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PREFERENCE STRUCTURES AND THE MEASUREMENT OF MILITARY WORTH

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CHAPTER I

INTRODUCTION

1. Economic Problems in the Military Establishment

In many important respects the military establishment is like a large multi-product business firm. Like the firm, it hires vast amounts of many kinds of goods and services and combines them in a bewildering variety of ways to "produce", not commercial products, but various kinds of commands, weapons systems, etc. Since many of the decision-making problems of the military organization have rather close analogues in the operations of commercial establishments, it is only natural that many of the techniques developed for the solution of business problems by economists, mathematicians, and others have been extensively studied for their applicability to the problems of military decision-making.

In spite of significant successes, however, these studies have frequently run up against a fundamental difficulty in the transference of techniques useful in the commercial setting to the problems of the military establishment. This difficulty has its roots in a basic difference between commercial firms and military organizations which is obvious after even the most superficial reflection on the analogy between the two types of organizations. The difference is simply that the business firm can refer to money as the measuring rod for the comparison of all or nearly all alternatives between which choices must be made. It is clear that for the military organization this is in large measure an impossibility.[1]

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1. The enormous importance of this difference has been emphasized by O. Morgenstern in The Question of National Defense (New York: Random House, 1959), pp. 203-205.

The absence of a monetary measuring rod profoundly complicates "intuitive" decision-making procedures and in many instances largely vitiates the usefulness of the optimizing techniques of the calculus, linear programming and so forth. While sophisticated techniques can, at least in principle, guide the allocation of given plant and equipment to the production of different commercial outputs, no such immediate application is possible in deciding, for example, the division of a given stock of bombers and tanks between two different theaters of operations. Since there is no revenue function associating different dollar values with different dispositions of the two items between the different operations, maximization procedures may be largely useless. Similarly much of the growing and highly sophisticated literature of inventory theory is largely inapplicable to military problems simply because there is no way of computing a monetary "loss function" in a very large number of cases.[1]

This does not mean that mathematical techniques are useless in all types of military problems, of course. Frequently the "costs" of alternative logistics policies are computable in monetary terms and therefore if the problem at hand happens to be one of minimizing the costs associated with some predetermined level of output, no special difficulties arise. Hence, for example, if one has decided in advance that a given number of bombers and tanks are to be routed to a given point of operations, such techniques can be used to determine the cheapest way of doing this within, perhaps, such other constraints as

1. This point has been explicitly recognized by T. Whitin in The Theory of Inventory Management (Princeton: Princeton University Press, 1953), pp. 265-75.

may be present. But if the levels at which the output activities themselves are to be carried on are among the variables the situation is quite different. The lack of the measuring rod of money to compare the "revenue" of alternative levels of different output activities and/or to compare these revenues with associated costs becomes a crucial difficulty where the levels of the outputs are not pre-determined. In short, it is the lack of some measure of the "value" of the outputs, some measure of the "revenue" derived from the operations of the military establishment which causes the trouble.[1]

The difficulties arising in the military establishment from the lack of a monetary measure of revenue are of course not confined to that type of organization. It is clear that a university, a hospital, or even a business firm for whom questions of "community prestige", etc. are important may face the same type of problem in using systematic mathematical techniques to solve decision-making problems. Finally, the socialist state itself faces decision-making problems of almost unimaginable complexity in an environment where there is no automatic price-making mechanism to establish the relative values of alternative outputs.[2]

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1. There is a different type of difficulty which we do not consider in this study which stems from the games-of-strategy aspects existing in many military decision-making situations. In such situations the relative values of alternative levels of different output activities depends upon the opponent's choices among the alternatives available to him. Thus the choices depend upon the opponent's choices. In this type of situation we cannot impute military worth values to alternative output levels except in the light of the solution of the game problem. See Whitin, *ibid*., p. 76.

2. See for example F. Hayek (ed.), *Collectivist Economic Planning* (London: G. Routlidge and Sons, 1928), 293 pp.; or G. R. Lange and F. M. Taylor, *On the Economic Theory of Socialism* (Minneapolis: U. of Minnesota Press, 1938), 128 pp.

In the study to follow we shall examine some systematic procedures for developing substitutes for a monetary revenue function in two specific types of problems: the so-called "assignment problem" and the inventory problem. All discussion will be developed in the specific context of the military establishment, but the relevance of the analysis to other types of institutions should be obvious.[1]

2. The Concept of Military Worth

The need for devising some sort of measure of the "revenue" or "utility" of alternative military outputs has of course been recognized for some time. It is at least implicit in any attempt to make up a military priority list. In the rather scattered literature of the problem the general concept of the revenue or utility of military output has frequently been referred to by the term "military worth", a usage to which we shall adhere in the present study. It is not our intention to present a comprehensive review of this literature or to attempt a general investigation of all aspects of the military worth problem. Such material is or shortly will be available elsewhere.[2]

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1. The general question of allocation procedures in the absence of monetary guides will be examined in a forthcoming research memorandum of the Econometric Research Program by W. G. Mellon.

2. See W. G. Mellon, "A Selected, Descriptive Bibliography of References on Priority Systems and Related Nonprice Allocators," Naval Research Logistics Quarterly, Vol. 5, No. 1 (March, 1958), pp. 17-27. A more extensive review of the literature and the problems it raises will shortly be available by the same author in an Econometric Research Program research memorandum.

We shall rather be concerned with the more limited task of exploring a particular kind of technique for developing military worth functions in the context of the two problems mentioned above, the assignment problem and the inventory problem.

As a preliminary to a more specific discussion of the task to be undertaken it will be helpful to make a few general comments on the concept of "military worth" which will be employed.

1. By a "military worth index" we shall mean no more than an index of the relative desirability of military alternatives where "relative desirability" is determined by a means to be specified. Such an index is thus a guide to choice.[1]

2. We shall be concerned only with the relative desirability of military alternatives in terms of each other, not in terms of any money costs which may be associated with them. Thus, for example, suppose we have two vectors A and B of inventory levels for different items. We seek to devise an index of military worth which tells us whether A is preferred to B, but not whether the preferred vector is "worth" the additional monetary cost that may be associated with it. In other words we are attempting merely to devise methods of comparing the values of different outputs, not a method of comparing the outputs with the value of the inputs.

For many practical purposes this is sufficient. At all but the highest levels of military decision-making the problem faced by the authorities is one of maximizing the use of given inputs. This may be a matter of maximizing the use of a fixed amount of money, distributing a given number of men, tanks, and planes among alternative uses or even

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1. See below, p. 112.

optimizing the content of an allowance list for a submarine with fixed space capacity. For none of these problems is it necessary to have a direct measure of the value of the alternatives in terms of money.[1]

3. We shall be concerned with situations where the basis of the military worth considerations is what may be called "subjective." The distinction between "objective" and "subjective" bases for military worth values may be illustrated by two examples. Suppose, to take the "objective" case first, one is considering alternative patterns of bombing missions whose aim is to destroy grounded enemy planes. In such a case the expected-number-of-planes-destroyed associated with a particular alternative constitutes an "objective" basis for ranking the desirability of the alternatives. Situations of this kind where the objective can be defined in some kind of uni-dimensional physical index present no particular conceptual problems with respect to the determination of the military worth of alternatives.

There are many other problems of a totally different character, however. Suppose one is considering alternative allocations of different kinds of hardware to different theaters of operations, or, again, alternative stocking policies where many different kinds of items are involved. In such cases there is no simple physical basis for a military worth index. The aim is rather to maximize some vague concept of over-all military "effectiveness." The key to problems of this kind is that their solution must inevitably call into play the "subjective" judgment of military authorities in determining the relative effectiveness of alternatives. We may therefore term any military worth problem

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1. Actually such a measure would really only become a necessity in deciding what the total level of expenditures should be. See A. Enthoven and H. Rowen, "Defense Planning and Organization," (RAND Corporation, P-1640, March, 1959), pp. 10-28.

where the valuation of alternatives is built up from the expression of preferences by military authorities a problem in determining a "subjective" military worth. Our concern will be exclusively with problems of this type. The expressed "subjective" preferences of military authorities may of course be related to some type of physical consideration of the sort noted in our discussion of the objective case, but the crucial characteristic of the "subjective" problem from our point of view is that the construction of military worth indices begins with only the preferences themselves. We seek to find systematic ways of organizing these expressions of preference to obtain the desired military worth indices.

3. Some Definitions and Notation

The three main chapters of this study (Chapters II-IV) will investigate the manipulation of preference information for the determination of military worth functions in two types of problems. The major item, the assignment problem, is dealt with in Chapters II and III. Here we shall assume that there are many items of different types and many uses or activities to which they may be assigned. We shall derive techniques of establishing an order on the various possible assignment patterns from "elementary" preference information. We are not concerned with techniques of computing optimal assignments from among feasible alternatives, but merely with methods of establishing preference relations among any possible pair of alternative assignment patterns. Thus in the linear programming context this would be equivalent to an attempt to determine the parameters of the objective function but not an attempt to compute optimal programs.

Chapter IV deals with the evaluation of alternative inventory positions. A detailed description of this problem is best left to that chapter. For the present it is sufficient to say that we wish to evaluate alternative stock levels in a system where there is more than one inventory point and more than one type of item being stocked at each point. In this chapter again we shall build up the military worth function on the "subjective" basis of certain types of elementary preference information discussed there in detail.

To give a somewhat clearer picture of what we mean by "obtaining military worth indicators by the systematic manipulation of elementary subjective preference information" it is desirable to establish some terminology. In a sense both the assignment problem and the inventory problem concern the assignment of "things to things", so to speak. In the case of the assignment problem itself the assignment of items to various types of uses, locations, ships, or activities is involved. In the inventory problem the pattern of stock levels depends, in part, upon the quantities of the different types of items assigned to the different inventory points. We shall call the types of items being assigned "models" and the types of things to which they are assigned, "activities". Neither of these terms represents more than, at best, a somewhat unhappy compromise among equally unsatisfactory alternatives. To avoid confusion the following points should be kept in mind in connection with the use of these terms in this study.

First, with respect to the "models": Two different model-types may represent simply two different types of the same sort of equipment, such as, for example, two different types of short wave receivers. On the other hand, in a somewhat different context, different model types may represent completely different sorts of items, such as tanks versus planes. In any event we shall at all times consider two physically distinct items as belonging to the same model-type if they are to be regarded as perfect substitutes for each other in the context of the particular problem under discussion. A model of the ith type shall be designated by m_i.

Secondly, with respect to the "activities": In some cases two different activity-types will represent what the term implies in its literal sense, for example, two different bombing missions. In other cases, however, different activity-types will be represented by different uses for the models. Thus in one problem we consider the assignment of equipment to different ship-types. here each ship-type represents a distinct activity type. In still other cases the different "activities" will refer to different theaters of operations or locations. Clearly the term "activity" is not equally appropriate for all of these uses but the formal similarities of the different situations require a common term, and we trust the reader will be clear as to exactly what is meant in each situation as it is described. An activity of the jth kind will be designated by v_j.

In actual practice it may not always be clear when two physically distinct activities, uses, locations, etc. should be regarded as belonging to the same activity-type. In some kinds of problems, such as, for example, assigning different kinds of radar sets to different kinds of ships, it might seem obvious that two ships of the same kind should be given the same "activity" designation. As we shall see, however, for our purposes two activities should be regarded as being "of the same kind" (and therefore designated by the same symbol) only if their military significance is the same. More precisely, two physically distinct activities, x and y should be thought of as belonging to the same activity type if and only if the military value of assigning some model m_i to x and to y is regarded as the same and if it is the same for all model types. In this case even though x and y may differ in many ways they are activities of the "same kind" from our point of view.[1]

A few more terms are introduced having primary application to the assignment problem. In general there will be many actually or potentially available "sets" of any given model-type.[2] Furthermore there will generally be many instances in the military establishment or in the problem under consideration of activities of the same kind. We shall designate the assignment of a model of the ith type to an activity of the jth type by a_{ij}. Any general scheme for allocating equipment to different activities is of course likely to involve more than one instance of assigning a particular type of equipment to a particular type of activity. The notation $x_{ij}a_{ij}$ will mean that some assignment plan includes x_{ij} instances of the assignment of an m_i to a v_j.

The interpretation of such symbolism as "$x_{ij}a_{ij}$" would not be altogether unambiguous without further discussion. Either one or both

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1. See R. E. McShane and H. Solomon, Letter to the Editor of the Naval Research Logistics Quarterly, Vol. V, No. 4 (December, 1958), pp. 363-67.

2. The reader is again cautioned to note the difference between the usual meaning of the word "model" and the broader one here in use. Thus in a problem of assigning tanks and fighter planes to different operations, "tanks" and "fighter planes" may be the two model-types involved. Thus the statement that there may be many "available sets of any given model" in this context means merely that there is more than one tank and one fighter plane available for assignment.

of two interpretations could be given. Thus "$5a_{ij}$" might stand for the installation of five pieces of radar of the ith type in a single submarine of the jth sort. It could also mean the installation of one piece of radar of the ith sort in each of five submarines of the jth sort (or, of course, some other distribution adding up to five). In any event it is desirable for a designation such as "$5a_{ij}$" to have a unique "utility" value so that the interpretations must be limited in such a way that this is possible. To speak loosely, if the "marginal utility" of a_{ij} is constant, then the complete range of possible interpretations given above is possible within our condition that the "utility" or "military worth" of x_{ij} have the same value within the permitted range of interpretations. If diminishing marginal utility exists, then obviously "$5a_{ij}$" interpreted as installing five sets on one sub will not have the same value as "$5a_{ij}$" interpreted as installing one set each on five different subs. Thus, as we shall see later, we shall find it necessary either to interpret our problem so that never more than one m_i is assigned to a particular instance of a v_j or else we shall simply have to assume that the marginal utility of the x_{ij}, however interpreted, is constant.[1]

In any event the logistics center is faced with the task of determining an over-all allocation of pieces of equipment of various model-types to activities of various types. Let us designate an "over-all assignment plan" by the following notation.

$$x_{11}a_{11} + x_{12}a_{12} + \cdots + x_{nm}a_{nm}$$

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1. At this juncture it may appear as though such limitations are only artificially forced by the choice of notation, but this is not the case. See below, p. 29.

This means that there will be x_{11} instances of the assignment of a type 1 model to a type 1 activity, x_{12} instances of the assignment of a type 1 model to a type 2 activity, and so on.[1]

We shall refer to an instance of the assignment of a particular model to a particular activity as an "element" of the over-all assignment plan.

4. The Nature of the Elementary Preference Information

Our task is to find techniques by which a logistics center can construct an ordering of the potential over-all assignment plans.[2] Since we are restricting ourselves to situations where the orderings must ultimately be derived from the "subjective" expressions of preferences on the part of competent military authorities, it would of course be possible merely to submit the over-all assignment plans directly to an authority for ranking. We shall assume, however, that for various reasons developed in the appropriate places (and summarized in Chapter V) that this direct or "naive" approach is impossible or undesirable. We shall assume instead that the logistics center has at its disposal, not direct rankings of the over-all assignment plans themselves but more elementary information.

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1. It would be more in conformity with standard usage to simply use the vector notation $(x_{11},\ldots,x_{nm})$ for the over-all assignment plan but our choice turns out to be more convenient later on.

2. We are of course also interested in obtaining an ordering of all possible distributions of stocks for different activities in the inventory problem, but the special features of that situation are better left for discussion in Chapter IV.

In clearing up the meaning of the phrase "more elementary information" a few examples may be helpful. (1) Suppose the logistics center must work from a priority list which in some sense purports to order the "importance" of the activities, the v_j. If at the same time it has technical information on the relative importance of the models to the different activities, how can this material be combined to obtain a ranking of the over-all assignment plans? (2) Suppose a direct ranking of the a_{ij} is available. Under what conditions can this information be used to solve the problem? (3) Suppose we have priority information for different subsets of the various activities and must combine them to obtain a "ranking of the priorities".[1] (4) Suppose the logistics center receives not merely an ordering on the a_{ij}, but what will be called a "difference order" in Chapter III, meaning, roughly, an ordering on the "differences" in importance of different items. What can be done with this "elementary" information?

We propose to consider a number of types of elementary preference information and in each case to characterize their contents in axiomatic terms. There will be some discussion about what kinds of preference information can appropriately be expected in different types of situations. Further, we shall consider various possible "combinatorial rules" for the manipulation of each type of preference information and characterize these also in axiomatic terms. While the setting in Chapter IV on the inventory problem contains certain special features,

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1. The need to rank priorities when a central supplying agency is dealing simultaneously with many activities, each of which has its own priority list is discussed by O. Morgenstern in "Consistency Problems in the Military Supply System," Naval Research Logistics Quarterly, Vol. 1, No. 3 (September, 1954), p. 271.

the same basic approach is developed there in an attempt to derive meaningful orderings of alternatives.

In each of Chapters II, III, and IV we shall consider the possibilities of summarizing the preference structures being discussed by a numerical "military worth function" similar in nature to the "utility functions" of the economics literature.[1] Such numerical summaries are of course useful in applying routine mathematical techniques for computing optimal alternatives under whatever constraints may be present. The entire technique of constructing military worth functions from preference information raises some important methodological questions which are best raised in the context of the more detailed discussion to follow. A summary of the main points underlying the method of approach used in this study is presented in Chapter V.

Chapter II deals with purely "ordinal" preference information, that is, preference information which can lead to military worth functions unique only up to a monotone transformation. Since all standard priority lists contain preference information of this kind some attention is given to the question of the design, interpretation, and uses of priority lists. The question of what will be called the "separability" of priority and efficiency information is discussed and we shall also consider the implications of situations requiring an amalgamation of preference information from different sources. A military worth function having certain useful properties is discussed.

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1. The relation between the utility problem and the military worth problem is discussed in Chapter III and again in Chapter V.

Chapter III deals with preference information of a more complex character. A method of constructing military worth indices developed by Aumann and Kruskal is subjected to analysis and interpretation. The so-called "multi-person amalgamation problem" is considered in the context of "cardinal" military worth indices.

Chapter IV is devoted to discussion of an adaptation of the well-known von Neumann-Morgenstern utility technique to the evaluation of alternative inventory stocking positions.

Chapter V is designed not to recapitulate results, but to gather together the various strands of a general theory of military worth which are scattered throughout Chapters II, III, and IV.

CHAPTER II

ANALYSIS OF THE USE OF PRIORITY AND EFFICIENCY ORDERINGS

1. Introduction

The problem dealt with in the following sections is the analysis of the possibilities of using certain types of ordering information to determine optimal over-all assignment plans. Problems of assigning models of different types to activities of different types are divisible into two basic kinds of situations, situations where the problem of ranking the "importance" of the activities can be separated from the problem of ranking the "efficiency" of the models, and situations where these problems cannot be thus separated. Consider, for example, the following two assignment problems.

Example 1 -- Assignment of different kinds of bombs to different kinds of bombing missions.

Activities

v_1 = Destroy one square mile of administrative real estate.

v_2 = Destroy one square mile of storage real estate.

v_3 = Destroy one square mile of industrial real estate.

Models

m_1 = Bomb capable of destroying one square mile.

m_2 = Bomb capable of destroying two square miles.

m_3 = Bomb capable of destroying three square miles.

