{\sum_f r_f{}^t \, {}^tz_f}{W^i(y^t)}\right]} \qquad (f = 1, \ldots, F),$$

$$x_s{}^t = \frac{x_s}{\max \left[1, \frac{\|^tz\|}{c}\right] \times \max \left[1, \frac{\sum_f r_f{}^t \, {}^tz_f}{W^i(y^t)}\right]} \qquad (s = 1, \ldots, S).$$

Then $(z^t, x_1^t, \ldots, x_S^t) \in \beta^i(y^t)$ and $(z^t, x_1^t, \ldots, x_S^t) \longrightarrow (z, x_1, \ldots, x_S)$ if $W^i(y) > 0$. Q.E.D.

We now define a demand correspondence γ^i corresponding to the budget correspondence β^i. Let

$$\gamma^i(y) = \begin{cases} \{(\hat{z},\hat{x}_1,\ldots,\hat{x}_S) \mid (\hat{z},\hat{x}_1,\ldots,\hat{x}_S) \in \beta^i(y) \\ \quad \text{and } \sum_s \pi_s{}^i U_s{}^i(\hat{x}_s) \geq \sum_s \pi_s{}^i U_s{}^i(x_s) \text{ for all} \\ \quad (z,x_1,\ldots,x_S) \in \beta^i(y)\} & \text{if } W^i(y) > 0, \\ \beta^i(y) & \text{if } W^i(y) = 0. \end{cases}$$

We define γ^i to be equal to β^i when $W^i = 0$ in order to make γ^i upper-semicontinuous. This technique is due to Debreu [4]. It will turn out that, for the y we are interested in, $W^i(y) > 0$ for all i.

Proposition 3: γ^i is upper-semicontinuous.

Proof: Suppose $(z^t,x_1{}^t,\ldots,x_S{}^t) \in \gamma^i(y^t)$, $y^t \to y$, and $(z^t,x_1{}^t,\ldots,x_S{}^t) \longrightarrow (z,x_1,\ldots,x_S)$. We show that $(z,x_1,\ldots,x_S) \in \gamma^i(y)$.

Case 1: $W^i(y) > 0$.

By Proposition 1, $(z,x_1,\ldots,x_S) \in \beta^i(y)$. Therefore, if $(z,x_1,\ldots,x_S) \notin \gamma^i(y)$, there must exist $(\tilde{z},\tilde{x}_1,\ldots,\tilde{x}_S) \in \beta^i(y)$ such that $\sum_s \pi_s{}^i U_s{}^i(\tilde{x}_s) > \sum_s \pi_s{}^i U_s{}^i(x_s)$. By Proposition 2, we may construct $(\tilde{z}^t,\tilde{x}_1{}^t,\ldots,\tilde{x}_S{}^t) \in \beta^i(y^t)$ such that $(\tilde{z}^t,\tilde{x}_1{}^t,\ldots,\tilde{x}_S{}^t) \longrightarrow (\tilde{z},\tilde{x}_1,\ldots,\tilde{x}_S)$. But, for large t,

$\sum_s \pi_s{}^i U_s{}^i(\tilde{x}_s{}^t) > \sum_s \pi_s{}^i U_s{}^i(x_s{}^t)$, which contradicts $(z^t, x_1{}^t, \ldots, x_S{}^t) \in \gamma^i(y^t)$.

Case 2: $W^i(y) = 0$.

Since $(z^t, x_1{}^t, \ldots, x_S{}^t) \in \beta^i(y^t)$, $(z, x_1, \ldots, x_S) \in \beta^i(y)$ by Proposition 1, and, therefore, $(z, x_1, \ldots, x_S) \in \gamma^i(y)$.

Q.E.D.

We are now in a position to define the excess demand correspondence for the economy ${}^bE$. For $y \in T$, let

$$\psi(y) = \Big\{\Big(\sum_f L_f - \sum_i \bar{L}^i,\ \sum_i z^i - (u_1, \ldots, u_F)\Big),$$

$$\sum_i x_1{}^i - \sum_i \sum_f z_f{}^i a_f{}^1,\ \ldots,\ \sum_i x_S{}^i - \sum_i \sum_f z_f{}^i a_f{}^S\Big|$$

$$L_f \in \alpha_f(y),\ u_f \leq g_f(L_f),\ r_f u_f = r_f g_f(L_f)$$

$$(f = 1, \ldots, F),\ (z^i, x_1{}^i, \ldots, x_S{}^i) \in \gamma^i(y)$$

$$(i = 1, \ldots, I)\Big\}.$$

ψ is not quite the excess demand correspondence of the economy, actually, since $\sum_i x_s{}^i - \sum_i \sum_f z_f{}^i a_f{}^s$ is the excess demand for outputs in state s only if the stock market is in equilibrium. It turns out to be more convenient to work with ψ than with the true excess demand correspondence.

ψ is a correspondence from T to a compact subset of

$R^{M+F} \times \prod_{s=1}^{S} R^{N}$. It may easily be verified that Propositions 1 and 3 imply that ψ is upper-semicontinuous and that $\psi(y)$ is convex for all $y \in T$. Also, $\psi(y) \neq \emptyset$ and, for every $y \in T$, $\Big(\Big(\sum_f L_f - \sum_i \bar{L}^i,\ \sum_i z^i - (u_1,\ldots,u_F)\Big),$ $\sum_i x_1^i - \sum_i \sum_f z_f a_f^1,\ \ldots,\ \sum_i x_S^i - \sum_i \sum_f z_f a_f^S\Big) \in \psi(y) \implies$

$$w\Big(\sum_f L_f - \sum_i \bar{L}^i\Big) + r\Big(\sum_i z^i - (u_1,\ldots,u_F)\Big) \leqq 0 \qquad (15)$$

and

$$p_s\Big(\sum_i x_s^i - \sum_i \sum_f z_f^i a_f^s\Big) \leqq 0 \quad (s = 1,\ldots,S) \ . \qquad (16)$$

Hence, the following proposition may be proved.

<u>Proposition 4</u>: There exists $y \in T$ such that $v \in \psi(y)$ and $v \leqq 0$.

This proposition is proved in Radner [20]. The proof is almost identical to the proof of the usual excess demand theorem (see Debreu [3]). The only difference is that, because ψ is defined on a product of simplices rather than a simplex, the fact that several Walras' laws ((15) and (16)) are satisfied is used in the proof.

Proposition 4 tells us that there exists $y \in T$ such that

$$L_f \in \alpha_f(y) \qquad (f = 1,\ldots,F) \ , \tag{17}$$

$$(z^i, x_1^i, \ldots, x_S^i) \in \gamma^i(y) \qquad (i = 1,\ldots,I) \ , \tag{18}$$

$$\sum_f L_f \leqq \sum_i \bar{L}^i \ , \tag{19}$$

$$\sum_i z^i \leqq (g_1(L_1),\ldots,g_F(L_F)) \ , \tag{20}$$

and

$$\sum_i x_s^i \leqq \sum_i \sum_f z_f^i a_f^s \qquad (s = 1,\ldots,S) \ . \tag{21}$$

It follows from (20) and (21) that

$$\sum_i x_s^i \leqq \sum_f g_f(L_f) a_f^s \qquad (s = 1,\ldots,S) \ , \tag{22}$$

so that $(L_1,\ldots,L_F,z^1,\ldots,z^I,x^1,\ldots,x^I) \in H$, the feasible set of ${}^bE$. Hence, by (14),

$$\|L_f\| < c \qquad (f = 1,\ldots,F) \ , \tag{23}$$

$$\|z^i\| < c \qquad (i = 1,\ldots,I) \ , \tag{24}$$

$$\|x^i\| < c \qquad (i = 1,\ldots,I) \ . \tag{25}$$

Since the bounds $\|L_f\| \leqq c$, $\|z^i\| \leqq c$ and $\|x^i\| \leqq c$ are not binding, it follows that, if we can show that $W^i(y) > 0$ for all i and that (19), (20) and (22) hold with equality, we will have proved the existence of a

type 3 equilibrium for ^{b}E.

We show first that $W^i(y) > 0$ for all i. Suppose $w = 0$. Then, since $(w,r) \in P^{M+F}$, $r \geq 0$ and so B2 implies that L_f will not satisfy (23) for all f. Therefore $w \geq 0$ and $W^i(y) > 0$ for some i by B8. It follows from B3 and B6 that $r > 0$ since otherwise consumer i's demanded portfolio z^i and consumption vector x^i will not satisfy (24), (25). But, if $r > 0$, B4 and (23) imply that $w > 0$, and, hence, by B7, $W^i(y) > 0$ for all i.