Example 2 -- Assignment of different types of radar sets to different types of ships.

Activities	Models
v_1 = A tanker.	m_1 = A set with 10 mile range.
v_2 = A submarine.	m_2 = A set with 15 mile range.
v_3 = A Coast Guard Cutter.	m_3 = A set with 20 mile range.

In Example 1 the "activities" involved are so defined as to have some clear military import independent of the type of equipment which is assigned to them. Under the circumstances it would seem reasonable to expect to be able to obtain a meaningful ranking of the relative "importance" of these activities from some authorized military authority. The nature of the activities is such that this ranking can be obtained without reference to installed models. In Example 2, however, quite the opposite appears to be the case. The definition of the "activities" involved is such that it does not appear to make sense to try to order their "importance" without knowledge of what models are installed in them. For example, can one say that a "submarine is more important than a tanker"? The answer to this depends on what the two kinds of ships can do and this, in turn, depends in part upon the equipment installed in them, on the m_i.[1]

When we are dealing with cases such as Example 1 where it is sensible to ask a competent military authority to rank the v_j in the abstract, so to speak--i.e., without any knowledge of the equipment installed--we shall make the following formal assumption. <u>Assumption 1: There exists an ordering on the activities.</u>[2] It is clear that Assumption 1 is not trivial and that in fact one would expect to find many

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1. The reader is again warned about the use of the word "activity." In Example 1 the v_j are activities in the literal sense of the word, whereas in the second example the v_j are really "uses" or "locations". In any event, in both examples the v_j are the "things" to which the models are to be assigned and therefore are "activities" for our purposes.

2. The definition of ordering involved is the usual one: i.e., for any activities a and b, either a is preferred to b (a P b), b is preferred to a, or a is indifferent to b (a I b). Also we assume that both the preference and indifference relations are transitive. We shall ignore the possibility of indifference relations in the rest of this chapter since its consideration merely complicated the exposition without adding anything of much interest.

real assignment problems, perhaps a majority, where it does not hold. We shall speak of situations where Assumption 1 holds as "separable" cases, that is, cases where the question of the importance of the activities can be separated from the question of the efficiency of the equipment installed in them. Conversely, cases where Assumption 1 does not hold, such as Example 2, are called "inseparable". The discussion which immediately follows deals only with the separable case. We shall return to a discussion of the inseparable case on page 34.

2. The Relative Efficiencies of the Models

In addition to Assumption 1 relating to the ranking of activities we need two additional assumptions dealing with the ranking of the models. <u>Assumption 2: For any given activity there exists an ordering of the models with respect to the given activity</u>. This assumption says merely that the effectiveness of the different models for any given use can be compared and ranked by some military technical expert. This assumption will be valid for any problem in which we are interested.[1]

<u>Assumption 3: The ordering of the models is the same for all activities</u>. This assumption states that in the problems to which it

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1. The reader might raise the question as to whether the order on the models would necessarily be the same for any two physically distinct examples of the same activity-type. The answer is "yes" and follows from our definition of an activity-type (p. 9). If we have two physically distinct activities in a problem of the separable type and if their technologies are not identical, then they belong to two different v_j categories even though they may be indifferent in the rank-order of the v_j. Thus two activities are placed in the same v_j category in the separable case if (1) they are indifferent and (2) they are indifferent when a set of any given model-type is installed in each, i.e., if they are technologically identical.

applies, the "technology" of the different activities is sufficiently similar so that the order of effectiveness of the different models is the same for all uses. It is clear that this assumption will not hold universally. For example, consider the following variant of the bombing mission problem.

Example 3 -- The v_j are the same as in Example 1, but the m_i are changed to the following:

m_1 = Destruction of 1 square mile of v_3 or v_1 or 3 miles of v_2.

m_2 = Destruction of 2 square miles of each type of real estate.

m_3 = Destruction of 3 square miles of v_3 or v_1 or 1 mile of v_2.

In this case it is clear that the order on the m_i is not the same for all activities.

3. The Multi-Person Amalgamation Problem

If the military organization confronted with a problem of the kind we are considering is such that one person has the sole responsibility for evaluating all the priorities and efficiencies involved, then the task of obtaining a complete ordering on the possible combinations of assignments of different models to different activities is a trivial one, at least in the formal sense (however painful and difficult the decision-making problem may be for the person responsible). In this case no more is involved than that the individual consider each possible pair of a_{ij} and decide which would make a greater military contribution. If his judgments are transitive, the task is complete. The whole existence of a "problem" in determining an order on the a_{ij} depends upon the supposition that the structure of authority is more complex than this. In particular the supposition that must be made is that there will be different military officers who are responsible for evaluating the relative merits of different subsets of the total set of activities and that the final ordering of all the activities should, in some unspecified way, "take account of" the expertise of each officer in his own realm.

The problem of how to "take account of" the expertise of individual subordinates will be examined further in a later section. The present aim is merely to indicate the nature of the different types of situations. It appears that this difficulty of obtaining an over-all ordering of the assignments is of two basic types. The first would stem from the possibility that, say, for example, of 9 possible activities (v_j) there are three groups of three items, the items in each group being evaluated by one of three different persons. For an over-all evaluation of an assignment program involving all 9 activities, these orderings must be meshed. The equivalent problem on the efficiency side is not likely to occur. Normally a single technical expert would be able to evaluate the relative effectiveness of all models being considered for any particular assignment. (Of course the ordering by different persons concerned with different assignments may be different, but then we merely have a case where Assumption III does not hold, not an amalgamation problem.)

The second major kind of problem stemming from the specialization and division of expertise, with its resulting organizational complexity is the fact that normally the evaluation of the relative efficiencies of different models in the various activities will be made by technical experts, while the relative military importance of the different activities will be made by a different person or persons. We turn to a consideration of this aspect of the problem in the following section.

4. Ranking the a_{ij}

In the present section we shall suppose that if a problem exists of meshing individual orderings of subsets of the v_j, that this problem has somehow been solved so that in fact we can assume the existence of a complete ordering on the activities. Let us assume at the same time that technical experts have provided a complete order on the relative efficiencies of the different model-types in the different activities. The question is now: how much use are the activity and model orderings in obtaining an ordering on the assignment elements themselves, that is, on the a_{ij}?

The situation we are envisioning here is the one mentioned in the previous section where a division of labor exists between purely technical experts and strategic experts. Thus we might imagine a central logistics office receiving two sets of information, information on the priorities of the activities from Washington, say, and information on the efficiencies of the models from testing laboratories. The task of the central office is to determine the relative order of the assignment elements without imposing additional judgments of its own. The task of obtaining a complete order on the a_{ij} "without imposing additional judgments" is impossible, as is easily seen, but the point will be examined in some detail since we are concerned not merely with whether or not a unique best solution to an assignment problem can be found, but also, if it cannot, what range of acceptable solutions may exist. It is also desirable to know the source and degree of any arbitrariness that may be involved in the determination of a solution. Let us therefore examine the applicability of the following three-part rule to the solution of the problem.

Rule 1a: For any pair of assignment elements a_{ij} and a_{rs}, a_{ij} is preferred to a_{rs} if $v_s = v_j$ (that is, if the two physically distinct activities involved belong to the same activity-type) and if m_i is preferred to m_r. Rule 1a states the innocuous conclusion that if two physically distinct activities belong to the same activity-type, the assignment of a more desirable model is to be preferred to the assignment of a less desirable model. It must be kept in mind, however, that even this statement is true only if the technology of all distinct activities in a given activity-type is identical. This is in fact the case by definition of an activity-type.[1]

Rule 1b: For any pair of assignment elements a_{ij} and a_{rs}, a_{ij} is preferred to a_{rs} if $m_i = m_r$ (that is, if the same model-type is installed in each) and if v_j is preferred to v_s. This rule states that if a model of the same type is installed in each of two activities, the installation in the more important activity would be preferred if a choice had to be made. It is important to notice that this rule makes sense only under a rather restrictive set of circumstances, namely that the technology of all physically distinct activities, regardless of the activity-type to which they belong, is identical. The rule would hold in the following type of problem: Suppose we are distributing radar sets to submarines which are technologically identical but located in different theaters of operations. In this case, physically distinct activities (different submarines) could be technologically identical but belong to different activity-types if located in different theaters.

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1. See above p. 9 and p. 18.

The rule <u>would not</u> hold in the following variant of the bombing mission problem.

<u>Example 4</u> -- The activities are the same as in Example 1 (p. 16) but the models have been changed to the following:

m_1 = Bomb capable of destruction of 3 square miles of administrative real estate, 2 miles of storage, and 1 mile of industrial real estate.

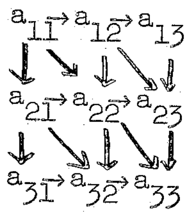
m_2 = Bomb capable of destruction of 4 square miles of administrative real estate, 3 miles of storage, and 2 miles of industrial real estate.

m_3 = Bomb capable of destruction of 5 square miles of administrative real estate, 4 miles of storage, and 3 miles of industrial real estate.

Applying Rule 1b to this situation, suppose activity v_3 (destruction of one mile of industrial real estate) is regarded as having a higher priority than v_2 (destruction of one mile of storage real estate). Now model m_1 destroys 2 miles of storage and one mile of industrial real estate. Rule 1b says that a_{13} must be preferred to a_{12}, i.e., it is more desirable to destroy one mile of industrial real estate than to destroy 2 miles of storage real estate. Such a conclusion cannot be drawn from the original information in the priority and efficiency lists.

<u>Rule 1c</u>: For any pair of assignment elements a_{ij} and a_{rs}, a_{ij} is preferred to a_{rs} if m_i is preferred to m_r and v_j is preferred to v_s. By similar reasoning the applicability of the rule is restricted to cases where all physically distinct activities, regardless of priority class are technologically identical.

For the case where all activity-types are technologically identical so that all three parts of Rule 1 apply we can picture the preference relationships obtainable by means of a matrix. In the matrix below movements down the rows represent movements toward superior models while movements to the right represent movements toward increasingly important activities. The arrows indicate the direction of preference. Also, since preference among the a_{ij} as determined by Rule 1 is transitive, any pair of a_{ij} connected by a chain of arrows pointing in the same direction are comparable. If technological identity does not hold, so that only Rule 1a applies, only elements in the same column can be compared.



It is of course clear that even when all of Rule 1 applies some pairs of elements (such as a_{21} and a_{12}) remain incomparable.[1] But the somewhat less obvious conclusion is that in those cases where technological uniformity among the activities does not prevail only horizontal comparisons can be made. One can therefore anticipate the conclusion that activity priority lists will be of very little assistance in separable problems which lack technological uniformity among activities.

5. Using the Ordering on the a_{ij} to Compare Alternative Assignment Plans

It is now necessary to show to what extent an ordering on the nm different individual possible assignments enables us to order different over-all assignment plans made up of different combinations of the

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1. Here and elsewhere in this study the relationship of "incomparability" between any two entities x and y means "neither x P y, nor y P x, nor x I y". This concept of incomparability is to be sharply distinguished from the indifference relationship. Nor is incomparability related to the absence of the so-called Archimedean property in any way. Furthermore it should be noted that the relationship of incomparability is not necessarily transitive. Thus in the above matrix we have a_{32} Inc a_{23} and a_{23} Inc a_{31}, but a_{32} P a_{31}. See J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (2d ed.; Princeton: Princeton University Press, 1947), p. 630 for a discussion of various interpretations of the concept of incomparability in preference systems. Note particularly that our concept does not satisfy their conditions A:16 and A:17.

individual a_{ij}. An over-all assignment plan is, to repeat a few definitions made earlier, simply a specification of how many of each type of individual assignments of a specific model to a specific activity are to be made. Also, we refer to an instance of the assignment of the ith model to the jth use as an "element" of an assignment plan.

We first require rules for comparing one assignment plan with another on the basis of the preference ordering of the a_{ij}. Two rules recommend themselves on the basis of the simplicity and lack of arbitrariness of the assumptions required to justify them in the general case.

(1) One assignment plan shall be considered preferable to another if each element in one plan is preferred or indifferent (not all indifferent) to each element in the other. A preferred assignment must contain at least as many elements as the plan to which it is preferred.

(2) One assignment plan shall be considered preferable to another if each element in one has corresponding to it an element in the other (different in each case) to which it is preferred or indifferent (not all indifferent). A preferred assignment plan must contain at least as many elements as the plan to which it is preferred.

To illustrate the meaning of these two different rules, consider the following four assignment plans.

a) $(1)a_{11} + (0)a_{12} + (1)a_{21} + (0)a_{22}$

b) $(0)a_{11} + (1)a_{12} + (0)a_{21} + (1)a_{22}$

c) $(1)a_{11} + (1)a_{12} + (0)a_{21} + (0)a_{22}$

d) $(0)a_{11} + (0)a_{12} + (1)a_{21} + (1)a_{22}$

Suppose that it has been determined that the four types of elements are preferred in the following order:

$$a_{11} \; P \; a_{12} \; P \; a_{21} \; P \; a_{22}$$

The use of Rule 2, as may be seen, orders the four plans in a complete, transitive way, i.e.,

$$c \; P \; a \; P \; b \; P \; d.$$

For example, a is preferred to b since a_{11} is preferred to a_{12} and a_{21} is preferred to a_{22}. On the other hand, if we use the stronger Rule 1 we get

$$c \; P \; d, \; c \; P \; b, \; c \; P \; a, \; a \; P \; d, \text{ and } b \; P \; d.$$

Since the assignment plans a and b are incomparable on the basis of this rule, however, a complete ordering of the alternatives is unobtainable. It could, of course, be assumed that if neither of two alternatives can be determined superior to the other, then the two alternatives should be judged indifferent. This assumption, however, is essentially arbitrary and may lead to intransitive results.

The first rule can evidently be safely rejected in favor of the second on the grounds that the second rule yields determinate results in more cases and does not seem to require stronger assumptions than the first. It does not yield determinate results in all cases, however. Consider the pair below, the alternatives of which must be regarded as incomparable,[1]

$$\text{a)} \quad (1)a_{11} + (0)a_{12} + (0)a_{21} + (1)a_{22}$$
$$\text{b)} \quad (0)a_{11} + (1)a_{12} + (1)a_{21} + (0)a_{22}$$

where the preference order is

$$a_{11} \; P \; a_{12} \; P \; a_{21} \; P \; a_{22} \; P \; a_{31}.$$

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1. See note on p. 24 above for definition of "incomparability".

It must be noted that both rules appear to involve the notion that the utilities (or military worths) of the individual elements are independent. Consider the pair of assignment plans c and d in the first set of examples, where c is preferred to d under both rules. The assertion that c is to be preferred to d because every element in c is preferred to every element in d depends upon the assumption that different activities do not co-operate with each other, on the one hand, or, on the other, that they do not compete with each other. If this were false, then a_{11} and a_{12} might interfere, or a_{21} and a_{22} co-operate in such a way that the latter pair in conjunction would be more valuable than the former in spite of the fact that each element of the d plan in isolation is less valuable than each individual element of the c plan in isolation. The assumption of the independence of the utilities of individual assignments or elements is a strong one, but it is clear that if it is invalid, no comparison between assignment plans simply on the basis of the order of the individual a_{ij} would be possible. We shall continue to explore the implications of the assumption in this paper, therefore.

One way of interpreting the meaning of Rule 2 derives directly from the notion of a preference order on the a_{ij}. What is the meaning of the statement that some element a_{rs} "is preferred to" some other element, a_{pq}? The most sensible interpretation seems to be that if confronted with an assignment plan consisting only of a single a_{rs} and one consisting of only a single a_{pq}, the latter would be ranked higher, i.e., "a_{rs} P a_{pq}" means that A_{rs} P A_{pq}, where

$$A_{rs} = (1)a_{rs} + (0)a_{pq}$$
$$A_{pq} = (0)a_{rs} + (1)a_{pq}.$$

Thus since an over-all assignment plan is a combination of a_{ij}, it is really a complex combination of such simple assignment plans. Rule 2 merely asserts, in effect, that if each of n such simple assignment plans is preferred or indifferent to one of m ($m < n$) simple assignment plans, then the n plans in combination must be preferred to the m plans in combination. Obviously this requires only the assumptions of "independent utilities" and, as we shall see below, "constant marginal utility" to make sense.

Two direct consequences of Rule 2 which will be useful in deciding between more complex situations than the ones illustrated above suggest themselves.

> 2a. If of two assignment plans, a and b, a is preferred to, indifferent to, inferior to, or incomparable with b, the same relationship will be preserved if identical or indifferent elements are added to both plans.

For example, consider the following pair of over-all assignment plans, a,b and a',b'.

a- $(1)a_{11} + (0)a_{12} + (1)a_{21} + (0)a_{22}$

b- $(0)a_{11} + (1)a_{12} + (0)a_{21} + (1)a_{22}$

a'- $(2)a_{11} + (0)a_{12} + (1)a_{21} + (0)a_{22} + (1)a_{31}$

b'- $(1)a_{11} + (1)a_{12} + (0)a_{21} + (1)a_{22} + (1)a_{31}$

The rule states that, for example, if a P b, then a' P b'.

> 2b. If of two assignment plans a and b, a is preferred to, indifferent to, inferior to, or incomparable with b, the same relationship will be preserved if all the coefficients in each plan are multiplied by the same number.

The meaning of this assertion may be illustrated by the following pairs of over-all assignment plans.

$$\begin{array}{ll} a- & (1)a_{11} + (0)a_{12} + (1)a_{21} + (0)a_{22} \\ b- & (0)a_{11} + (1)a_{12} + (0)a_{21} + (1)a_{22} \\ a'- & (3)a_{11} + (0)a_{12} + (3)a_{21} + (0)a_{22} \\ b'- & (0)a_{11} + (3)a_{12} + (0)a_{21} + (3)a_{22} \end{array}$$

This rule states that if, for example, a P b, then a' P b'.

It is clear that both 2a and 2b are merely special cases of Rule 2. Rule 2a seems to raise no difficulties on intuitive grounds. Rule 2b, however, brings into relief another assumption upon which the reasonableness of Rule 2 depends. It has already been remarked that Rule 2 makes sense if the contribution to utility or military worth of different activities can be regarded as independent. Rule 2b also suggests that the validity of Rule 2 depends upon the assumption of something like constant marginal utility of elements of the same kind, since it implies that if $(1)a_{ij}$ P $(1)a_{rs}$, then $(n)a_{ij}$ P $(n)a_{rs}$. As in the case of independent utilities, this is a strong empirical assumption without which little or no progress can be made. If the assumption were not valid, then the knowledge of an ordering on the a_{ij} would permit comparison of assignment plans only if all the coefficients in the plans were either 0 or 1. This would clearly be very restrictive in practical application. One might conjecture that in actual allocation problems the constant marginal utility assumption could be made approximately true if n were not too large and if a sufficiently large number of distinctions were made to produce many priority classes of activities.