This argument establishes that $W^i(y) > 0$ for all i and also that $w > 0$, $r > 0$. In addition, $p_s > 0$, since otherwise B6 implies that the x^i cannot satisfy (25). Hence, since B3 and B6 imply that the budget constraints (15), (16) must hold with equality, (19), (20) and (22) must also hold with equality.

We have thus proved that ^{b}E has an equilibrium if all the $\pi_s{}^i$ are positive, and Step 1 is completed.

The assumption that the $\pi_s{}^i$ are positive was used implicitly in the last part of the proof when we argued that z^i is an equilibrium portfolio and x^i is an equilibrium consumption vector for consumer i at prices y if $(z^i, x_1{}^i, \ldots, x_S{}^i) \in \gamma^i(y)$ and $W^i(y) > 0$. We saw in Section 3, however, that this may not be true if

some of the π_s^i are zero.

In Step 2, we show that ${}^bE$ will still have an equilibrium if some of the π_s^i are zero.

Step 2: ${}^bE$ has an equilibrium when the π_s^i are non-negative.

Choose a sequence of positive numbers $\{{}^t\pi_s^i\}$ such that $({}^t\pi_1^i, \ldots, {}^t\pi_S^i) \in P^S$ for each i and t and ${}^t\pi_s^i \to \pi_s^i$ for each s and i as $t \to \infty$. Step 1 establishes that the economy ${}^bE^t$, in which the ${}^t\pi_s^i$ replace the π_s^i, has an equilibrium. Let ${}^ty \in T$ be the equilibrium price vector of ${}^bE^t$, $({}^tL_1, \ldots, {}^tL_F, {}^tx^1, \ldots, {}^tx^I)$ the equilibrium allocation and $({}^tz^1, \ldots, {}^tz^I)$ the equilibrium portfolios. Since the sequences $\{{}^ty\}$, $\{{}^tL_1, \ldots, {}^tL_F, {}^tx^1, \ldots, {}^tx^I\}$, $\{{}^tz^1, \ldots, {}^tz^I\}$ are bounded, we may assume without loss of generality that they have limits. Let

$$ {}^ty \to y \ ,$$

$$ ({}^tL_1, \ldots, {}^tL_F, {}^tx^1, \ldots, {}^tx^I) \longrightarrow (L_1, \ldots, L_F, x^1, \ldots, x^I) \ ,$$

$$ ({}^tz^1, \ldots, {}^tz^I) \longrightarrow (z^1, \ldots, z^I) \ .$$

It is now a relatively easy matter to show that y is an equilibrium price vector, and that $(L_1, \ldots, L_F, x^1, \ldots, x^I)$ and

$(z^1,\ldots,z^I)$ are the corresponding equilibrium allocation and equilibrium portfolios, for ${}^bE$. This follows from the upper-semicontinuity of $\alpha_f(y)$, the lower semi-continuity of $\beta^i(y)$ when $W^i(y) > 0$, the fact that B2, B3, B4, B6, B7, B8 and B10 $\Rightarrow$ $y > 0$ and $W^i(y) > 0$ for all i, and the fact that

${}^tx_s{}^i$ maximises $U_s{}^i(x)$ subject to ${}^tp_s x \leq {}^tp_s(\sum_f {}^tz_f{}^i a_f{}^s)$

and

$${}^tp_s \to p_s > 0, \quad {}^tx_s{}^i \to x_s{}^i, \quad {}^tz_f{}^i \to z_f{}^i$$

imply that

$x_s{}^i$ maximises $U_s{}^i(x)$ subject to $p_s x \leq p_s(\sum_f z_f{}^i a_f{}^s)$.

This proves Step 2. Q.E.D.

We now remove the bounds $z^i \geq -\underset{\sim}{b}$.

Step 3: E has an equilibrium.

Let ${}^by = (({}^bw, {}^br), {}^bp_1,\ldots,{}^bp_S)$ be an equilibrium price vector for ${}^bE$ and let $({}^bL_1,\ldots,{}^bL_F, {}^bx^1,\ldots,{}^bx^I)$, $({}^bz^1,\ldots,{}^bz^I)$ be the corresponding equilibrium allocation and portfolios. Consider the sequence of equilibria of ${}^bE$ as $b \to \infty$. If, for some b, ${}^bz^i > -\underset{\sim}{b}$ for all i,

${}^{b}y$ is clearly an equilibrium price vector for E and Step 3 is completed. We may confine ourselves therefore to the case where, for each b,

$$ {}^{b}z^{i} \not> -\underset{\sim}{b} \tag{26} $$

for some i. We show that, in this case, for large b, we may replace $({}^{b}z^{1},\ldots,{}^{b}z^{I})$ by alternative equilibrium portfolios $({}^{b}\hat{z}^{1},\ldots,{}^{b}\hat{z}^{I})$ which do satisfy ${}^{b}\hat{z}^{i} > -\underset{\sim}{b}$, thus proving again that ${}^{b}y$ is an equilibrium for E. To do this, we use a technique from Hart [10].