Rules 2a and 2b considerably simplify the task of comparing complex alternatives. Suppose, for example, we are to rank the following pair of over-all assignment plans, a and b.

a- $(17)a_{11} + (13)a_{12} + (20)a_{21} + (16)a_{22} + (11)a_{13}$

b- $(14)a_{11} + (\ 7)a_{12} + (29)a_{21} + (19)a_{22} + (\ 8)a_{13}$

The first rule tells us that for comparative purposes we may safely reduce the alternatives by appropriate subtraction to the following forms:

a' $(3)a_{11} + (6)a_{12} + (0)a_{21} + (0)a_{22} + (3)a_{13}$

b' $(0)a_{11} + (0)a_{12} + (9)a_{21} + (3)a_{22} + (0)a_{13}$

The second rule enables us to further simplify the alternatives to

a'' $(1)a_{11} + (2)a_{12} + (0)a_{21} + (0)a_{22} + (1)a_{13}$

b'' $(0)a_{11} + (0)a_{12} + (3)a_{21} + (1)a_{22} + (0)a_{13}$

Generally, as in the above case, even after simplification of the alternative assignment plans by application of 2a and 2b, the reduced plans will contain coefficients greater than 1. In such cases it is desirable to have a systematic procedure for comparing the plans in accordance with the test of Rule 2. Of course two plans may turn out to be indifferent or incomparable, but if one is superior to the other it must have a larger number of instances of the highest a_{ij} on the preference scale. A possibly preferable plan should first be identified by this feature. Suppose in the above case the order is

$$a_{21} \ P \ a_{12} \ P \ a_{11} \ P \ a_{22} \ P \ a_{13},$$

we can pick b'' as potentially preferable to a'' on this basis. After b'' has been thus identified, its possible superiority to a'' can be checked by taking the elements in b'' in order of their appearance on the preference scale and matching them off against elements in a'', matching each element in b'' in turn against the element remaining in a'' which is highest on the preference scale. Thus the $(3)a_{21}$ of b'' matches against the $(2)a_{12}$ and $(1)a_{11}$ of a'', while $(1)a_{22}$ of b'' matches against $(1)a_{13}$ of a''. Since each element of b'' is preferred to a different element in a'', assignment b must be preferred to assignment plan a--provided, of course, that the assumptions of independent utilities and "constant marginal utility" makes sense in the particular application.

Actual assignment problems will of course normally involve comparison of many more than two pairs of plans. Four kinds of situations can be envisaged: (1) There are exactly as many pieces of equipment available as there are activities and they are to be distributed in the best possible way. (2) Less than sufficient numbers of pieces are available. (3) More than sufficient numbers of pieces are available. (4) There is no specific restriction on the potential numbers of pieces of equipment but it is desired to evaluate all the different possible ways of fulfilling given activity needs.

Suppose, as an example of the first case, that there are three instances each of two types of activities, v_1 and v_2 to be supplied. Suppose further that there are three available pieces of equipment of type m_1 and three of type m_2. There are then three distinct assignment plans which could be employed.

a- $(3)a_{11} + (1)a_{12} + (0)a_{21} + (2)a_{22}$

b- $(2)a_{11} + (2)a_{12} + (1)a_{21} + (1)a_{22}$

c- $(1)a_{11} + (3)a_{12} + (2)a_{21} + (0)a_{22}$

With four distinct types of elements there are (excluding the possibility of indifference) six possible preference orders and three different pairs of plans. Of course among the six preference orders some would enable us to determine a complete, transitive order on the plans, while others would leave all pairs of plans on an incomparable basis. In other words in any given situation, the ordering on the a_{ij} may convey all the information that is needed for any decision among alternative plans, or it may be of no use at all. Something in between these two extremes would probably be most typical.

6. A Numerical Military Worth Function

We have considered some assumptions about the elementary preference information and in Rule 2 we have developed a method of combining this information to obtain preferences between the members of (some) pairs of complex over-all assignment plans. It is now possible to define a military worth function which will be of use in the mechanical solution of certain types of problems.

Let us consider a set of numbers $u(a_{ij})$ such that $u(a_{ij}) > u(a_{rs})$ if and only if a_{ij} P a_{rs}. This numerical index is obviously "ordinal" in the sense that it reflects only the order of preferences on the a_{ij} and any monotonically increasing transformation would yield an equally acceptable set of values. Now let us define the military worth of any over-all assignment plan A as follows:

$$MW(A) = \sum x_{ij} \cdot u(a_{ij}).$$

It is then true that if an over-all assignment plan A is preferred to an over-all assignment plan B by application of Rule 2,

$$MW(A) > MW(B).$$

The proof of this assertion is obvious. 1. If an element in one plan is preferred to an element in another, the utility of the first is greater than the utility of the second, by the construction of the

$u(a_{ij})$ index. 2. If plan A is preferred to plan B, then by the definition of Rule 2, for each element in A there exists, corresponding to it, a particular element in B to which it is preferred or indifferent, not all indifferent; also there are at least as many elements in A as in B. 3. Therefore the sum of the utilities of the elements in A is greater than the sum of the utilities of the elements of B. Notice that this result does not depend upon the particular choice of the $u(a_{ij})$ index.

The converse of the above proposition is, of course, false. If $MW(A) > MW(B)$, it does not follow that A is preferred to B, where preference is determined by Rule 2. Thus preference is a sufficient, but not a necessary condition for one plan to have a higher military worth value than another.

The significance of this military worth function is explained by the following consideration. Suppose we are dealing with an actual assignment problem and that there are constraints which limit the set of feasible plans. For example, the number of models of any given model-type may be limited, or the number of models which can be assigned to activities of a given type may be fixed. If we attempt to solve the problem as an integer linear programming problem, using the military worth function as an objective function, we can be sure that the computed solution will be optimal in the following sense: there will be no other feasible assignment plans which are preferred to the solution which maximizes military worth. There of course may be other feasible plans which are incomparable with the plan picked by the programming technique, and the choice of the plan which maximizes military worth as against these other (incomparable) plans is, in a sense, purely arbitrary. Nevertheless, if the nature of the real-life situation is such that only a rank-order on the a_{ij} can meaningfully be obtained, then this "arbitrariness" is really only an indeterminacy inherent in the nature of the situation itself and not a fault of the technique.

Let us call a feasible over-all assignment plan "optimal" if there are no other feasible plans which are preferred to it. We can then say the following about the optimal plans: 1. There may be more than one optimal plan. 2. The optimal plans are incomparable on the basis of Rule 2. 3. The plan which maximizes military worth will be among the optimal plans. 4. An optimal plan may be incomparable with some of the non-optimal feasible plans. 5. There may exist an optimal plan x, different from the plan which maximizes military worth, and a non-optimal plan y such that x is preferred to y but y is incomparable with the plan which maximizes military worth.[1]

The proof of 2 is trivial and it has already been shown that no plan can be preferred to another plan which has a higher military worth value. Hence 3 is proved. All the propositions are illustrated by the following example, which, incidentally, also proves 1, 4, and 5.

Suppose we have the following preference order on a set of four a_{ij}:

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1. The reader familiar with welfare economics will immediately notice the strong similarity between the present situation and that described by the familiar Edgeworth box diagram. This diagram, it will be recalled illustrates all possible divisions of fixed totals of two goods between two different persons. There the Pareto criterion, like our Rule 2, creates a partial ordering of the alternatives. In our case the optimal assignment plans form "points" equivalent to the points on the contract curve. Our non-optimal plans are equivalent to points off the contract curve. The relation between optimal and non-optimal plans and between optimal plans is exactly the same as the relationship between points on and points off the contract curve. For a description of the box diagram analysis see Tibor Scitovsky, Welfare and Competition (Chicago: Richard D. Irwin, Inc., 1951), pp. 51-55.

$$a_{11} \text{ P } a_{12} \text{ P } a_{22} \text{ P } a_{21}.$$

Now let us arbitrarily assign the following order-preserving utilities to the assignment elements:

$$u(a_{11}) = 4 \qquad u(a_{22}) = 2$$
$$u(a_{12}) = 3 \qquad u(a_{21}) = 0.$$

We can then set up an objective function for the value of the over-all assignment plans:

$$MW = 4(x_{11}) + 3(x_{12}) + 2(x_{22}) + 0(x_{21}).$$

Suppose we seek to maximize the military worth function subject to the following constraints:

$$x_{11} + x_{12} \leq 3$$
$$x_{22} + x_{21} \leq 3$$
$$x_{11} + x_{22} \leq 4$$
$$x_{11} + x_{12} + x_{22} + x_{21} = 6.$$

The values of the variables must of course be non-negative integrals.

The maximum feasible value of military worth turns out to be 16 and the over-all assignment plan which maximizes military worth is the following:

$$A = (1)a_{11} + (2)a_{12} + (3)a_{22} + (0)a_{21}.$$
$$MW(A) = 16.$$

This plan is optimal in the sense that by the test of Rule 2 no other feasible plan is preferred to it. But the following plan is also optimal.[1]

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1. This can be shown by a tedious process of inspection.

$$B = (2)a_{11} + (1)a_{12} + (2)a_{22} + (1)a_{21}.$$
$$MW(B) = 15.$$

The reader may verify for himself that A and B are incomparable in terms of Rule 2. Further, plan B is preferred to the following non-optimal plan:

$$C = (2)a_{11} + (1)a_{12} + (1)a_{22} + (2)a_{21}.$$
$$MW(C) = 13.$$

Finally, the reader may also verify that the non-optimal plan C and the optimal plan A, which maximizes military worth, are themselves incomparable.[1]

7. Cases Where Priority and Efficiency Aspects are Inseparable

We now turn to situations in which Assumption 1[2] cannot be meaningfully applied to the problems in question, i.e., problems where the military significance of the activities is not sufficiently well-defined to be able to rank them without respect to installed equipment.

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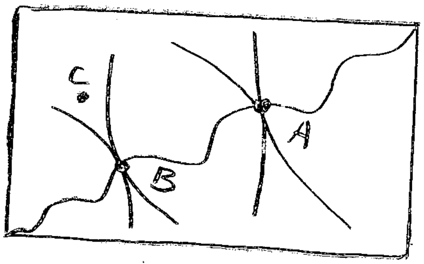
1. The relationship between plans A, B, and C may be illustrated in terms of the box diagram as follows. The wiggly line is the contract locus while the convex lines are indifference curves. See the note on p. 34 above.



2. See above, p. 17.

Since Assumptions 2 and 3[1] deal only with the possibility of evaluating the technical effectiveness of different models in a given activity, they are equally relevant in the type of problem now under discussion. We can state two new assumptions designed to replace Assumption 1 and to capture the spirit of the idea that our different activities can only be meaningfully compared if some assumption is made as to what equipment is installed in them. For the moment we shall be content merely to explore their formal implications. Assumption 4: *For any given m_i, there exists an ordering on the a_{ij}.* This proposition merely states that it would be possible for some competent person to order the activities if it were assumed that the same type of equipment is installed in all of them. Assumption 5: *For any given m_i there exists an ordering on the a_{ij}, and the order is the same for all m_i.* This assumption merely adds to 4 the idea that the order of the activities with given model installed is the same regardless of what model is considered.

8. Obtaining an Order on the a_{ij}.

To examine what basis may be found for ordering the a_{ij} in the "inseparable" cases under discussion let us first for simplicity take the case of a problem where Assumptions 3 and 5 apply. Example 2 given above[2] is intended to be intuitively characterizable by these two propositions. Suppose we again arrange the a_{ij} into a matrix in the following way.

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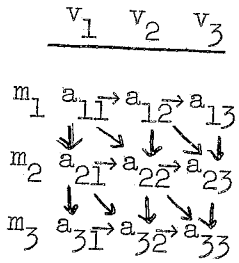
1. See above, p. 18.

2. See above, p. 16.



Now because of Assumption 3, the order within any given column must be the same for all columns. By appropriate labeling of the models, therefore, the matrix can be set up so that any movement vertically down would represent an improvement. On the other hand, the assumption of Proposition 5 implies directly that we have an order within any given row and that this order is the same for all rows. Again, by suitable labeling of the v_j the matrix can be set up so that any movement horizontally to the right is an improvement.

The following information on the ranking of the a_{ij} is thus available. 1. The order within any given column is determined by the technical information on the ranking of the models for given activities. The situation here is the same as in Rule 1a[1]. 2. The ranking of the a_{ij} within any given row is obtained directly, by assumption. 3. The ranking along the northwest-southeast diagonals of the matrix of a_{ij} is obtained by the assumption of transitivity. Thus the information in the present case is exactly the same as in the separable case where all three parts of Rule 1 are applicable.[2]

Where Assumptions 3 and 5 apply, the number of incomparable pairs is determined once the number of a_{ij} is known, being the number of elements related by a southwest-northeast line in the matrix. Thus there

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1. See above, p. 22.

2. Compare the matrix above with the matrix on p.24.

are 7 incomparable pairs in the three-by-three case. When only Assumptions 2 and 4 apply, however, almost anything is possible, depending upon the exact structure of the preferences in the particular problem. In some cases the incoming preference information may be connected by the transitivity assumption to form a complete order on the elements, as in the following example:

<u>Information Under Assumption 4</u>	<u>Information Under Assumption 2</u>
a_{11} P a_{12} P a_{13}	a_{13} P a_{23} P a_{33}
a_{23} P a_{21} P a_{22}	a_{11} P a_{31} P a_{21}
a_{33} P a_{32} P a_{31}	a_{12} P a_{32} P a_{22}.

By assuming transitivity the above information can be combined to give the following complete order on the assignment elements:

$$a_{11} \text{ P } a_{12} \text{ P } a_{13} \text{ P } a_{23} \text{ P } a_{33} \text{ P } a_{32} \text{ P } a_{31} \text{ P } a_{21} \text{ P } a_{22}.$$

What is somewhat surprising, however, is that in other cases involving Assumptions 2 and 4 the application of the transitivity assumption to the elementary preference information may not yield consistent results, i.e., we get both x P y and y P x. This may occur even though the incoming information from the technical experts and the incoming information from the "strategic" experts, taken separately, is entirely free from intransitivity. Thus suppose under Assumption 2 we get the following technical information on the relative efficiencies of the models in the three activities:

1. a_{11} P a_{31} P a_{21}
2. a_{22} P a_{32} P a_{12}
3. a_{13} P a_{23} P a_{33}.

At the same time let the incoming information under Assumption 4 from the "strategic" authority be:

4. $a_{11} \; P \; a_{12} \; P \; a_{13}$
5. $a_{23} \; P \; a_{21} \; P \; a_{22}$
6. $a_{33} \; P \; a_{32} \; P \; a_{31}$.

These two sets of preference information, consistent when taken separately, are not consistent when combined. Hence if we use transitivity to combine the information obtained in lines 1, 3, 4, 5, and 6, we get the same complete order obtained in the previous example. On the other hand, the information contained in line 2 is inconsistent with this so that, for example, while combining the other lines gives us $a_{12} \; P \; a_{22}$, line 2 has $a_{22} \; P \; a_{12}$.[1]

Comparing the difficulties of obtaining an order on the assignment elements in the inseparable case with the difficulties in the separable case, we find that although in some cases a complete ordering may be possible there is the new difficulty that inconsistencies not attributable to intransitivities in the original preference information may turn up.

Quite aside from the dangers of inconsistencies, however, there seems to be a good argument to be made for the position that inseparable problems should be handled by having a single authority directly rank the entire set of a_{ij}. In the inseparable case the information coming into the logistics center must in any case contain direct comparisons of a_{ij} involving different activities. By definition,

1. In cases involving Assumptions 2 and 4, rather than 3 and 5 we may also run into cases which (1) involve no intransitivities or inconsistencies, (2) do not completely order the elements, and (3) involve fewer incomparable pairs than would a problem containing the same number of a_{ij} to which Assumptions 3 and 5 applied.

inseparable problems are problems in which only comparisons which take into account the installed equipment are meaningful. Thus Assumptions 4 and 5, at least one of which must apply to any inseparable case on which any progress can be made, both contemplate that some official will be able to answer questions of the following type: "Given model m is installed in both an x activity and a y activity, which contributes more to the military program?" But if this kind of question can be meaningfully answered by the authority, why arbitrarily assume that his knowledge stops at the ability to order pairs that happen to lie within the same row of the matrix of a_{ij}?

In short, it would seem that if we must expect to get informed answers to questions of the type required for the satisfaction of Assumptions 4 and 5, we could just as well hope to get direct expressions of preference between the members of any pair of elements. Thus there is a strong a priori case for setting up solutions to assignment problems involving inseparability so that all preference information comes from the same source. In other words the case for a division of labor is weak.

Once the a_{ij} are ordered, the inseparable case is exactly like the separable situation. Rule 2 can then be applied as well as the military worth function associated with it. Since the heart of the analysis of the distinction between the two situations involves the nature of the incoming information in the form of priority lists and like documents, we shall next turn to a discussion of their possible uses and interpretations.

9. The Interpretation and Use of Priority Lists

While we are not primarily concerned here with the actual uses to which priority lists have been put in solving allocation problems, a brief digression on some discussion in the literature may illuminate a few of the problems and issues.

A first point to be noted is the failure of some priority lists to come to grips with the inseparability problem. As was indicated above, when such a problem exists a meaningful priority list must attach rank labels to the activities with models installed, rather than to the activities *in vacuo*. The practical difficulties of this are fairly obvious. Such a ranking of the a_{ij} requires that the ranker have a combination of detailed technical competence and at the same time, the authority to evaluate the military significance of different elements.

The Navy's Materiel Improvement Plan (MIP) is an instance of a priority list which deals with what looks intuitively like an "inseparable" case as though it were "separable". We cite this example because the only really careful analysis of the use of priority indicators extant concerns itself with the MIP.[1] Its purpose has been described as follows:

> Because of the limitation of procurement funds and changing requirements, the total available number of preferred (electronic) sets and suitable substitutes combined often is not sufficient to outfit all ships. Furthermore, installation funds often are limited. To guide the materiel bureaus in the allocation of limited sets and limited installation funds, the Chief of Naval Operations provides the... MIP. This document lists groups of requirement-ship class combinations in priority order. Within each requirement ship class such as surface search radar might be group priority number one for certain ship classes and group priority number 16 for other ship classes. Thus each requirement-ship class combination has its own unique place in the over-all priority structure indicated by the MIP.[2]

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1. R.J. Aumann and J.B. Kruskal, "Assigning Quantitative Values to Qualitative Factors in the Naval Electronics Problem," *Naval Research Logistics Quarterly*, Vol. VI, No. 1 (March, 1959), pp. 1-16.

2. J.W. Smith, "A Plan to Allocate and Procure Electronic Sets by the Use of Linear Programming Techniques and Analytical Methods of Assigning Values to Qualitative Factors," *ibid.*, Vol. III, No. 3 (September, 1956), p. 152.

Thus the form of the MIP is as follows:

Priority	Requirement	Ship Class
1	Surface Search Radar	CVA DDR CA, etc.
2	VHF Transmitters	DDR DM APD, etc.
.		
.		
.		
16	Surface Search Radar	AGC APA BB, etc.

The kinds of activities enumerated above seem to suggest an inseparable case. Now it is possible that although, to use our own terminology, the above list ranks only v_j, the list is really to be treated as an implicit ranking of the various a_{ij} themselves.[1] This could come about in situations where the order on the a_{ij} depends on the v_j alone. In such cases the ranking of an activity x over an activity y in a priority list such as the MIP may mean that with <u>any</u> model installed, x is to be regarded as of more military value than y, also with <u>any</u> model installed.

We could rephrase this possibility as <u>Assumption 6</u>: <u>The order on the a_{ij} depends only on the priority designation of the v_j, except that if the v_j are the same for some pair of elements, the preferred element</u>

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1. There seems to be some reason to believe that the Navy's own conception of the nature of the problem is insufficiently developed to attribute to them any definite intentions on the use of the MIP as an allocation device. The distribution of the MIP list has evidently been accompanied by a vague injunction that it should somehow "guide" the logistics system in the rationing of equipment without specification of in just what the "guiding" should consist. According to Smith's own statement the actual allocation scheme in use prior to the Naval Basin Project was a combination of "good judgment" and somewhat arbitrary rules which took account of the MIP in various ways under various circumstances. <u>Ibid.</u>, p. 155.

depends upon the order of the m_i. Notice that this rule establishes a lexicographic ordering on the a_{ij}.[1] Notice also that it does not logically depend for its reasonableness on the possibility of evaluating assignments *in vacuo* so far as potentially assigned equipment is concerned.

It has been suggested that, in effect, the MIP and similar priority lists are not to be regarded in either of the two senses so far distinguished--i.e., (1) as an ordering on v_j, or (2) as an ordering on a_{ij}. One view of priority lists is that they implicitly constitute rules-of-thumb for evaluating the different over-all assignment plans themselves. Evidently this conception of the import of the MIP guides McShane and Solomon's suggestions for alternative interpretations of it; for their "alternative interpretations" are, in effect, alternative rules for evaluating the over-all assignment plans. They ask the following questions about the intended significance of the Navy's priority list:

> (a) Is it the intent of the MIP that if it can be done, all priority 1 requirements be fulfilled before priority 2? Or, (b) is it the intent to maximize the total "effectiveness" of all the ships covered by the particular MIP with due consideration given to the specified 'priorities'?[2]

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1. That is, an ordering of entities each of which is ordered in more than one dimension and where the dimensions themselves are ordered, so that the order of the entities depends upon their order in the first dimension, unless it is the same, in which case it depends upon their order in the second dimension, and so on.

2. R. E. McShane and H. Solomon, Letter to the Editor of the *Naval Research Logistics Quarterly*, Vol. V, No. 4 (December, 1958), p. 365.

The meaning of the second (b) clause is further explained as follows:

> ....if priority 1 required two 'models' with a given set of characteristics and priority 16 required two 'models' with the same set of characteristics, an available model might be assigned to fill the first of priority 16 rather than the second of priority 1.[1]

Of these two interpretations it appears that the first is itself somewhat ambiguous. Evidently it could be taken to imply one of two slightly different rules for deciding between alternative assignment plans:

a-1) When there is a fixed number of models available, all the higher priority requirements should be filled before the lower ones are even partially filled.

To illustrate the meaning of this rule, assume for the moment that only one model type is available (though we shall return later to this assumption), and consider the following pair of assignment plans:

$$\text{a-}\quad (3)a_{11} + (0)a_{12}$$
$$\text{b-}\quad (2)a_{11} + (1)a_{12}.$$

Now if v_1 has a higher priority than v_2, the rule a-1 would pick plan (a) over plan (b). Consider now a second possible reading of McShane and Solomon's "(a)" interpretation:

a-2) A plan is to be preferred to another plan if it contains a higher number of assignments of equipment to the higher priority activity regardless of how many assignments to the lower priority activity are involved.

Now suppose we compare plan (a) above with a new plan, (c):

$$\text{a-}\quad (3)a_{11} + (0)a_{12}$$
$$\text{c-}\quad (1)a_{11} + (1)a_{12}$$

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1. _Ibid._, p. 366.

Under this interpretation of the meaning of the MIP, plan a would be ranked above plan c, although rule a-1 would be silent when confronted with this particular pair of plans since it deals with a fixed number of models. McShane and Solomon's (b) interpretation can be illustrated without comment by the following pair of assignment plans:

$$d- \quad (2)a_{11} + (0)a_{12}$$
$$e- \quad (1)a_{11} + (1)a_{12}$$

Under this interpretation of the meaning of the MIP, plan e could be preferred to d, while this is impossible under either reading of their (a) interpretation.

Before evaluating these rules-of-thumb it is necessary to see how they could be extended to the more general case where there are n, rather than one available model type. Since the rules do not specify any assumption about the number or relative technical effectiveness of the potentially available model types, these facts evidently are not to be allowed to influence the result. This could make sense only if the order of the a_{ij} did not depend upon the m_i, except where i is the same for a given pair of elements. This is the case of Assumption 6 where the priorities and efficiencies determine a lexicographic ordering on the a_{ij}. Hence, if Assumption 6 is really implicit (and, what is quite another question, reasonable) the rules a-1, a-2, and b would not have to be reworded to apply to the n-model case. If Assumption 6 is not assumed, the rules are simply incomplete and cannot be used to decide between different assignment plans.

In commenting on these three rules-of-thumb let us deal first with McShane and Solomon's interpretation (b). This interpretation leaves open the possibility that (e) might be preferred to (d), although this directly contradicts Rule 2. Now it should be fairly evident that since a_{11} P a_{12}, plan (e) could be preferred to (d) only if the a_{ij} are defined in such a way as to admit the possibility of diminishing marginal utility in the sense suggested in Chapter I[1]. There it was pointed out that a_{ij} might be defined, for example, such that $5a_{ij}$ means the installation of 5 type i radar sets in 5 different type j ships, or the installation of 5 sets in one ship or some other distribution adding up to five, or, all of these. The point here is simply that unless the a_{ij} are defined in such a way as to rule out the possibility that e could ever be preferred to d, then the original priority list is virtually useless in the evaluation of alternative possible assignment plans. What we are in effect doing, is to insist that the a_{ij} never be defined in such a way that plan (e) could ever possibly be preferred to plan (d). This is not a trick, however, but merely a minimal requirement for a useful priority list.

A description of the operation of the machine tool priority system in 1941 suggests that this system broke down precisely because it failed to eliminate the diminishing marginal utility problem noted above--or at least so the following passage seems to suggest.

> While affording temporary relief, the designation of subdivisions within the A-1 (priority)rating band was no solution to the problem of channeling machine tools to defense suppliers. With ordnance items generally rated below Navy and Air Corps programs, ordnance contractors were placed at a serious disadvantage in their quest for tools. The situation was even worse in the case of contractors...holding still lower priorities. Since demand greatly exceeded supply, these lower rated orders were deferred in favor of the higher priority consumers. Deferments of the same order were often repeated. As a result, low-rated orders were either prevented from obtaining tools at all, or else tools were furnished

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1. See above, p. 10.

> on such an unpredictable schedule as to prevent orderly planning by the manufacturer. Unless corrected the situation threatened to result in an unbalanced defense program.[1]

To correct this situation it was found necessary to change the priority system. Translating into our own terminology, this seems to be a situation where plan (d) was followed rather than plan (e) with unfortunate results because of diminishing marginal utility in the higher priority activities. (I take it that this is the implication of "unbalanced" in the passage cited above.) The point is that if the priority list is to serve as an automatic guide for allocation (so that it need not become a matter of so-called "judgment" as to when diminishing marginal utility sets in), then the "activities" itemized in the list have to be distinguished with great care--and it is true that this problem might cause considerable practical difficulties in some cases.

This may be the place to emphasize again a point first raised on page 27. A priority list, if it is to be of any use in setting up more or less automatic allocation formulae must be defined for activities which display a minimum of complementarity or substitutability. If these possibilities cannot be eliminated in defining the "activities" to which priority labels are assigned, little useful information about the relative value of over-all assignment plans can be gleaned from a knowledge of the "priorities".

Of the remaining interpretations of the meaning of the MIP, it is evident that a-1 is equivalent to our Rule 2 for those cases where a-1 can be applied. Rule a-2 treats all cases treated by a-1 in the same

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1. The Industrial College of the Armed Services, Department of Research, Study of Experience in Industrial Mobilization in World War II: Priorities and Allocations: A Study of the Flow of Materials to War Suppliers (Washington: Industrial College of the Armed Services, 1946), p. 18.

way as a-1 but also ranks two different plans where the number of models assigned in the two plans is different. For example, look again at the pair of plans (a) and (c). It is clear that such pairs of plans would be regarded as incomparable on the basis of our Rule 2.

Is the rule-of-thumb for ranking plans given by a-2 reasonable? I think it must be evident that from the point of view developed in this paper, rule a-2 is quite arbitrary. Two possibilities may be considered. Suppose we have two activities, v_1 P v_2 and two model types, m_1 P m_2 in both types of activities. Suppose this information is intended as the basis of a lexicographic ordering on the a_{ij} of the type envisioned by Assumption 6. Then we have

$$a_{11} \text{ P } a_{21} \text{ P } a_{12} \text{ P } a_{22}.$$

Now consider the following pair of assignment plans:

$$\text{i)} \quad (3)a_{11} + (0)a_{21} + (1)a_{12} + (0)a_{22}$$
$$\text{ii)} \quad (0)a_{11} + (3)a_{21} + (0)a_{12} + (3)a_{22}.$$

Rule 2-a would pick plan i over plan ii. But this clearly expresses an arbitrary preference for having a few first class models installed over having a larger number of comparatively ineffective models installed. Certainly no a priori basis for this prejudice can be found in the ordering on the a_{ij}.

Now consider a second possible situation. In this case we assume that we are confronted with the following pair of alternative plans:

$$\text{iii)} \quad (3)a_{11} + (0)a_{21} + (0)a_{12} + (0)a_{22}$$
$$\text{iv)} \quad (2)a_{11} + (0)a_{21} + (0)a_{12} + (3)a_{22}.$$

Here iii would be picked over iv according to a-2. This conclusion says that it is always better to fill the higher priority activity even if by sacrificing some assignment to such activities one could obtain the assignment of models to several additional low priority activities. Again, as a general rule this seems quite arbitrary. It could of course be true in any particular case, but the information conveyed by the ordering on the a_{ij} does not permit us to say in which cases it would be true and therefore it does not seem admissable as a general rule.[1]

To return to a point made earlier, it may be that the MIP or other priority list is merely to be handed down to the logistics system together with one of these rules-of-thumb and with no further questions expected. In this case the logistics system may simply be directed to (1) treat the MIP as determining, together with the efficiencies, a lexicographic ordering on the a_{ij}, and (2) to rank the plans according to a dictated rule-of-thumb. While such a procedure is of course possible, it hardly seems desirable. In the first place, even such a program (if based on the use of a-1, a-2, or b) would be insufficient to obtain a ranking on all conceivable alternative allocation plans. Secondly, where these rules-of-thumb differ from Rule 2 developed in the earlier part of this chapter, they are likely to lead to absurd results. A more basic point is that such a procedure would exhibit hopeless confusion of viewpoint. In this paper we have taken the line that officials would be expected to rank individual simple alternatives, such as different models

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1. It might be objected that the above example is artificial since it would be possible to redistribute the models to get a plan clearly superior to iv--i.e., $(2)a_{11} + (3)a_{21} + (0)a_{12} + (0)a_{22}$. But it could be that type 2 models cannot be used in type 1 activities. In that case "a_{21}" would not really exist.

or different activities, and that the more complex comparisons must be made in a manner consistent with these comparisons.

10. Conclusion

A brief summary may help to draw together the results of the present chapter. We have attempted to present a logical foundation for the use of strategic and technological "priorities" in ranking assignment plans. The first step in obtaining such a ranking is to obtain a ranking on the a_{ij}. As we have seen it is important to distinguish between "separable" and "inseparable" cases in this aspect of the problem. Completely separate priority and efficiency information cannot be combined to obtain a complete order on the a_{ij} without the imposition of additional judgments in the form of new preference information not available from the original sources. In the inseparable case we have seen that the incoming information may lead to a complete order on the a_{ij} or may be completely inconsistent. We have argued that in the separable case the a_{ij} should be ranked directly by a single authority.

Turning to the ranking of the over-all assignment plans, we have developed Rule 2 which partially orders the over-all plans. The rule leads to the identification of optimal plans and to an ordinal military worth indicator which can facilitate the selection of an optimal plan from among feasible alternatives through programming techniques.

Finally, we have argued that since standard priority lists in use seem to be an attempt to facilitate the solution of "subjective" military worth problems of the type under study here, they should be designed and interpreted in conformity with the logical basis here uncovered.

CHAPTER III

EXTRA-ORDINAL MILITARY WORTH FUNCTIONS

1. Introduction

In earlier chapters we have restricted ourselves to discussion of situations in which the incoming information is in the form of simple orderings. We shall now turn to more complex cases, more complex with respect to the structure of the elementary preference information and consequently more fruitful with respect to the solution of allocation problems.

In particular we shall assume that the incoming information contains, in addition to the usual ordering information discussed earlier, information on what we shall call "difference orders". We shall make use of the difference order concept in part because of its naturalness. That is, we expect to find many actual situations in which the preference information from which a difference order is constructed can, in fact, be made available by suitable questioning of relevant military personnel. A second reason for detailed consideration of the difference order concept is its use in a procedure for obtaining numerical military worth values developed by Aumann and Kruskal.[1] Since this is virtually the only work in the field that can lay any real claim to sophistication it will be the subject of intensive analysis.

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1. R. J. Aumann and J. B. Kruskal, "The Coefficients in an Allocation Problem," Naval Research Logistics Quarterly, Vol. V, No. 2 (June, 1958), pp. 111-123.

The program in this chapter is accordingly as follows. We shall first define and explore the use of the difference order concept in the context of our previous discussion. Next we shall detail the use of this concept by Aumann and Kruskal and explore some methodological questions raised by their work. Finally, we shall take another look at the multi-person amalgamation problem discussed earlier.

2. The Concept of a "Difference Order"

Consider the following type of question placed to an appropriate military official. "Would you rather replace a w with an x, or replace a y with a z"? Although we shall return to the question of interpretation of "w", "x", "y", and "z" in a moment, let us regard this question as meaning, for example, "would you rather replace an m_1 in a v_2 position with an m_2, or replace an m_3 in a v_2 position with an m_3?" Suppose the answer to the question at the beginning of the paragraph is, "I would rather replace w with x than replace y with z", let us express this by means of the following notation:

$$(x \underline{*} w) \text{ P } (z \underline{*} y).$$

We shall call $(x \underline{*} w)$, i.e., "replace w with x", a "difference" and the preference relation between such differences, a "difference order." In what follows we shall assume that the preference relation has the usual property of transitivity. Our terminology has been suggested by the loose similarity to the idea of utility differences.

To see the significance of difference orders in the ranking of assignment plans let us assume that in addition to the above preference information we also know the preference order on the four elements to be as follows:

$$x \text{ P } z \text{ P } w \text{ P } y.$$

Now let us first consider an assignment plan that contains the two inferior elements, i.e.,

$$A \quad (1)w + (1)y + (0)x + (0)z.$$

Now if we "replace w with x" we get

$$A' \quad (0)w + (1)y + (1)x + (0)z.$$

On the other hand, we get the following when we replace y in the original plan with z:

$$A'' \quad (1)w + (0)y + (0)x + (1)z.$$

It is evident that on this interpretation "would rather replace w with x than replace y with z" means that assignment plan A' is preferred to plan A''. Notice also that on this interpretation, information on difference orders conveys knowledge about orderings on assignment plans which would not be obtainable from mere ordering information. For example under Rule 2 developed in Chapter II, A' and A'' would be incomparable given the preference order on w, x, y, and z alone.

It is also important to notice that on the interpretation given here $(x \,\underline{*}\, w)$ P $(z \,\underline{*}\, y)$ is equivalent to $(x + y)$ P $(z + w)$, that is, the one implies the other.[1] The significance of this is that when we obtain answers to the kinds of questions that give us a difference order we obtain a preference relationship between pairs of elements. In what follows we shall assume that for any collection of assignment elements, in the sense defined in Chapter II we always have a complete,

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1. The "+" is of course to be interpreted as "union". A similar situation is discussed by E. W. Adams and R. F. Fagot in "A Model of Riskless Choice," Applied Mathematics and Statistics Laboratory, Stanford University, Technical Report No. 4, August, 1956.

transitive order on the elements and also that we have a complete, transitive order on all possible pairs of elements. We shall not inquire further as to how the two kinds of orders were obtained as there are no new difficulties beyond those discussed earlier in connection with simple orders.

3. Use of Difference Orders to Compare Complex Assignment Plans

Since the basic problem which concerns us is still the ranking of complex over-all assignment plans on the basis of the ranking of simple alternatives, we must define an additive rule which enables us to use the new information to make comparisons between the complex alternatives. As in the case of the additive rule discussed on pages 24-32 of Chapter II, we shall again assume "independence of utilities" and "constancy of marginal utility", i.e.,

(1) If $(w + x)$ P $(y + z)$ and K is any allocation plan, then $(w + x + K)$ P $(y + z + K)$, and also

(2) If $(w + x)$ P $(y + z)$, then $m(w + x)$ P $m(y + z)$ for any positive m.

Now let us state Rule 3 for the comparison of the military worth of two over-all assignment plans:

Rule 3: An assignment plan A is preferred to an assignment plan B if the elements in A and the elements in B can be partitioned off in such a way that each partition in A has corresponding to it a partition in B to which it is preferred or indifferent, not all indifferent.

Notice that Rule 2, presented in Chapter II, is a special form of Rule 3 where the partitioning is done in such a way that each partition contains only one element. Note also that Rule 3 implies the two assertions noted above. Rule 3 in conjunction with information on the order of differences will decide many cases where Rule 2 would be silent. It will not solve all, however, This is perhaps not obvious in view of the fact that one has a rather large choice of partitionings of the elements in a plan since by hypothesis we have information on the relative size of every conceivable pair of elements. The following counter-example shows that not all pairs of over-all plans can be ordered as to preference by Rule 3.

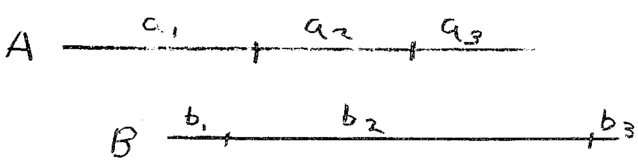
Consider the following collection of six assignment elements (a_{ij}), a_1, a_2, a_3, b_1, b_2, and b_3. Suppose we consider two over-all assignment plans composed of these six elements as follows:

$$A = a_1 + a_2 + a_3$$
$$B = b_1 + b_2 + b_3.$$

Let us assume that the preference order for the six elements is

$$b_2 \; P \; a_1 \; P \; a_2 \; P \; a_3 \; P \; b_1 \; P \; b_3.$$

Assignment plan A will be found incomparable to plan B by Rule 3 for any ordering on the two-element combinations of six elements which is consistent with the following pair-wise comparisons: (1) The combination $(b_1 + b_3)$ is inferior to any pair of elements in A. (2) The combination $(b_2 + b_1)$ is preferred to any pair of elements in A. (3) The combination $(b_2 + b_3)$ is preferred to any pair of elements in A. The following pictorization in which the relative length of line segments is consistent with the above preference information will make the situation clear.