Consider, for each i, the sequence $\left\{\frac{{}^{b}z^{i}}{b}\right\}$, where b = 1,2,... . This sequence is bounded since ${}^{b}z^{i} \geqq -\underset{\sim}{b}$ for all i, $\sum_{i} {}^{b}z^{i} = (g_1({}^{b}L_1),\ldots,g_F({}^{b}L_F))$, and the ${}^{b}L_f$ are bounded. Therefore, without loss of generality, we may assume that the sequence has a limit, δ^{i} say.

Since

$$ \sum_{i} {}^{b}z^{i} = (g_1({}^{b}L_1),\ldots,g_F({}^{b}L_F)) , $$

it follows that

$$ \frac{\sum_{i} {}^{b}z^{i}}{b} = \left(\frac{g_1({}^{b}L_1)}{b},\ldots,\frac{g_F({}^{b}L_F)}{b}\right) , $$

and, therefore, taking limits, we obtain

$$\sum_i \delta^i = 0 \ . \qquad (27)$$

We may assume also that the sequence $\{{}^b p_s\}$ has a limit, p_s say $(s = 1,\ldots,S)$. B6 and B10 imply that $p_s > 0$ $(s = 1,\ldots,S)$ since equilibrium demands are bounded. For each b,

$${}^b p_s \, {}^b x_s = \sum_f {}^b z_f^i ({}^b p_s a_f^s) \qquad (s = 1,\ldots,S) \ ,$$

since budget constraints hold with equality in equilibrium, and so, dividing each side by b and taking limits,

$$0 = \sum_f \delta_f^i (p_s a_f^s) \qquad (s = 1,\ldots,S) \ . \qquad (28)$$

That is, at the limit prices $(p_1,\ldots,p_S)$, the portfolio δ^i provides zero income in each state of the world.

We show next that, as a consequence of B11, there exists, for each b, a portfolio ${}^b\delta^i$ which provides zero income in each state at prices $({}^b p_1,\ldots,{}^b p_S)$, that is,

$$0 = \sum_f {}^b\delta_f^i ({}^b p_s a_f^s) \qquad (s = 1,\ldots,S) \ , \qquad (29)$$

and which satisfies

$$\sum_i {}^b\delta^i = 0 \qquad (30)$$

and

$$ {}^{b}\delta^{i} \to \delta^{i} \quad \text{as} \quad b \to \infty \,. \tag{31} $$

Let ${}^{b}R$ be the matrix $\begin{bmatrix} {}^{b}p_1a_1{}^{1} & \cdots & {}^{b}p_Sa_1{}^{S} \\ \vdots & & \vdots \\ {}^{b}p_1a_F{}^{1} & \cdots & {}^{b}p_Sa_F{}^{S} \end{bmatrix}$ and

R the matrix $\begin{bmatrix} p_1a_1{}^{1} & \cdots & p_Sa_1{}^{S} \\ \vdots & & \vdots \\ p_1a_F{}^{1} & \cdots & p_Sa_F{}^{S} \end{bmatrix}$. Consider the equation

$$ {}^{b}\varepsilon^{i}\, {}^{b}R = -\delta^{i}\, {}^{b}R \,. \tag{32} $$

If we can show that this equation has a solution ${}^{b}\varepsilon^{i}$ satisfying

$$ \sum_{i} {}^{b}\varepsilon^{i} = 0 \tag{33} $$

and

$$ {}^{b}\varepsilon^{i} \to 0 \quad \text{as} \quad b \to \infty \,, \tag{34} $$

then, by (27), ${}^{b}\delta^{i} = \delta^{i} + {}^{b}\varepsilon^{i}$ satisfies (29), (30) and (31), and is what we are looking for. Now, (32) has a solution ${}^{b}\varepsilon^{i} = -\delta^{i}$. Since $p_s > 0$ we may choose b large enough so that ${}^{b}p_s > 0$ $(s = 1,\ldots,S)$. For such b, the rows $f_1,\ldots,f_\varphi$ span the rows of ${}^{b}R$ by assumption B11, and so

(32) has a solution ${}^{b}\hat{\varepsilon}^{i}$ satisfying ${}^{b}\hat{\varepsilon}^{i} = 0$ for $f \neq f_1, \ldots, f_\varphi$.

We show that ${}^{b}\hat{\varepsilon}^{i}$ satisfies (33) and (34). Since

$$ {}^{b}\hat{\varepsilon}^{i}\, b_R = -\delta^{i}\, b_R \,, \tag{35} $$

$$ \Big(\sum_i {}^{b}\hat{\varepsilon}^{i}\Big) b_R = -\sum_i \delta^{i}\, b_R{}^{i} $$

$$ = 0 \,, $$

by (27). It follows that $\sum_i {}^{b}\hat{\varepsilon}^{i} = 0$ since the rows $f_1, \ldots, f_\varphi$ of ${}^{b}R$ are linearly independent by assumption B11. In order to show that ${}^{b}\hat{\varepsilon}^{i} \to 0$, we show first that the sequence $\{{}^{b}\hat{\varepsilon}^{i}\}$ is bounded. If not, we may assume that $\dfrac{{}^{b}\hat{\varepsilon}^{i}}{\|{}^{b}\hat{\varepsilon}^{i}\|}$ tends to a limit, η_i say, and that $\|{}^{b}\hat{\varepsilon}^{i}\| \to \infty$. Clearly, $\|\eta^{i}\| = 1$, and, dividing (32) by $\|{}^{b}\hat{\varepsilon}^{i}\|$ and taking limits as $b \to \infty$, we obtain

$$ \eta^{i} R = 0 \,, $$

which contradicts the fact that the rows $f_1, \ldots, f_\varphi$ of R are linearly independent.

Hence the sequence $\{{}^{b}\hat{\varepsilon}^{i}\}$ is bounded. Let $\hat{\varepsilon}^{i}$ be any limit point of the sequence. By (32) and (28),

$$ \hat{\varepsilon}^{i} R = -\delta^{i} R = 0 \quad . $$

Therefore $\hat{\varepsilon}^i = 0$ since the rows $f_1,\ldots,f_\varphi$ of R are linearly independent, and (34) is established.

We have proved the existence of a portfolio ${}^{b}_{\delta}i$ satisfying (29), (30) and (31). Consumer i is indifferent between the portfolio ${}^{b}_{z}i$ and the portfolio $({}^{b}_{z}i - {}^{b}_{\delta}i)$. We show next that

$$ {}^{b}_{z}i - {}^{b}_{\delta}i > -\underset{\sim}{b} \qquad (36) $$

for large enough b.

Suppose not. Then we may assume that

$$ {}^{b}_{z_f}i - {}^{b}_{\delta_f}i \leqq -b \qquad (37) $$

for some f and all b. Dividing by b, taking limits and using the definition of ${}^{i}_{\delta}$, we obtain

$$ {}^{i}_{\delta_f} \leqq -1 \ . \qquad (38) $$

However, ${}^{b}_{z_f}i \geqq -b$ since ${}^{b}_{z}i$ is an equilibrium portfolio, and so (38) implies that ${}^{b}_{z_f}i - {}^{b}_{\delta_f}i > -b$ for large b, contradicting (37).

If we can show that the cost of ${}^{b}_{\delta}i$ is zero at prices ${}^{b}y$, that is, ${}^{b}r\,{}^{b}_{\delta}i = 0$ for all i, it will follow from (30) and (31) that $({}^{b}_{z}1 - {}^{b}_{\delta}1, \ldots, {}^{b}_{z}I - {}^{b}_{\delta}I)$ are equilibrium portfolios for ${}^{b}E$ for large b. Since, by (36), the boundedness constraints on the portfolios are

not binding, this establishes that ^{b}y is an equilibrium price vector for E for large b.