In this case there exists no way of partitioning A and B so that Rule 3 applies either to show A preferred to B, B preferred to A, or A and B indifferent. The two plans are thus incomparable on this basis.

4. Other Uses of Difference Orders

In Chapters I and II the problem of obtaining rankings of over-all allocation plans was divided into several stages. Thus in the "separable" case we begin by attempting to obtain an order on the "models" and the "activities" separately. These orders are then to be combined into an order on the a_{ij}, this order in turn being combined with an additive rule to obtain orders on combinations of a_{ij}. In the discussion of difference orders above, the concept has been interpreted solely in terms of the a_{ij}. We shall briefly explore the natural thought that the concept might be useful at other stages of the problem.

Let us first take up the possibility of using difference orders in connection with the m_i and v_j in the separable case. To recall briefly the discussion developed there, it will be remembered that the pragmatic significance of an ordering on the v_j lies in its potential use in obtaining an ordering of the a_{ij}. Three rules relating orders on m_i and v_j to orders on the a_{ij}, were noted there.[1] (1) a_{ij} P a_{rs} if $j = s$, if m_i P m_r and if all the individual items that are indifferent with respect to priority (i.e., within a given v_j designation) are also technologically identical. (2) a_{ij} P a_{rs} if $i = r$, if v_j P v_s and if all activities, regardless of priority class are technologically identical (e.g., two subs, identical from a technological point of view might be located in two operations theaters of different importance and therefore belong to

1. See above, p.22.

different priority indifference classes.) (3) a_{ij} P a_{rs} if m_i P m_r, if v_j P v_s, and if all activities, regardless of priority class are technologically identical. It can thus be seen that if in fact all activities, regardless of priority class are not technologically identical, not much can be said.

Suppose that this technological assumption is valid for some problem. In this case what would be the possible significance of a difference order defined over the m_i and v_j? For notation we might say that for any two pairs of v_j, (a + b) and (c + d), either (a + b) P (c + d) or (c + d) P (a + b). The same sort of relationships could be asserted to exist between the various m_i. If any sense could be made of such difference orders, one might hope to use them either (1) for obtaining more information on the order on the a_{ij}, or (2) for obtaining some information on the difference order for the a_{ij}. It is difficult to conceive of any meaningful interpretation for these difference orders, however, and the concept of difference orders evidently makes sense only as applied directly to the a_{ij} themselves. In other words even in cases where priority and efficiency aspects are separable there is no meaningful way of asking questions to obtain orders on the differences except with direct reference to the a_{ij}.

It would be possible to use difference orders in connection with lexicographic orderings of the type mentioned on page 44. Suppose we assume, as was done there that we have an order on the v_j which in connection with the ordering on the m_i gives a lexicographic ordering on the a_{ij}. Now let us assume in addition that we obtain an ordering on the differences of the activities of the usual form, i.e., for example, (v_a + v_b) P (v_c + v_d). Let this have the following interpretation: $(a_{pa} + a_{qb})$ P $(a_{rc} + a_{sd})$ for any p, q, r, and s. This means that the orders on the differences of the a_{ij} depend only upon the activities and not at all on installed models. To make a complete rule for ordering the a_{ij} we must also consider the case where the activities are indifferent, as for example, the pair $(a_{pj} + a_{qj})$ and $(a_{rj} + a_{sj})$. It will be possible to make such a comparison for all items in the j priority class only if they are technologically identical. If this is the case we have a complete rule.

5. The Aumann and Kruskal Method of Ranking Over-all Assignment Plans

Aumann and Kruskal, in what is probably the most exciting paper yet produced in the military worth field, have attempted to develop a numerical utility for the solution of allocation problems which can, with inessential simplifications be reduced to the sort of thing we have been considering.[1] The problem, thus reduced, and its solution as proposed by these authors can be described as follows. There are several types of activities and several types of equipment. Each model may be used in any activity. The technological requirements of all instances of a given type of activity are identical. It is desired to find a way of ordering different assignments of available models to activities.

Their method of solution also proceeds in general along the lines considered by this paper. A "board" of competent military officials is asked questions concerning their preferences among simple alternatives. If their preferences display a certain kind of consistency, they can be represented by numerical values attached to the a_{ij} by a method to be described shortly. The military worth of an over-all assignment plan (they do not use this language or notation) is given

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1. Aumann and Kruskal, Naval Research Logistics Quarterly, Vol.V, pp.111-123.

numerically by the sum of the values of its component a_{ij} and a plan is assumed to be of higher worth if the resulting numerical value associated with it is higher than that associated with some other plan.

It is the present author's opinion that the Aumann and Kruskal result is of considerable interest and importance. The best way to explain the method is to give an example of its use, and accordingly we have constructed the following numerical example from the author's description of their method.[1] In what follows the notation has been changed to conform with that of the previous chapters. Certain other minor changes and translations have been made. It is hoped that these alterations will result in a considerable improvement in clarity over the discussion in the original paper.

6. A Numerical Example

In the following problem assume that there are two types of activities, v_1 and v_2 and four types of models, m_1, m_2, m_3, and m_4. All activities of a given type are identical in their technological requirements. It is desired to obtain numerical values for each a_{ij} so that a military worth function

$$MW = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \cdot u(a_{ij})$$

can be formed to evaluate the worths of different over-all assignment plans. The numerical values $u(a_{ij})$ are to be obtained by operating with preferences among the a_{ij} expressed by a competent military authority.

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1. *Ibid.*, pp. 117-120.

Step 1: Consider any activity indifference class, v_j. Obtain by questioning a "goodness order", i.e., the order on the m_i for the given activity. Suppose in this case we get the following:

For v_1: a_{11} P a_{21} P a_{41} P a_{31}.

For v_2: a_{32} P a_{22} P a_{12} P a_{42}.

Step 2: Now let us assume the existence of a difference order for the assignment of different models within a given type of activity, i.e., for fixed j. Let us also consider among the a_{ij} the assignment of no equipment to a given activity, so that we also have an a_{01} and an a_{02} to consider. Now ask questions in order to obtain the two difference orders, one for v_1 and one for v_2. Suppose we get the following. (Note that only "positive" differences are considered, for example, in $(a_{pj} \overset{*}{-} a_{qj})$, a_{pj} is always preferred to a_{qj}.)

For v_1: $(a_{11} \overset{*}{-} a_{01})$ P $(a_{21} \overset{*}{-} a_{01})$ P $(a_{41} \overset{*}{-} a_{01})$ P $(a_{11} \overset{*}{-} a_{31})$ P $(a_{21} \overset{*}{-} a_{31})$ P $(a_{31} \overset{*}{-} a_{01})$ P $(a_{11} \overset{*}{-} a_{41})$ P $(a_{21} \overset{*}{-} a_{41})$ P $(a_{41} \overset{*}{-} a_{31})$ P $(a_{11} \overset{*}{-} a_{21})$

For v_2: $(a_{32} \overset{*}{-} a_{02})$ P $(a_{32} \overset{*}{-} a_{42})$ P $(a_{22} \overset{*}{-} a_{02})$ P $(a_{32} \overset{*}{-} a_{12})$ P $(a_{22} \overset{*}{-} a_{42})$ P $(a_{22} \overset{*}{-} a_{12})$ P $(a_{12} \overset{*}{-} a_{02})$ P $(a_{32} \overset{*}{-} a_{22})$ P $(a_{12} \overset{*}{-} a_{42})$ P $(a_{42} \overset{*}{-} a_{02})$

Step 3: Assign the number zero to a void, i.e., to an a_{0j}[1].

Step 4: Assign numbers $r(a_{ij})$ to the other a_{ij} such that

a-if a_{rj} P a_{sj}, then $r(a_{rj}) > r(a_{sj})$

b-if $(a_{rj} \overset{*}{-} a_{sj})$ P $(a_{pj} \overset{*}{-} a_{qj})$, then $\left[r(a_{rj})-r(a_{sj})\right] > \left[r(a_{pj})-r(a_{qj})\right]$.

Aumann and Kruskal assert, and promise to prove in a later paper, that if for any given j the set of a_{ij} is infinitely large, the utility numbers consistent with requirements a and b above are unique up to a positive linear transformation, and, in this case, since we have assigned zero to a_{0j}, they are unique up to a factor of proportionality.[1] Since the number of a_{ij} for both j = 1 and j = 2 is finite, the numbers consistent with the requirements are not strictly a unique set of numbers, but a collection of "bands", so to speak, unique up to a linear transformation. In any event, the following numbers, for example will be found to meet the requirements for the given preferences in our sample problem.

For v_1	For v_2
$r(a_{11}) = 100$	$r(a_{12}) = 21$
$r(a_{21}) = 97$	$r(a_{22}) = 85$
$r(a_{31}) = 42$	$r(a_{32}) = 100$
$r(a_{41}) = 63$	$r(a_{42}) = 10$
$r(a_{01}) = 0$	$r(a_{02}) = 0$

Notice that for convenience we have arbitrarily set the value of the top item in each column equal to 100. The top values need not be the same and in any case their choice does not alter the final result.

Step 5: The numbers obtained above are called "relative goodness values" by Aumann and Kruskal--so-called because the value for any a_{ij} should be relative to the priority of importance of the activity, a factor yet to be taken into account. Let us call the priority value

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1. Ibid., p. 118.

of an activity p (v_j) and then define the utility of an assignment element, $u(a_{ij})$ as

$$u(a_{ij}) = r(a_{ij})\cdot(p(v_j))$$

so that

$$p(v_j) = u(a_{ij})/r(a_{ij}).$$

Now to obtain the values of the $p(v_j)$ proceed as follows. Consider the two priority groups, v_1 and v_2 with, for example, m_2 installed in v_1 and m_4 installed in v_2. Suppose one model of m_1 type is available. Question the relevant military authorities to determine in which of the two activities they would prefer to install it. Thus before installation we would have

$$(1)a_{21} + (0)a_{11} + (1)a_{42} + (0)a_{12}.$$

The official is then, in effect, asked to choose between the following;

$$A = (0)a_{21} + (1)a_{11} + (1)a_{42} + (0)a_{12}$$
$$B = (1)a_{21} + (0)a_{11} + (0)a_{42} + (1)a_{12}.$$

Since the military worth of an assignment plan is defined as $\sum x_{ij}\cdot u(a_{ij})$ the values of the two plans will be given as follows:

$$MW(A) = p(v_1)\cdot r(a_{11}) + p(v_2)\cdot r(a_{42})$$
$$MW(B) = p(v_1)\cdot r(a_{21}) + p(v_2)\cdot r(a_{12}).$$

Now if in fact B is chosen over A (i.e., B P A), then $MW(B) > MW(A)$. This means that

$$p(v_1)\cdot r(a_{21}) + p(v_2)\cdot r(a_{12}) > p(v_1)\cdot r(a_{11}) + p(v_2)\cdot r(a_{42})$$

or

$$p(v_1)\left[r(a_{11}) - r(a_{21})\right] < p(v_2)\left[r(a_{12}) - r(a_{42})\right]$$

or

$$p(v_1)/p(v_2) < \frac{r(a_{12}) - r(a_{42})}{r(a_{11}) - r(a_{21})}.$$

Let us now consider all possible pairs of plans of the A,B sort, that is, where the relevant $r(a_{ij})$ are such that both numerator and denominator of the fraction of the right hand side of the above inequality are positive. Table 1 below gives all such pairs and the associated values for the right-hand fraction. By asking preference questions for each (A,B) pair we can determine between what values the ratio $p(v_1)/p(v_2)$ lies. If one of the $p(v_j)$ is arbitrarily assigned some value, then the value of each $u(a_{ij})$ is almost uniquely determined up to a factor of proportionality. The word "almost" refers to the fact that the $r(a_{ij})$ are not unique due to the fact that the sets of a_{ij} are finite and also the fact that the ratios of priority values can be determined only within limits. Note that the resulting values for $u(a_{ij})$ are not changed by any proportional transformation of any one or all of the $r(a_{ij})$ scales.

TABLE 1

Pairs of Assignment Plans	Related Values of $\frac{r(a_{p2})-r(a_{q2})}{r(a_{r1})-r(a_{s1})}$
1. $A = a_{11} + a_{02}$ $B = a_{21} + a_{12}$	7.00
2. $A = a_{11} + a_{42}$ $B = a_{21} + a_{12}$	3.67
3. $A = a_{21} + a_{42}$ $B = a_{01} + a_{22}$	2.50

Pairs of Assignment Plans	Related Values of $\frac{r(a_{p2})-r(a_{q2})}{r(a_{r1})-r(a_{s1})}$
4. $A = a_{31} + a_{02}$ $B = a_{01} + a_{32}$	2.38
5. $A = a_{21} + a_{42}$ $B = a_{41} + a_{22}$	2.21
6. $A = a_{31} + a_{42}$ $B = a_{01} + a_{32}$	2.14
7. $A = a_{21} + a_{12}$ $B = a_{41} + a_{22}$	1.88
8. $A = a_{31} + a_{12}$ $B = a_{01} + a_{32}$	1.74
9. $A = a_{11} + a_{42}$ $B = a_{41} + a_{12}$	1.57
10. $A = a_{21} + a_{02}$ $B = a_{31} + a_{22}$	1.55
11. $A = a_{21} + a_{42}$ $B = a_{31} + a_{22}$	1.36
12. $A = a_{21} + a_{12}$ $B = a_{31} + a_{22}$	1.16
13. $A = a_{21} + a_{02}$ $B = a_{01} + a_{22}$	.88
14. $A = a_{21} + a_{42}$ $B = a_{01} + a_{22}$	.77
15. $A = a_{21} + a_{22}$ $B = a_{01} + a_{22}$	.66

Pairs of Assignment Plans	Related Values of $\frac{r(a_{p2})-r(a_{q2})}{r(a_{r1})-r(a_{s1})}$
16. $A = a_{11} + a_{02}$ $B = a_{41} + a_{12}$	.57
17. $A = a_{41} + a_{02}$ $B = a_{31} + a_{42}$	.48
18. $A = a_{11} + a_{02}$ $B = a_{31} + a_{12}$	.36
19. $A = a_{31} + a_{22}$ $B = a_{01} + a_{32}$	.36
20. $A = a_{11} + a_{02}$ $B = a_{01} + a_{12}$	.21
21. $A = a_{11} + a_{42}$ $B = a_{31} + a_{12}$	.19
22. $A = a_{41} + a_{02}$ $B = a_{01} + a_{42}$	.16
23. $A = a_{11} + a_{42}$ $B = a_{01} + a_{12}$	.11

By asking questions to determine whether A or B in a given pair is preferred, we can determine whether $p(v_1)/p(v_2)$ is greater or less than the expression on the right hand side. Thus for example if in pair No. 12, B P A, then the ratio $p(v_1)/p(v_2) < 1.16$. If in pair No. 13, A P B, then we know the value of the ratio must lie between 1.16 and .88. The reader may notice that this method implies a rather definite pattern of answers to the preference questions put to the military authority. We shall comment in detail on this matter in a moment.[1]

7. The "Methodology" of the Aumann and Kruskal Technique

It is very difficult at first sight to evaluate the meaning and significance of the utility numbers for military worth that result from the above construction. That numbers will in fact result, there can be no doubt, but whether they will correctly measure a reasonable intuitive notion of military worth is a much more difficult question to decide. In attempting to answer this question it is instructive to turn first to the authors' own attitude to their technique. The authors view their problem as one of finding a method of combining judgments about small-scale allocation problems to yield evaluations of enormous allocation problems which would be too complex for military judgment to weigh meaningfully. This is the very problem with which we have been concerned.

> We say we are trying to optimize military effectiveness, but this is obviously a subjective idea rather than an objective, physical, measurable utility. One can always count his dollars to see how much profit he's made; but there's no way of measuring how much "military effectiveness" is contributed by some radio transmitter...
>
> The navy handles this problem in the following way: it appoints someone familiar with navy objectives and experienced with the equipment and the ships involved, it helps him out with a few directives and it tells him to make up an allocation plan. Although a quantitative statement of the objective function and the restraints is out of the question, the responsible person can qualitatively weigh all the factors and come up with a "reasonable" or "acceptable" plan...

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1. It should be noted that in each case the "B" member of the pair has a preferred model installed in v_2 while the "A" member is accorded better treatment with respect to equipment installed in v_1. This consistency of labeling is required so that in each case answers to questions of preference restrict the value of $p(v_1)/p(v_2)$ and never its reciprocal.

> This process works very well when the choice the responsible person must make is a clear-cut one with more or less clear-cut implications. Thus if the allocation problems under discussion involved but a few sets and ships, there would be no point in trying to improve current methods. Actually, these allocation problems involve hundreds of electronic sets and ships; and at this point, the responsible person is no longer able to exercise his judgment soundly, because he is overwhelmed by the combinatorial difficulties inherent in problems of this size.
>
> The job of solving large allocation problems really is two jobs; the combinatorial job and the naval judgment job. Under current methods these two are hopelessly mixed together, with the result that probably neither one is being done as well as it could be...[1]

This, then, is the authors' conception of the problem they are trying to solve. Their attitude towards their solution is a modest one.

> Subjective problems of the kind considered here can have no unique correct solutions. The technique outlined in this paper is just *a* way to go about finding acceptable plans, by no means *the* way.[2] (Italics in original.)

The position outlined by the authors raises some questions, however, which become rather more insistent as one delves into the details of their technique. If problems of the type under discussion "can have no unique correct solutions", why is the particular solution arrived at by their method more desirable than some other method of deriving a military worth or utility function? By what criterion are we to judge the reasonableness of the six assumptions required by their technique and how is the result to be judged? Exactly what properties of a utility function are to be desired? It is, in short, very difficult to evaluate the details of the method without a fairly complete methodology.

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1. *Ibid*., pp. 112-113.

2. *Ibid*., p. 122.

In the construction of useful and meaningful utility functions by the method of observing preferences, the work of von Neumann and Morgenstern in defining a utility function for game theory can be taken as a model.[1] In fact certain general aspects of the game theory utility discussion are directly relevant to an analysis of the present issues. Von Neumann and Morgenstern show that if five axioms can be taken to characterize "rational" choice in the presence of risky alternatives, then a utility index can be found for pure alternatives which permits the calculation of utility values for probability combinations of the pure alternatives that have the following property: if the calculated utility value of one "lottery ticket" exceeds that of another, a "rational" person (in the sense of the axioms) would be known to pick the first ticket if the choice were in fact presented to him.

Thus the function of the game theory utility index is to permit "prediction" of choices by the "rational" person even though the person is not directly questioned about these alternatives, but about others.[2] Likewise in the Aumann and Kruskal problem the military worth function "predicts" the military choice between complex over-all assignment plans although the officials are directly questioned only about elementary ones.

Now the term "prediction" may be understood in two different senses in the context of the von Neumann-Morgenstern utility problem. One may say that people are in fact "rational" in the sense of their axioms and that therefore the "predictions" would be confirmed by actual behavior under ideal conditions. Another treatment is to

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1. J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (2d ed.; Princeton: Princeton University Press, 1947), pp.15-31.

2. Game theory and the utility theory associated with it are of course logically independent of each other.

suppose that the axioms characterize desirable decision-making and that therefore the "predictions" which follow from the axioms should be treated as "good advice" to the decision-maker whether or not he would violate their implications in the absence of explicit direction.

Although Aumann and Kruskal do not use the word "prediction" it is legitimate to ask in what sense their military worth function can be said to "predict" choices between complex over-all allocation plans. The situation their problem presents is in some respects rather paradoxical. On the one hand the authors point out that in the absence of "objective" criteria, a choice between allocation plans should rest on naval judgment. The difficulty is that while naval judgment can be expected to rank satisfactorily some plans (say, consisting of only two activities and two different model types), it boggles at the complexity of real-life problems. Thus while naval choice is the only criterion, this criterion is useless for the complexities of real-life problems--or if not useless, at least it is not to be trusted. Hence it would make no sense to say that their military worth function is designed to "predict" what a naval decision between complex alternatives would in fact be. It is obvious that "prediction" in the second sense noted above must be involved. That is, given knowledge of choices among pairs of simple assignment plans, the resulting military worth function "predicts" what ought to be the rational choice among the more complex alternatives.

This being the case we would like to know explicitly what definition of "rationality" is involved. In von Neumann and Morgenstern "rationality" is completely and explicitly defined by the axioms asserting the nature of choice among alternatives of the kind they are considering. No such definition is available here. What is required is a definition of "rationality" such that the indexed military worth of one assignment plan is greater than the indexed military worth of another if and only if the first would be preferred to the second by a person using rational "combinatorial" rules. The difficulty with Aumann and Kruskal's work is that while it turns out to have a definition of "rational combinatorial rules" implicit in it, it is difficult to see what they are and therefore difficult to judge the relevance of the resulting function. It is almost as though the von Neumann-Morgenstern utility theory consisted of nothing but the definition of the utility of a lottery ticket and instructions for finding the numerical value of this utility. Even then, of course, if experience showed that people behaved as predicted by the resulting utility numbers, one would have something of value. In the present case, however, it is not even proposed that the "predictions" of the military worth function be tested by experience. Rather we are to take them as normative. In such a case it seems most essential to know what axioms about behavior would be consistent with the predictions of the military worth function.

Our task in the next few sections will therefore be to indicate the choice axioms implicit in the Aumann and Kruskal paper. The first need is to set forth the "assumptions" stated by Aumann and Kruskal themselves in deriving their method for computing a military worth index.

8. The Aumann and Kruskal Assumptions

In the following presentation of the Aumann and Kruskal assumptions we shall retain our own notation. Also the order of presentation differs from that of the authors.[1]

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1. Aumann and Kruskal, Naval Research Logistics Quarterly, Vol.V, pp. 114-119.

Assumption 1: With each assignment plan T and relevant military official to be questioned, O, there is associated a numerical value $u(T,O)$.

Assumption 4: If of two assignment plans T_1 and T_2, an official prefers T_1 to T_2, then $u(T_1,O) > u(T_2,O)$. These two assumptions together merely assert that u is an order-preserving function and are of course unobjectionable. The assignment plans referred to are both (1) the simple plans about which the official is directly questioned and (2) the over-all assignment plans which result from combinations of the simple ones.

Assumption 3: Let T_1 and T_2 be two assignment plans involving the assignment of exactly the same number of models m_i to each given activity v_j. Then $u(T_1,O) = u(T_2,O)$. This assumption is of course both necessary and entirely harmless.

Assumption 5: $r(a_{ij})$ is independent of $p(v_j)$. This assumption asserts that the relative goodness values of models installed in activities of a given priority value do not depend upon the priority value of the activity. This assumption also seems highly plausible.

Assumption 2: $u(T,O) = MW(T) = \sum x_{ij} \cdot u(a_{ij})$.

It is obvious that Assumption 2 is the heart of the matter, for it asserts that there exist numbers $u(a_{ij})$ associated with any allocation plan such that the sum of them gives the value of the allocation plan; and it further asserts that the values for the allocation plans thus obtained correctly reflect the order of preference as noted in Assumptions 1 and 4. The result of the paper is to show that if such numbers exist, they can be "almost" uniquely determined by the method outlined in the sample problem. The big question is, then, "what assumptions must be made about the structure of preferences so that such numbers will in fact exist?" We shall attempt to shed some light on this question in the following section.[1]

9. Implications of the Aumann-Kruskal Assumptions

We now move directly to the nature of the preference structure required by the Aumann and Kruskal assumptions and the method of computing military worth derived therefrom. First of all it is perfectly clear that the authors assume the existence of a complete and transitive "goodness order" for all activities. Thus for any fixed j and for any models p, q, and r, either a_{pj} P a_{qj} or a_{qj} P a_{pj}, and if a_{pj} P a_{qj} and a_{qj} P a_{rj}, then a_{pj} P a_{rj}.[2]

It is equally obvious that the scheme proposed by the authors requires a complete and transitive difference order for all activities. Thus for any fixed j and for any models a, b, c, d, e, and f, either

$$(a_{aj} \overset{*}{-} a_{bj}) \text{ P } (a_{cj} \overset{*}{-} a_{dj})$$

or

$$(a_{cj} \overset{*}{-} a_{dj}) \text{ P } (a_{aj} \overset{*}{-} a_{bj}).$$

Also, if

$$(a_{aj} \overset{*}{-} a_{bj}) \text{ P } (a_{cj} \overset{*}{-} (a_{dj})$$

and

$$(a_{cj} \overset{*}{-} a_{dj}) \text{ P } (a_{ej} \overset{*}{-} a_{fj}),$$

---

1. Aumann and Kruskal remark of Assumption 2 that "from the strictly logical viewpoint Assumption 2 could equally well have been stated as a definition. It involves important conceptual assumptions, however, and is used not only in theoretical work but in the deduction of numerical values from the (military authority's) decisions. It therefore seems more honest to state it as an assumption". Aumann and Kruskal, Naval Research Logistics Quarterly, Vol. V, p. 114.

2. See above, p. 61. Of course the scheme admits of the possibility of indifference, but we have ignored this possibility throughout the discussion on the grounds that it merely introduces expositional complications without adding anything to an understanding of the method.

then

$$(a_{aj} \overset{*}{-} a_{bj}) \text{ P } (a_{ej} \overset{*}{-} a_{fj}).$$[1]

The above restrictions on the permissible kinds of preferences are both fairly trivial and quite obvious on casual inspection. The method for computing the priority values, however, contains implicit in it another restriction on permissible preferences which is neither so trivial nor so obvious. To see how the method of computing the $p(v_j)$ implies such a restriction recall the discussion of the method contained on pages 62-67 above and the table of hypothetical data contained therein.

Consider, for example, the pair of two-element assignment plans

$$A = a_{21} + a_{12}$$
$$B = a_{31} + a_{22},$$

pair number 12 on page 65. Note that if the relevant authority prefers B to A it is possible to conclude that the ratio $p(v_1)/p(v_2) < 1.16$. Now consider the pair numbered 11 in the table:

$$A = a_{21} + a_{42}$$
$$B = a_{31} + a_{22}.$$

If the authority prefers B to A we find that $p(v_1)/p(v_2) < 1.36$, which raises no questions. If, on the other hand, the authority prefers A to B then this implies that the ratio $p(v_1)/p(v_2) > 1.36$. Such a preference would obviously lead to nonsense results, however. For the preference of B to A in pair number 12 implies that the ratio of the priority values is smaller than 1.16 and consequently the preference of A to B in pair 11 would imply that the ratio is at the same time smaller than 1.16 and larger than 1.36, which is nonsense. Consequently

1. See above, p. 61.

if such a pattern of preferences exists, the method will not work.

In more general terms the method clearly requires the following kind of "consistency" in the preferences of the authority: Suppose that for some pair of two-element assignment plans

$$\frac{r(a_{p2}) - r(a_{q2})}{r(a_{r1}) - r(a_{s1})} = K.$$

In this case if the "B" member of the pair is preferred to the "A" member (see page 67 for definitions of "A" and "B" members), then the "B" member of any pair for which

$$\frac{r(a_{p2}) - r(a_{q2})}{r(a_{r1}) - r(a_{s1})} > K$$

must be preferred to the "A" member. Likewise if A is preferred to B in the original pair, then A must be preferred to B in any pair the value of whose ratio is smaller than K.

Now it is by no means intuitively clear what assumptions about the preference structure are necessary to ensure that preferences among the A-B pairs of two-element plans will generate consistent and meaningful values for the ratios of the $p(v_j)$. Certainly the original paper by Aumann and Kruskal provides no clue and in fact does not even raise the question. At an earlier stage of research we had approached the problem by laying down, essentially, the following three requirements for a "rational" preference structure for preferences among single a_{ij} and among two-element pairs.

1. Transitivity of preferences among the single elements, that is, among the individual a_{ij}.
2. Transitivity of preferences among the two-element pairs.

3. Independence of utilities, that is, for example, in the two two-element pairs X and Y

$$X = (a + b)$$
$$Y = (c + d),$$

if a is preferred to c and b is preferred to d, then X is preferred to Y.

The first two of these requirements are obviously elementary. The third, like Rule 2 to which it is closely related,[1] rules out the possibility of complements and substitutes among the a_{ij}.

Now the question is, will any set of preferences which satisfies the above three requirements yield a consistent set of numerical values by the Aumann and Kruskal method? It is clear that the three requirements constitute at least a necessary condition for the existence of an Aumann-Kruskal military worth function--i.e., it is clear that any set of preferences which does not conform to the requirements can not be made to yield a consistent set of numbers by their technique. Indeed, the set of requirements seems to constitute the weakest definition of "rational choice" which would have any hope of yielding a utility function derived by the Aumann and Kruskal procedure. It is therefore of some interest to determine whether the requirements are a sufficient, as well as a necessary condition for the existence of answers to the preference questions which yield consistent values for the ratios of the $p(v_i)/p(v_j)$.

We had originally conjectured and proved[2] that in fact it is possible to have a set of preferences for the single a_{ij} and two-element pairs which (1) are consistent with the three requirements, and (2) at the same time yield the nonsense result that $p(v_i)/p(v_j)$ is both greater than and less than some number. The issue involved, however, has been considerably clarified as the result of new, unpublished work by Professor

1. See above, p. 25.
2. See below, p. 80.

Aumann and correspondence between the writer and Aumann.

To see what is involved let us first consider a new set of axioms about the underlying preference structure. The preference structure we are interested in now concerns not just the single a_{ij} and two-element pairs, but a set X whose elements consist of all possible combinations of the original a_{ij}. In other words the elements of X are all possible assignment plans of any degree of complexity, whether single elements, or highly complex combinations of them. We now state three axioms about the preferences among the elements of X.[1]

1. The preferences are transitive, but not necessarily complete.
2. The preferences are "consistent", i.e., if x P y, then (x + z) P (y + z) where x, y, and z may be any elements of X.
3. The preferences are "finitely generated", that is, there is a finite number of elements of X from the preferences between which all other preferences can be generated by repeated application of the consistency axiom.

Comparing the two sets of axioms, the "consistency" axiom is an extension of the "independence of utilities" axiom to complex pairs of a_{ij}. Thus the independence of utilities axiom, applying as it does only to the two-element pairs (about which the authority would be directly questioned), says, for example, that if

$$a_{11} \text{ P } a_{12}$$
$$a_{22} \text{ P } a_{34}$$

then

$$(a_{11} + a_{22}) \text{ P } (a_{12} + a_{34}).$$

---

1. These axioms are to be found in an unpublished paper by R. J. Aumann, "A Non-Probabilistic Theory of Utility" (1960), p. 16.

On the other hand since the consistency axiom extends the notion to any degree of complexity, we could conclude that if a_{78} P a_{56},

$$(a_{11} + a_{22} + a_{78}) \text{ P } (a_{12} + a_{34} + a_{56}),$$

and so on. Thus the consistency assumption of the second set of axioms is clearly stronger than the independence assumption of the first set.

The finite generation axiom need cause no concern since in any allocation problem of the type we have been dealing with, any set of preferences satisfying the consistency axiom will obviously also satisfy the finite generation axiom.

The significance of the new set of three assumptions is that Aumann asserts that any set of preferences which satisfies these assumptions can be made to yield a military worth index having the requisite properties by the method of Aumann and Kruskal.[1] The basis of this belief appears to be the following set of considerations.

1. Professor Aumann has proved that if the preference order on the elements of X is transitive, consistent, and finitely generated, then there exist utility numbers u(x) and u(y) such that for any x and y in X u(x) > u(y) if, but not only if, x P y. Furthermore the utility of any element $x = a_{11} + \dots + a_{mn} = \sum u(a_{ij})$.[2]

2. As we have seen, the Aumann and Kruskal paper assumes that (1) there exists a function u(x) defined for all x in X such that (2) $u(x) = \sum u(a_{ij})$ and (3) if x P y, u(x) > u(y). The paper goes on

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1. Correspondence between the writer and Professor Aumann.

2. Aumann, "A Non-Probabilistic Theory of Utility," p. 17.

to prove that if in fact such a function exists it can be found by the method discussed above.

3. Taking paragraphs 1 and 2 together, (1) says that if the preference structure is transitive, consistent, and finitely generated a utility function of the desired type exists, while (2) says that if it exists, it can be found by the Aumann and Kruskal method. Therefore if the preferences are transitive, consistent and finitely generated the Aumann and Kruskal method will always work.[1]

10. An Example

A concrete example may be offered to clarify several of the points made in the above section. Suppose we are dealing with a situation in which there are two activities, v_1 and v_2 and three model types, m_1, m_2, and m_3. For this 3-by-2 case there are 21 possible two-element assignment plans which we list below with identifying numerals for future reference.

1. $(a_{11} + a_{12})$	7. $(a_{01} + a_{12})$	12. $(a_{21} + a_{01})$	17. $(a_{21} + a_{21})$
2. $(a_{11} + a_{22})$	8. $(a_{01} + a_{22})$	13. $(a_{12} + a_{22})$	18. $(a_{01} + a_{01})$
3. $(a_{11} + a_{02})$	9. $(a_{01} + a_{02})$	14. $(a_{12} + a_{02})$	19. $(a_{12} + a_{12})$
4. $(a_{21} + a_{12})$	10. $(a_{11} + a_{21})$	15. $(a_{22} + a_{02})$	20. $(a_{22} + a_{22})$
5. $(a_{21} + a_{22})$	11. $(a_{11} + a_{01})$	16. $(a_{11} + a_{11})$	21. $(a_{02} + a_{02})$
6. $(a_{21} + a_{02})$			

Let us assume the following set of preferences among the a_{ij} to be held by some relevant military authority:

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1. Professor Aumann may have a more direct proof in mind, but if so it has evidently not been made available for distribution.

$$a_{11} \; P \; a_{12} \; P \; a_{21} \; P \; a_{22} \; P \; a_{02} \; P \; a_{01}.$$

Also assume the following preferences to hold between two-element pairs:

$$16 \; p \; 1 \; p \; 19 \; p \; 10 \; p \; 2 \; p \; 4 \; p \; 3 \; p \; 11 \; p \; 13 \; p \; 17 \; p \; 5 \; p \; 14$$
$$p \; 7 \; p \; 6 \; p \; 12 \; p \; 20 \; p \; 15 \; p \; 8 \; p \; 21 \; p \; 9 \; p \; 18.$$

About the above set of assumed preferences we may note the following: (1) it is obvious that any set of pair-wise preferences extracted from the chains would be transitive. (2) Each pair-wise preference for the two-element pairs is consistent with the "independence of utilities" assumption. The reader may verify this for himself.

Now the interesting thing about this set of preferences is that although it is complete, transitive, and satisfies the independence of utilities condition it will not yield a consistent set of utility numbers by the Aumann and Kruskal method. For example, the reader may satisfy himself that the following relative goodness values meet the requirements set for them.[1]

$$r(a_{11}) = 4 \qquad r(a_{12}) = 8$$
$$r(a_{21}) = 1 \qquad r(a_{22}) = 1$$
$$r(a_{01}) = 0 \qquad r(a_{02}) = 0.$$

Now let us consider the following A-B pairs:

$$5: \; A = a_{21} + a_{22} \qquad 3: \; A' = a_{11} + a_{02}$$
$$7: \; B = a_{01} + a_{12} \qquad 4: \; B' = a_{21} + a_{12}$$

The value of $\dfrac{r(a_{r2}) - r(a_{s2})}{r(a_{p1}) - r(a_{q1})}$ for the A-B pair is 7.00 and since

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1. See above, p. 61.

by assumption A is preferred to B (5 is preferred to 7), $p(v_1)/p(v_2) > 7.00$. On the other hand, the value of the fraction for the A'-B' pair is 2.67 and since, by hypothesis, B' is preferred to A' (4 is preferred to 3), we have that $p(v_1)/p(v_2)$ is both less than 2.67 and greater than 7.00. Thus the first set of preference axioms is not a sufficient condition for the existence of the Aumann and Kruskal function.

The second interesting thing about the set of preferences is that if we attempt to extend it by applying the consistency axiom, we are unable to get a set of preferences which is both consistent and transitive, not only for two-element pairs, but for more complex pairs. For example, by assumption, 5 is preferred to 7 and 2 is preferred to 4. Thus by the consistency assumption (5 + 2) P (7 + 4), i.e.,

$$(a_{11} + a_{22} + a_{21} + a_{22}) \text{ P } (a_{01} + a_{12} + a_{21} + a_{12}).$$

However it may be seen that the left-hand combination may also be thought of as the sum of 10 + 20 since

$$10 = (a_{11} + a_{21})$$
$$20 = (a_{22} + a_{22})$$

while the right-hand side may be thought of as the sum of 12 and 19 since

$$12 = (a_{21} + a_{01})$$
$$19 = (a_{12} + a_{12}).$$

But by assumption, 19 is preferred to 10 and 12 is preferred to 20, so that by the consistency rule (19 + 12) P (10 + 20), which contradicts the conclusion that (5 + 2) P (7 + 4).

Thus we may sum up the significant conclusions as follows:

1. Not every set of preferences which is transitive and obeys the independence of utilities assumption will yield a consistent set of utility numbers by the Aumann and Kruskal method. 2. Not every set of preferences which is transitive and obeys the independence of utilities assumption can be extended to yield a set of preferences over all possible pairs which is transitive and consistent. 3. Every set of preferences which is transitive, consistent and finitely generated yields a consistent set of utility numbers by the Aumann and Kruskal method.[1]

11. Significance of the Result

From the above it can be seen that not just any responses to the preference questions put to the military authority will work in the Aumann and Kruskal schema. A natural question to ask is whether actual responses could be expected to be of the desired type and what would happen if they were not? It is clear that all difficulties enter in at the attempts to determine the values of the $p(v_i)/p(v_j)$. Let us suppose that some military official actually has in mind a set of preferences like those in the example, i.e., transitive and obeying the "independence of utilities", but not the "consistency" assumption. There would be nothing "irrational" about such a set of preferences from the intuitive point of view. It would simply be a set where the consistency assumption would fail to predict choices among more complex plans. Perhaps, for example, the inapplicability of the consistency assumption to this set of preferences merely reflects the authority's

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1. Of course all problems in which the preferences are generated from a finite number of a_{ij} and two-element pairs will obey the finite generation assumption--in other words all cases in which we are interested.

belief in the existence of complementarities that do not show up until we get to complex combinations of the a_{ij}. In any event, if the consistency assumption is not met, the scheme simply will not work, as we have pointed out, because nonsense numerical results will occur.