In order to show that $^{b}r^{b}\delta^{i} = 0$, we suppose the contrary. Since $\sum_{i} {}^{b}\delta^{i} = 0$, this implies that $^{b}r^{b}\delta^{i} > 0$ for some $i=i_o$. Therefore

$$^{b}r(^{b}z^{i_o} - {}^{b}\delta^{i_o}) < {}^{b}r^{b}z^{i_o} \leqq W^{i_o}(^{b}y) \ .$$

Consider the portfolio z which is greater than $(^{b}z^{i_o} - {}^{b}\delta^{i_o})$ by a small amount in every component. z satisfies i_o's budget constraint, and, by B3 and B6, is preferred to $(^{b}z^{i_o} - {}^{b}\delta^{i_o})$ by i_o. This, however, contradicts the fact that $^{b}z^{i_o}$, which yields the same income in every state as $(^{b}z^{i_o} - {}^{b}\delta^{i_o})$, is optimal for i_o at prices ^{b}y.

This shows that $^{b}r^{b}\delta^{i} = 0$ for each i, and completes the proof of Step 3 and Theorem 3.2. Q.E.D.

The argument given in Steps 1 and 2 of the proof of Theorem 3.2 also proves the existence of a type 1 equilibrium if we put $b = 0$. In order to prove the existence of a type 2 equilibrium, we redefine the feasible set of the economy to be

$$\Big\{(L_1,\ldots,L_F,z^1,\ldots,z^I,x^1,\ldots,x^I) \mid L_f \in R_+^M \quad (f = 1,\ldots,F),$$

$$z^i \in R^F \quad (i = 1,\ldots,I), \quad \sum_f z_f^i a_f^s \geq 0$$

$$(s = 1,\ldots,S;\ i = 1,\ldots,I), \quad x^i \in \prod_{s=1}^{S} R_+^N \quad (i = 1,\ldots,I),$$

$$\sum_f L_f \leq \sum_i \bar{L}^i, \quad \sum_i z^i \leq (g_1(L_1),\ldots,g_F(L_F)),$$

$$\sum_i x_s^i \leq \sum_f g_f(L_f) a_f^s \quad (s = 1,\ldots,S)\Big\} .$$

This set can be shown to be bounded, restricted demand and supply correspondences can be defined in terms of it, and the argument of Steps 1 and 2 can again be applied. Assumption B11 is no longer required.

Theorem 4.2: Under assumptions B1 - B11, a constrained Pareto-optimum exists in regime 3.

Proof: Consider the following maximisation problem:

$$\text{Max} \quad \sum_i \sum_s \pi_s^i U_s^i(x_s^i)$$

subject to the constraint that, for some $(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$ satisfying

$$L^i \in R_+^M \quad (i = 1,\ldots,I) , \tag{39}$$

$$\sum_i L^i = \sum_i \bar{L}^i ,$$

$$\theta^i \in R_+^F \qquad (i = 1,\ldots,I) \ , \tag{40}$$

$$\sum_i \theta_f^i = 1 \qquad (f = 1,\ldots,F) \ , \tag{41}$$

$(L_1,\ldots,L_F,x^1,\ldots,x^I)$ is a type 3 equilibrium allocation for the economy $E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$.

If we can prove that this maximisation problem has a solution, we will have proved the existence of a constrained Pareto-optimum. Let

$$\lambda = \sup \Big\{ \sum_i \sum_s U_s^i(x_s^i) \;\Big|\; (L_1,\ldots,L_F,x^1,\ldots,x^I) \text{ is a type 3 equilibrium allocation for some } E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I), \text{ where } L^1,\ldots,L^I,\theta^1,\ldots,\theta^I \text{ satisfy (39), (40) and (41)} \Big\} .$$

By Theorem 3.2, this supremum is well-defined.

Choose $({}^tL_1,\ldots,{}^tL_F,{}^tx^1,\ldots,{}^tx^I)$ and $({}^tL^1,\ldots,{}^tL^I,{}^t\theta^1,\ldots,{}^t\theta^I)$ so that $({}^tL_1,\ldots,{}^tL_F,{}^tx^1,\ldots,{}^tx^I)$ is an equilibrium allocation for $E({}^tL^1,\ldots,{}^tL^I,{}^t\theta^1,\ldots,{}^t\theta^I)$ at prices ${}^ty \in T$ and

$$\sum_i \sum_s \pi_s^i U_s^i({}^tx_s^i) \longrightarrow \lambda \quad \text{as} \quad t \to \infty \ .$$

Since the sequence $\{{}^t y\}$ is bounded, we may assume without loss of generality that it has a limit, y say. It is not difficult to see that $y > 0$ since equilibrium allocations are bounded. Hence, ${}^t y > 0$ for large t. It follows from assumption B11 that, for large t, there exist unique numbers ${}^t\mu_1{}^f, \ldots, {}^t\mu_\varphi{}^f$ such that

$$R_f({}^t p) = \sum_{j=1}^{\varphi} {}^t\mu_j{}^f R_{f_j}({}^t p) \qquad (f = 1, \ldots, F),$$

where ${}^t y = ({}^t w, {}^t r, {}^t p_1, \ldots, {}^t p_S)$ and ${}^t p = ({}^t p_1, \ldots, {}^t p_S)$.