Thus there remains the practical question of whether or not in fact military preferences are likely to have the kind of structure required for the applicability of the numerical utility model suggested by Aumann and Kruskal. These authors evidently ran into no difficulty in the particular problem to which they applied the method.[1]

The question of the structure of actual preferences is touched upon by Aumann and Kruskal in both their papers. It is also raised in a critical note by McShane and Solomon and in a reply to this note by Kruskal.[2] This discussion is rather unsatisfactory, however. There are several points in the procedure where "inconsistency" of preferences in one sense or another might show up in the answers to the preference questions and the discussion of the problem in this literature frequently fails to specify what type of "consistency" is under discussion. Somewhat ironically, there seems to be no evidence that any of the four authors involved have in mind the type of potential numerical inconsistency we have noted which results from the violation of what Aumann himself later called the "consistency axiom".

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1. R. J. Aumann and J. B. Kruskal, "Assigning Quantitative Values to Qualitative Factors in the Naval Electronics Problem," Naval Research Logistics Quarterly, Vol. VI, No. 1 (March, 1959), pp. 1-16. This paper discusses an application of the technique first presented in the earlier paper. See particularly p. 15.

2. R. E. McShane and H. Solomon, Letter to the Editor of the Naval Research Logistics Quarterly, Vol. V, No. 4 (December, 1958), pp. 363-367. J. B. Kruskal, Letter to the Editor, ibid., Vol. VI, No. 3 (September, 1959), pp. 161-162.

In their critical note on the first Aumann and Kruskal paper, McShane and Solomon remark that "for those...cases where a particular (model) is applicable to two or more (activities), intransitivity could result."[1] To this, Kruskal replies as follows:

> With regard to (this) criticism, of course we might run into intransitivity. Whenever you go up to a human being and ask "which do you prefer, a or b; which do you prefer, b or c; which do you prefer, c or a?" it is possible he will say "a better than b, b better than c, c better than a"...This risk exists as soon as you consult expert opinion on anything.[2]

Now it is not clear just exactly what the "a", "b", and "c" refer to here. The authors may have any one of three kinds of intransitivity in mind: (1) Intransitivity of expressed preferences in the goodness orders, (2) intransitivity of expressed preferences with respect to the difference orders, and (3) intransitivity of expressed preferences with respect to the two-element assignment plans. But as we have seen, intransitivities of these kinds may be entirely absent, and yet the preferences still lack the kind of "consistency" required for the applicability of this additive utility model. None of the above possible types of intransitivity is necessary to get a situation where preferences among the A-B pairs are such as to determine that $p(v_i)/p(v_j)$ is at the same time greater and less than the same value.

Some discussion by Aumann and Kruskal superficially suggests that they have anticipated this problem, but closer examination reveals that they are concerned with yet another problem.[3] To see what they

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1. McShane and Solomon, Naval Research Logistics Quarterly, Vol. V, p. 365.

2. Kruskal, Naval Research Logistics Quarterly, Vol. VI, p. 262.

3. Aumann and Kruskal, Naval Research Logistics Quarterly, Vol. VI, p. 9.

have in mind let us go back to our sample problem (pp. 64-66). Suppose that for A-B pairs 1-6 we find that B p A and therefore conclude that $p(v_1)/p(v_2) < 2.14$. Further, suppose that for pairs 10-24 we find that A p B and therefore conclude that $p(v_1)/p(v_2) > 1.55$. Now Aumann and Kruskal suggest that in the intermediate range of values, pairs 7-9, A and B may be so "close" in value that the authority is unable to make consistent distinctions and as a result we get numerically inconsistent preferences for this intermediate range of pairs simply because the military worths of the A and B plans involved are so "close" in magnitude. This seems to be a different kind of "inconsistency" again, and it may of course have considerable practical importance.

12. The Combinatorial Problem

In the Aumann and Kruskal method a military authority is asked to express preferences with regard to some pairs of single a_{ij} and some two-element pairs. If his preferences among all single a_{ij} and two-element pairs are transitive and consistent, the resulting military worth numbers will correctly "predict" his choices among both the remaining pairs of a_{ij} and pairs of two-element plans. The real purpose of the procedure for computing military worth values, however, is of course to "predict" choices among complex combinations of the a_{ij}. Now the only comparisons among "complex", more-than-two-element, plans that can be directly constructed are those which can be built up from repeated application of the consistency assumption to their components, for example, if x, y, z, a, b, and c represent two-element plans where x P a, y P b, and z P c, the consistency rule says, for example, that (x + y + z), consisting of six elements, must be preferred to (a + b + c), also consisting of six elements. In other words, complex plans are comparable only where our Rule 3 (see p. 55) is applicable. But we can be sure that if a complex plan M is preferred to a complex plan M' on the basis of this rule, the computed military worth of M will be greater than the computed military worth of M', i.e.,

$$\sum u(a_{ij}) > \sum u(a'_{ij}).$$

The proof of this is obvious.

1. If a_{ij} P a_{rs}, $u(a_{ij}) > u(a_{rs})$, by the construction of the utility index.
2. If $(a_{ij} + a_{pq})$ P $(a_{rs} + a_{fg})$, then $u(a_{ij} + a_{pq}) > u(a_{rs} + a_{fg})$, also by the construction of the utility index.
3. If M is preferred to M', then M and M' can be partitioned into single and double-element components such that for every component in M there is a component in M' to which it is preferred or indifferent, not all indifferent. (This is Rule 3.) This follows from the fact that M is preferred to M' if and only if the two plans can be built up from elementary components by repeated application of Aumann's consistency axiom.
4. Therefore if M is preferred to M', M and M' can be divided into single and double-element components in such a way that for every component in M' there is a component in M with a higher or equal, not all equal, utility number.
5. It follows that the sum of the utilities of the components of M is greater than the sum of the utilities of the components of M'.

The difficulty is that the converse of the above proposition is not true, in general. Thus if M and M' are two complex plans, the mere fact that the utility of M is greater than the utility of M' does not ensure that M is preferred to M'. The two plans may be incomparable even though the utility maximization rule grinds out a choice between them.

13. The Multi-Person Amalgamation Problem

We shall conclude this chapter by taking another look in the light of the immediately preceding discussion at what we earlier (p. 19) called the "multi-person amalgamation problem". In the present chapter we have assumed up to this point that the preferences of only one person (or the concensus of a board) were involved in the determination of the military worth function. In Chapter II we pointed out that in many cases it might be necessary or desirable to consult the preferences of more than one authority because of a division of expertise. Two kinds of divisions were mentioned there. One of these, occuring in the "separable" case, calls for technical experts to rank the relative desirability of different models in any given activity while strategic experts rank the military significance of the activities. We have seen, however, that except in the case of a lexicographic ordering scheme of a rather arbitrary sort,[1] the "separability" concept is probably not very meaningful in connection with the use of difference orders. The second type of multi-person problem due to division of expertise

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1. See above, pp. 57-59.

might occur where there are several military authorities, each responsible for ranking alternatives only insofar as they affect some subset of the activities.

More specifically suppose we have the following type of situation. Assume that there are m types of equipment and n activities. Suppose that each of the n activities is directed by a different individual. Suppose further that each individual is questioned to obtain a ranking and a difference order on the a_{ij} for the particular activity j for which he is responsible. If, as before, we include among the a_{ij} the assignment of voids (i.e., the a_{0j}), then we can represent the preferences obtained by n different utility functions, each unique up to a factor of proportionality. Let us represent the numerical values thus obtained by $r_j(a_{ij})$, using the subscript r_j to remind us that the value is relative to the preferences of the jth authority.[1]

Now, as before, we would like to find a numerical index $u(a_{ij})$ which correctly "predicts" a "rational" preference between any two a_{ij} and among all possible two-element assignment plans. Given the $u(a_{ij})$ we could then compute military worth of an over-all assignment plan by $MW = \sum x_{ij} \cdot u(a_{ij})$, as before, and with, of course, the same reservations.[2]

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1. Note that we have used the letter j as a subscript in two different places since the preferences among the a_{ij} for the "jth" activity are expressed by the "jth" authority.

2. This problem of course bears a strong family resemblance to the so-called "social welfare" problem and related types of multi-person amalgamation problems. For a very general introduction to this problem see L. A. Goodman, "On Methods of Amalgamation," Decision Processes, ed. R. Thrall, C. Coombs, and R. L. Davis (New York: John Wiley, 1954), pp. 15-19. For other references see R. Luce and H. Raiffa, Games and Decisions (New York: John Wiley, 1957), pp. 327-370. The most extensive bibliography will be found in a forthcoming research memorandum by W. G. Mellon for the Econometric Research Program, Princeton University.

It seems at the outset that at least two properties should be required of any solution to this problem. First of all, the $u(a_{ij})$ and the preferences reflected thereby should somehow "take account of" the preferences indicated by the individual authorities. In the somewhat similar problem of mapping individual preferences into a social preference Arrow has stated a "non-imposition" requirement for the social welfare function which is similar to our requirement.[1] In our context "imposition" would imply that the central authority determines a difference order on the a_{ij} without any regard to the preferences expressed by the individual authorities through the $r_j(a_{ij})$. There is, of course, nothing a priori wrong with imposition in a military context, except that it wastes the expertise of the individual authorities within their own spheres.

A second property which should be required by a solution is that it should not involve "interpersonal comparisons of utility".[2] By this we mean here neither more nor less than that the $u(a_{ij})$ should be invariant up to a factor of proportionality with respect to a proportional change in any or all of the individual $r_j(a_{ij})$ scales. An elaborate philosophical defense of this requirement would probably be futile and certainly beside the point here. The only defense we shall offer is the point that as long as the $r_j(a_{ij})$

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1. K. J. Arrow, Social Choice and Individual Values (New York: John Wiley, 1951), p. 28.

2. The most relevant discussion of this concept as applied to ordinary utility situations is found in Luce and Raiffa, Games and Decisions, p. 33.

do not have fixed units there is no reason why arbitrary changes in these units should affect the $u(a_{ij})$. That is not to say that there may not be some conceivable circumstances in which the central authority could find a meaningful way to fix the units for the individual $r_j(a_{ij})$ functions; but this would be a different problem and one that we do not intend to explore further here. Thus, in a sense, this second requirement should really be regarded as part of the definition of the problem.[1]

It should be noted, however, that in general we would not expect the $u(a_{ij})$ to be invariant with respect to any linear transformations on some or all of the $r_j(a_{ij})$ scales. This is so because the existence of the voids among the a_{ij} establishes a natural zero for each of the $r_j(a_{ij})$ scales.

14. Some Possible Solutions

With the above definitions and requirements in mind we shall now explore a few obvious methods of solution.

(1) For the sake of completeness we list a method already rejected whereby the central authority simply ignores the preferences expressed by the individual authorities.

(2) An opposite policy would be one which sets the values of the $u(a_{ij})$ mechanically without regard to any preferences by the central authority. Thus for example one could arbitrarily let

1. Not every reader may feel that invariance with respect to proportional changes in the $r_j(a_{ij})$ scales captures the full intuitive content of the phrase "no interpersonal comparisons of utility". However, we feel that this invariance is at least at the heart of a rather elusive idea.

all the $r_j(a_{ij})$ scales run between zero and one and set the resulting numbers equal to $u(a_{ij})$. Thus while the first procedure obviously and explicitly violates the first requirement of an acceptable solution, its "opposite" violates the second, since in this case the solution depends upon particular choices of the units for the $r_j(a_{ij})$ scales.

(3) A third possibility in which the $u(a_{ij})$ depend both upon the $r_j(a_{ij})$ and the preferences of a central authority is as follows: Consider the two-element assignment plans

$$A = a_{wi} + a_{xj}$$
$$B = b_{yi} + b_{zj}.$$

Now let A p B if

$$\left[r_j(b_{zj}) - r_j(a_{xj})\right] - \left[r_i(a_{wi}) - r_i(b_{yi})\right] < k_{ji},$$

where both differences are positive and where k_{ji} is a figure picked by the central authority, one for each pair of activities, i and j, and which reflects, or rather defines, its attitude to the "relative importance" of the two activities. This method, however, like (2) obviously is not invariant with respect to changes in scale in one or more of the $r_j(a_{ij})$ functions. It therefore implicitly depends upon an "interpersonal comparison of utility" on our definition of that phrase.

(4) A fourth possibility which like the third makes the $u(a_{ij})$ depend both on the $r_j(a_{ij})$ and the judgment of a central authority, but lacks the defect of method (3) can be sketched as follows: Define

$$u(a_{ij}) = p(v_j) \cdot r_j(a_{ij})$$

following Aumann and Kruskal. Let the central authority then express its preferences by choice of the $p(v_j)$. The difficulty with this method is that in reversing the Aumann and Kruskal procedure by letting determination of $p(v_j)$ determine choices rather than letting choices fix the value of $p(v_j)$, it offers no guide to the choice of the $p(v_j)$. Since choices among assignment plans would be implicit in choices of $p(v_j)$ by the central authority, it seems much more natural to deal directly in terms of choices among the two-element assignment plans.

(5) The following method makes use of direct choices by the central authority, takes account of the $r(a_{ij})$ and does not involve interpersonal comparison of utility. It requires, however, a special situation which might make it of limited applicability to actual problems. Let us suppose, as in (4) above that we have a matrix of a_{ij} with n activities and n corresponding utility functions $r_j(a_{ij})$ and let us again define

$$u(a_{ij}) = p(v_j) \cdot r_j(a_{ij}).$$

In this case, however, we suppose that the elements within certain rows in the matrix can be compared by the central authority without reference to the values of the corresponding $r_j(a_{ij})$. Let us then obtain preferences from the central authority on pairs of two-element assignment plans for elements in these rows, using the preferences to obtain the values of the $p(v_i)/p(v_j)$ ratios as in the Aumann and Kruskal method. With the values of the $p(v_j)$ thus obtained, the values for the $u(a_{ij})$ in the entire matrix are determined.

To make this a little clearer, consider the following matrix of assignment elements where the rows represent model-types and the columns, activities:

$$\begin{matrix} (a_{11}) & (a_{12}) & a_{13} \\ a_{21} & (a_{22}) & (a_{23}) \\ (a_{31}) & a_{32} & (a_{33}). \end{matrix}$$

Now let us suppose that assignment pairs involving elements in parentheses can be compared by the central authority without reference to the corresponding $r_j(a_{ij})$ values. For example assume that a preference can be thus obtained for the following A-B pair:

$$A = a_{11} + a_{22}$$
$$B = a_{31} + a_{22}.$$

But the values of the $r_j(a_{ij})$, although having no direct influence on the central authority's choice between such A-B pairs, can be independently computed from the preferences of the local authorities. Thus, for example, local authority 1 directing activity 1 gives preference information from which the $r_1(a_{i1})$ are determined. Now the values of the $r_j(a_{ij})$ can be combined with knowledge of preferences among the A-B pairs obtained from the central authority to yield the values of the $p(v_i)/p(v_j)$ ratios and thence the values of the $u(a_{ij})$ themselves.

To give an admittedly somewhat crude interpretation of the setting required for the use of this technique, suppose that the "activities" are military bases and the "models" are different items to be assigned to the bases. Suppose among possible items to be assigned to the "activities" are guided missiles with nuclear warheads and poison gas. Now it is reasonable to suppose that the relative desirability of supplying such items as these to any given activity will depend entirely upon general policy known only to the central authority and not at all on the relative "importance" of these items to the individual activities as compared with other items. Thus the assignment of such items to the various activities would correspond to the a_{ij} in parentheses in the matrix presented above.[1]

It may appear to some readers that this method implicitly involves the determination of a common unit for the $r_j(a_{ij})$ scales, but this is not the case. The reader can convince himself that any scalar transformation of one of these functions will leave the $u(a_{ij})$ unchanged since if, for example, we multiply each $r_j(a_{ij})$ by 10, for some given j, the corresponding $p(v_j)$ is automatically divided by 10 and hence $u(a_{ij})$ is unchanged. At the same time it is clear that the determination of the $p(v_j)$ and hence $u(a_{ij})$ depends upon the central authority's preferences among only a limited subset of the a_{ij}. Thus the ultimate computed "preference" among the remaining elements depends in part upon these preferences.[2]

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1. It must be emphasized that the particular expertise of the local authorities enters into the ultimate determination of the $u(a_{ij})$ through the determination of the $r_j(a_{ij})$.

2. Hence the procedure violates Arrow's "independence of irrelevant alternatives" condition. See Arrow, Social Choice and Individual Values, pp. 26-28.

CHAPTER IV

INVENTORY PROBLEMS AND THE MAXIMIZATION OF EXPECTED MILITARY WORTH

1. Introduction

The previous chapters of this study have dealt with problems of choice among "sure prospects"--as they have come to be known in the literature of utility theory. That is, there have been no probabilistic elements involved in the items among which choices are to be made. We now want to examine briefly an extremely important military worth problem where probabilistic elements must come to the center of attention. The importance of the problem, its comparative neglect, and the potential fruitfulness of the general line of approach we shall take, rather than the practical usefulness of the particular conclusions reached is the chief justification for the discussion that follows.[1]

Our problem is the evaluation of alternative inventory positions in the military setting. Inventory theory treats three types of costs associated with inventories: the costs of procurement, holding costs, and the costs associated with having insufficient inventories--the costs of runouts. In ordinary commercial operations all of these costs can, conceptually at least, be calculated in money terms and compared. In the military establishment, however, while procurement and holding costs are computable in money terms, the costs of being caught short are not.[2]

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1. The difficulties inherent in applying inventory theory to the military establishment are noted in K. J. Arrow, T. Harris and J. Marschak, "Optimal Inventory Policy," Econometrica, Vol. XVIV, No. 3 (July, 1951), p. 250. See also T. M. Whitin, The Theory of Inventory Management (Princeton: Princeton University Press, 1953), pp. 165-75.

2. Except in such comparatively rare cases where runouts can always be prevented by "rush orders" which have some known extra monetary cost as compared with regular orders.

It is the latter with which we shall be concerned. Thus although two different inventory policies may differ in the money costs of obtaining and holding the inventory, our concern is with differences in the ability to meet demands. We shall view the military worth of an inventory position, as opposed to its cost, as determined by its ability to meet the demands for the various items stocked and by the military importance, in a sense to be determined, of the different items themselves. In what follows we are going to define "ability to meet demands" in terms of the runout probability associated with a given policy for a given military item in a given period. Other definitions of "ability to meet demands"--such as expected percentage of time the system is out of stock--are possible. However, the one used here is most often considered in the literature of inventory theory[1] and turns out to be most convenient for our purposes.