Consider the equilibrium portfolios $({}^t z^1, \ldots, {}^t z^I)$ at prices ${}^t y$. The sequence $\{{}^t z^1, \ldots, {}^t z^I\}$ may be unbounded. We use the μ's to define new equilibrium portfolios which are bounded. Let

$$\hat{{}^t z}_f{}^i = 0 \qquad \text{for } f \neq f_1, \ldots, f_\varphi,$$

$$\hat{{}^t z}_{f_j}{}^i = \sum_{f=1}^{F} {}^t\mu_j{}^f \, {}^t z_f{}^i \qquad (j = 1, \ldots, \varphi) \quad \text{for } i = 1, \ldots, I-1,$$

$$\hat{{}^t z}_f{}^I = g_f({}^t L_f) \qquad \text{for } f \neq f_1, \ldots, f_\varphi,$$

$$\hat{{}^t z}_{f_j}{}^I = \sum_{f=1}^{F} {}^t\mu_j{}^f \, {}^t z_f{}^I - \sum_{f \neq f_1, \ldots, f_\varphi} {}^t\mu_j{}^f g_f({}^t L_f) \qquad (j = 1, \ldots, \varphi).$$

$\hat{t_z}^i$ yields the same income in every state as $t_z{}^i$ and therefore costs the same, since otherwise there would be a possibility for arbitrage. Furthermore, by construction,

$$\sum_i \hat{t_z}^i = (g_1(t_{L_1}),\ldots,g_F(t_{L_F})) ,$$

and, therefore, $(\hat{t_z}^1,\ldots,\hat{t_z}^I)$ are equilibrium portfolios at prices t_y.

Consider now the sequences $\{(t_{L_1},\ldots,t_{L_F},t_x{}^1,\ldots,t_x{}^I)\}$, $\{(\hat{t_z}^1,\ldots,\hat{t_z}^I)\}$ and $\{(t_L{}^1,\ldots,t_L{}^I,t_\theta{}^1,\ldots,t_\theta{}^I)\}$. $\{(t_{L_1},\ldots,t_{L_F},t_x{}^1,\ldots,t_x{}^I)\}$ and $\{(t_L{}^1,\ldots,t_L{}^I,t_\theta{}^1,\ldots,t_\theta{}^I)\}$ are obviously bounded and it is straightforward to show that $\{(\hat{t_z}^1,\ldots,\hat{t_z}^I)\}$ is also bounded as a consequence of the linear independence of $R_{f_1},\ldots,R_{f_\varphi}$ (see Step 3 of the proof of Theorem 3.2). Without loss of generality, then, we may assume that these sequences have limits. Let

$$(t_{L_1},\ldots,t_{L_F},t_x{}^1,\ldots,t_x{}^I) \longrightarrow (L_1,\ldots,L_F,x^1,\ldots,x^I) ,$$

$$(t_L{}^1,\ldots,t_L{}^I,t_\theta{}^1,\ldots,t_\theta{}^I) \longrightarrow (L^1,\ldots,L^I,\theta^1,\ldots,\theta^I) ,$$

$$(\hat{t_z}^1,\ldots,\hat{t_z}^I) \longrightarrow (\hat{z}^1,\ldots,\hat{z}^I) .$$

It can now be shown as in Step 2 of the proof of Theorem 3.2 that y is an equilibrium price vector for

$E(L^1,\ldots,L^I,\theta^1,\ldots,\theta^I)$, with the corresponding equilibrium allocations and portfolios given by $(L_1,\ldots,L_F,x^1,\ldots,x^I)$ and $(\hat{z}^1,\ldots,\hat{z}^I)$. Therefore, since

$$\lambda = \lim_{t\to\infty} \sum_i \sum_s \pi_s{}^i U_s{}^i({}^t x_s{}^i) = \sum_i \sum_s \pi_s{}^i U_s{}^i(x_s{}^i) \ ,$$

the maximisation problem has a solution and a constrained Pareto-optimum exists.

Q.E.D.

Theorem 4.1 is proved in a similar way.