Let us suppose that a military system is concerned with stocking n items, $a_1,..,a_n$. Any given inventory policy will result in a certain level of stocks at the beginning of a period for each of the n items. Given the stochastic character of demand this means that associated with any inventory policy there are n probability numbers, $p_1,...,p_n$ indicating the probability that all demands made on the stock of the n items during the period will be met. For our purposes we shall characterize an inventory plan by the set of probabilities associated with it. Thus if we are considering two different inventory plans, we shall treat choice between them (from the military worth point of view alone) as being a choice between two probability vectors.

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1. I am indebted to Professor D. Orr of Amherst College for this observation.

How should the inventory authority go about ranking two inventory plans, M and M'? To assert that a problem exists at all here really implies again a desire to separate the "judgment" from the "combinatorial" aspects of military worth determination.[1] This is so because the "problem" could of course be "solved" simply by appointing an authority familiar with the technical characteristics of the items to be stocked and their use in military operations to make a choice between the two plans M and M' on the basis of his expert judgment. This "solution" to the problem of ranking the two plans is objectionable for two main reasons. The first reason is the same as that presented in Chapter III in a similar context: the most astute judgment is not reliable when the complexity of the alternatives to be judged is very great. For this reason it would be desirable to limit the direct use of expert judgment to the making of decisions among simple alternatives, the decisions among the complex ones being constructed through the use of mechanical combinatorial rules.

A second objection to the "hand" or "naive" solution is that it does not permit easy, immediate determination of the relative merits of any pair of inventory plans. In view of the fact that the runout probabilities for each of the n items can take on any value between zero and one, the number of potential plans to be ranked is limitless. It would be nice to have a technique which makes military worth values emerge as a function of the runout probabilities so that the relative values of any two plans can be computed directly, thus making it unnecessary to consult expert judgment every time. The point here is similar to the one which motivated the development of a utility theory

1. See above, p. 67.

for game theory. The von Neumann-Morgenstern utility permits the game theorist to compute the preferences of the players for all possible probability combinations of outcomes without the necessity of constant reference to the players themselves.[1]

Superficially the resemblance between the problem discussed in this chapter and the von Neumann-Morgenstern utility problem would appear to be fairly close, but actually the similarity breaks down at several crucial points. In the following section we shall present a proposal for a method of evaluating alternative inventory policies which obviously derives its inspiration from the work of von Neumann and Morgenstern. Like these authors, we shall axiomatize our choice assumptions and then derive a utility index. However, we should like to emphasize that in spite of the occasional parallelism, the system of entities under discussion is different and the axioms covering the relations among them is different, both from the original von Neumann-Morgenstern axioms and from all subsequent alternative versions of them.[2]

2. A Method of Evaluating Alternative Inventory Positions

The following section attempts to do two things. First we shall develop a set of assumptions which (1) specifies the nature of the elementary preference information we shall require and (2) provides rules which enable us to combine the elementary preferences into a complete order on the set of all possible inventory plans. After this has been done we shall discuss a military worth function which has the property

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1. Game theory and the utility theory associated with it are of course logically independent of each other.

2. Unfortunately an earlier attempt to work from the more satisfactory von Neumann-Morgenstern axioms proved unsuccessful.

that the computed military worth of one alternative is greater than that of another if and only if it is preferred by the criterion provided by the set of preference assumptions.[1]

To obtain a complete order of the inventory positions we shall require six preference assumptions to be introduced below. The technique to be presented is no stronger than the validity of these six assumptions. It must be admitted at the start that five of the six are distinctly non-trivial and in fact may be objectionable under some circumstances. It is to be hoped that the reader will feel that the technique is suggestive enough to make the exploration of the assumptions worthwhile. In any event the limitations of each of the assumptions will be discussed as they are presented.

Our assumptions deal with the following set of entities: (1) There are n different items $a_1,\ldots,a_n$, representing the types of items to be stocked. (2) If p_i represents one minus the runout probability for the ith item when the stock of that item is placed at a given level, we can consider all inventory plans M as being defined by the runout probabilities associated with it, so that

$$M = p_1a_1 + \cdots + p_na_n.$$

This means that the stocking policy M involves stocks of the n different items such that p_i is the probability that all demands can be met during a specified period for the ith item. (3) We also consider a

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1. The general line of argument and the techniques of proof were inspired by roughly similar procedures employed in a different context by W. J. Baumol in "The Cardinal Utility Which is Ordinal," The Economic Journal, Vol. LXVIII, No. 272 (December, 1958), pp. 669-672.

"standard prize" (e.g., a monetary sum) R such that R p a_i for all i. This means that the prize R is preferred to the certainty of a sufficient supply of any of the items taken separately. Write p(R) for the probability p of getting the prize R.[1] We now state the six assumptions about preferences among these entities.

<u>Assumption 1</u>: For any two stocking plans M and M' either M p M', M' p M, or M I M'. Also for any a_i and any probability p_i $(0 \leq p_i \leq 1)$ and any probability p $(0 \leq p \leq 1)$, either $p_i a_i$ is preferred to p(R), p(R) is preferred to $p_i a_i$, or the two alternatives are indifferent. This means (1) that a preference or indifference relation exists between any pair of inventory plans; and (2) that a preference or indifference relation exists between any probability of a sufficient supply of a given item (including a probability of 1) and any given probability of getting the standard prize R.

<u>Assumption 2</u>: For any inventory item a_i there exists a probability number q_i $(0 \leq q_i \leq 1)$ such that a_i I q_i(R). This is a continuity-of-preference-as-a-function-of-probability assumption. It means that there is some probability which makes the authority just indifferent between getting a certainty of having enough of the ith item, on the one hand, or a q_i chance of R, but an insufficient supply of the ith item, on the other hand. Some discussion of this assumption is necessary. If a military authority were asked to choose, for example, between the certainty of an adequate supply of item a_i and a .5 chance of R, his choice would presumably depend upon what happens

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1. The probability of not getting R is of course (1-p). If we represent "not getting R" by R', then p(R) in the more customary "lottery ticket" notation would be (p,R,R'). However it is superfluous to keep the R' and the more elaborate notation for our purposes.

to a_i in the event that he picks $.5(R)$ rather than a_i. All we specify is that there will be a shortage of the ith item, not how large a shortage or how long-lasting. The authority's assumptions about the nature of the hypothetical shortage would surely affect his answer. This ambiguity is a definite weakness in the scheme. We can defend Assumption 2 on the following grounds, however. (1) This ambiguity is inherent in any attempt to evaluate inventory policies where "ability to meet demands" is defined solely in terms of runout probabilities. (2) Even though the ambiguity is present, the value of q_i is a rough measure of how much the authority would be willing to pay for the certainty that all demands on the ith item are met.

<u>Assumption 3</u>: If $a_i \; I \; q_i(R)$, then $p_i a_i \; I \; p_i \cdot q_i(R)$. This assumption says that if the guarantee of a sufficient supply of a_i is worth, for example, a .5 chance of R, then a .5 chance of such a guarantee is worth half as large a chance of R, i.e., a one-quarter chance. Two remarks concerning Assumption 3 are necessary. First of all, the proposition that a one-half chance of a one-half chance of R equals a one-quarter chance of R is similar in spirit to von Neumann and Morgenstern's algebra-of-combining axiom.[1]

Secondly, Assumption 3 may appear to be absurd if one follows a natural inclination to draw a superficially appealing, but incorrect, parallel with a similar situation in von Neumann and Morgenstern. The apparent parallel stems from the following consideration. A one-half chance of a supply sufficient to meet all demands for a_i is also a one-half chance of a deficient supply of a_i. Let us call a deficient

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1. See <u>The Theory of Games</u>, p. 26.

supply (a runout) of this item a_i'. Hence in "lottery ticket" notation our $.5a_i$ would be $(.5, a_i, a_i')$.

Now if a_i and a_i' were typical "prizes" in the context of the von Neumann and Morgenstern problem, one would evaluate such a lottery ticket as follows. Find probabilities p and q such that

$$a_i \ I \ (p, R, R')$$

and

$$a_i' \ I \ (q, R, R').$$[1]

Then from their discussion it would follow that

$$(.5, a_i, a_i') \ I \left[.5(p, R, R'), (q, R, R') \right],$$

whereas, on the contrary, our Assumption 3 says that

$$(.5, a_i, a_i') \ I \left[.5(p, R, R') \right].$$

Thus Assumption 3 seems implicitly to make

$$a_i' \ I \ (0, R, R').$$

It is perfectly true that such an implicit assumption is absurd when dealing with the sort of "prizes" ordinarily encountered in the context of discussions of the von Neumann-Morgenstern utility theory. Thus it is absurd to say that if a Ford is indifferent to a .5 chance of R, then a 50-50 chance of a Ford or an MG is indifferent to a (.5)(.5) chance of R. This treats the .5 chance of the MG as worthless. It is not absurd, however, to say that if a sufficient supply of x is indifferent to a .5 chance of R, then a 50-50 chance of a sufficient supply

1. See note 1 on p. 100 for definition of R'.

of x is indifferent to a (.5)(.5) chance of R. The plausibility of the assumption as applied to the inventory situation, if it be plausible, stems from the reflection that in evaluating a_i, the sufficient supply, one must implicitly take account of a_i', the deficient supply, whereas in evaluating a Ford against R, one does not implicitly or otherwise take account of the value of the MG.

However, it must be admitted that even though Assumption 3 is not absurd and to the present writer, at least, seems perfectly plausible, it is not certain that one would want to include it among the sine qua nons of "rational" behavior. In any event it is essential in getting the desired result and so we consider it.

Assumption 4: For any entities w, x, y, and z in our system, if w I x and y I z, then (w + y) I (x + z). This is another use of the idea of independent utilities encountered earlier in Chapters II and III. Here, as earlier, the assumption is far from harmless and in fact may seriously falsify the facts when there are strong complementarities among the different items.

Assumption 5:

$$\left[r_1(R) + \cdots + r_n(R)\right] \; I \; \left[(r_1 + \cdots + r_n)(R)\right]$$

To explore the meaning of Assumption 5 and of 4 and 5 taken together we consider an interpretation and an implication. First, imagine that there exists a drawing with 1000 numbers for an automobile. The acceptance of a single free ticket is equivalent to the acceptance of a "lottery ticket" where the winning prize (R) is getting the car, the losing prize is not getting it (R'), and the winning odds are 1/1000. Now suppose a person is offered a choice between a sure prize x and a certain number of free tickets to the drawing. Suppose the offer of 7 free tickets would make him just indifferent between the two alternatives. Now we withdraw this offer and make a similar offer of a certain number of tickets or a prize y and discover the number of free tickets indifferent to y. Suppose it is 3. Suppose also that 4 free tickets are indifferent to a prize z. Assumptions 4 and 5, then, say that a person should be indifferent between an offer of x and y and z, on the one hand, and (7 + 3 + 4) = 14 free tickets to the drawing, on the other.

Again, consider the following lottery ticket, (p, \$10,\$0). Suppose we find that the following indifference relations hold for some particular individual:

x I (.1,\$10,\$0)

y I (.2,\$10,\$0)

z I (.3,\$10,\$0).

From Assumptions 4 and 5 it follows that the individual is indifferent between (x + y + z) and (.6,\$10,\$0). In other words if x is indifferent to an expected value of \$1, y is indifferent to an expected value of \$2, and z, to an expected value of \$3, then the three items together are indifferent to an expected value of \$6.

Would or should such an indifference actually exist? It is clear at least that it need not be so under all circumstances. In any particular case the answer would depend in part upon the nature of the x, y, and z, and in part on one's reaction to the probabilities of getting the standard prize. These are distinct questions. Suppose, for example, to use the classic illustration, x, y, and z are three right shoes of different makes. In this case there may exist no a, b, and c, such that if x I a, y I b, and z I c, then (x + y + z) I (a + b + c). But even if the x, y, and z "satisfy different wants" so that they are neither substitutes nor complements of each other, it need not necessarily follow that one could sum up the probabilities of getting some other item to which each is indifferent, taken separately, and then assume that the whole collection must be indifferent to the sum of the probabilities in the way that we have done. Nevertheless the assumptions may hold under a fairly wide range of circumstances and since they help to get a result which would be nice to have, we explore them here.

Assumption 6: $p(R)$ is preferred to $p'(R)$ if and only if $p > p'$. This assumption is entirely harmless since it says merely that a larger chance of getting the prize R is preferred to a smaller chance. Note that R must of course be chosen so that getting R is preferred to not getting it, other things being equal.

Having completed our discussion of the assumptions we show how 1-6 can be used to compare any two inventory plans M and M' where

$$M = p_1a_1 + \ldots\ldots + p_na_n$$

and

$$M' = p_1'a_1 + \ldots\ldots + p_n'a_n.$$

(1) First find n probability numbers such that, for example,

$$a_i \text{ I } q_i(R).$$

This is made possible by Assumptions 1 and 2.[1]

1. The values of the q_i are the only "elementary preference information" required.

(2) It follows that $p_i a_i$ is indifferent to $p_i \cdot q_i(R)$, by Assumption 3.

(3) Next we have that

$$M \; I \; p_1 q_1(R) + \ldots\ldots + p_n q_n(R)$$

$$M' \; I \; p_1' q_1(R) + \ldots\ldots + p_n' q_n(R),$$

by Assumption 4. Now let

$$r_i = p_i \cdot q_i$$

and

$$r_i' = p_i' \cdot q_i .$$

(4) Then it follows that

$$M \; I \; (r_1 + \ldots + r_n)(R)$$

and

$$M' \; I \; (r_1' + \ldots + r_n')(R),$$

by Assumption 5.

(5) Finally, M is preferred to M' if and only if

$$\sum_{i=1}^{n} r_i > \sum_{i=1}^{n} r_i',$$

by Assumption 6.

Now let us define the military worth of any inventory plan M as follows:

$$MW(M) = u(p_1 a_1) + \ldots\ldots + u(p_n a_n).$$

We shall adopt the following conventions with respect to the utility numbers u:

(1) If x I y, $u(x) = u(y)$ and if x P y, $u(x) > u(y)$.

(2) Define $u(R) = k$, where $k > 0$.

(3) Define $u(p_i a_i) = p_i \cdot u(a_i)$.

(4) Define $u\left[q_i(R)\right] = q_i \cdot u(R)$.

We wish to show that the military worth of an inventory plan M is greater than the military worth of an inventory plan M' if and only if M p M', where the preference relation is established by application of Assumptions 1-6.

First find the values of the q_i such that a_i I $q_i(R)$.

Then $u(a_i) = u\left[q_i(R)\right]$, by convention (1).

But $u(a_i) = q_i \cdot u(R)$, by convention (4).

Also, the utility of the individual items in M is $u(p_i a_i) = p_i \cdot u(a_i)$, by convention (3).

By substitution we get $u(p_i a_i) = p_i \cdot q_i \cdot u(R)$.

We define $r_i = p_i \cdot q_i$, as before.

Therefore $u(p_i a_i) = r_i \cdot u(R)$.

But by definition $MW(M) = u(p_1 a_1) + \dots + u(p_n a_n)$.

Therefore

$$MW(M) = r_1 \cdot u(R) + \dots + r_n \cdot u(R).$$

Collecting terms we get

$$MW(M) = (r_1 + \dots + r_n) \cdot u(R).$$

Likewise, by similar reasoning

$$MW(M') = (r_1' + \dots + r_n')\cdot u(R).$$

But since $u(R) > 0$, by convention (2), $MW(M) > MW(M')$ if and only if

$$\sum_{i=1}^{n} r_i > \sum_{i=1}^{n} r_i'.$$

But we have already shown that M p M' if and only if the above inequality holds. Hence, $u(M) > u(M')$ if and only if M p M'.[1]

3. Evaluation of the Solution

It is obvious that it cannot be predicted with certainty that the above solution to the inventory problem will prove to be desirable or feasible in actual practice. Probably no *a priori* determination of its feasibility and desirability can be made. Practical experience would surely be needed, although a few of the relevant issues can be explored here. There seem to be two sorts of questions that would be natural to ask: (1) What potential advantages does such a system have as compared with evaluations arrived at by ordinary "competent judgment" of inventory alternatives? (2) How can the "accuracy" or "reliability" of the military worth judgments arrived at by the utility scheme be appraised?

As compared with more intuitive methods of evaluating inventory policies the present system has only two advantages, but they are

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1. Note that owing to the nature of the preference assumptions, only one arbitrary constant is involved in the construction of the index. The index is thus unique up to a factor of proportionality and .0(R) is the "natural" zero point.

important ones. In the first place the system does permit, to use Aumann and Kruskal's language, "separation of the combinatorial from the judgment" aspects of evaluation. The "judgment" aspect enters in when we ask authorities to evaluate the importance of having a sufficient supply of a given item as compared with a given probability of obtaining a "standard prize" If the six assumptions are valid, such comparatively simple matters of expert judgment can be combined to yield judgments about quite complex alternatives. The second advantage of the utility scheme is exactly analogous to the advantage the von Neumann-Morgenstern utility confers on the game theorist. In that theory the utility numbers permit the theorist to know the responses of the players to any conceivable probability combination of pure outcomes without the necessity of constant recourse to questioning of the players. Likewise here, the authorities responsible for inventory control can compute the relative merits of any pair of inventory policies that may be under consideration without the necessity of always having to call in the strategic authorities for an evaluation. Presumably those who would be most expert in weighing the relative importance of avoiding shortages in different items would not be those in charge of the routine operations of the logistics system. Since each runout probability could in principle range anywhere between zero and one, the range of possible inventory policies which could come under consideration is infinite. In view of these two essential facts the advantages of enabling the inventory authorities to compute the merits of all possible plans is obvious.

How "accurate" will the evaluations generated by the utility scheme under consideration be? Let us first consider the case where each of the six assumptions required to get the result is in fact deemed to be applicable to some particular military situation. In this case we are sure that the evaluations generated by the application of utility numbers will be "correct". An alternative to the use of the method of course always exists in simply asking some relevant authority whether or not, for example, plan M should or should not be preferred to plan M'. If the authority's answer is "correct" it will be the same as that given by the utility numbers of the two plans. If his answer is different, then his judgment must somewhere be in implicit violation of one or more of the six assumptions. One might say of this situation that the combinatorial complexities of the problem have been too great. It is in just such cases that it is desirable to "separate the combinatorial from the judgment problems".

On the other hand suppose one or more of the six assumptions is considered implausible in a particular context. In this case the utility numbers of a set of inventory positions $M_1, \ldots, M_n$ would in all likelihood not correspond to the "true" ranking of the alternatives. In such a case the ranking produced by direct questioning of a military "authority" might or might not be nearer to the "true" answer than that generated by the application of the utility numbers. If the rankings produced by the authority are simply <u>defined</u> to be the true ones, then of course it is trivial that the utility number rankings could only be worse. If no other <u>a priori</u> standard of "correctness" is provided, then a comparison between the computed and the intuitive judgment is meaningless. To the present author this type of dilemma demonstrates the great advantage of the axiomatic approach to military worth problems. If you can characterize any complex choice system by a set of simple axioms, then you have a meaningful way of checking the validity of your comparisons. Conversely, if the comparisons you make are derived from such a set of axioms, the rationale behind your decisions is always out in the open. You can check the validity of the judgment by inspection of the validity of the axioms.[1]

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1. There is, of course, always the possibility that a different and valid set of axioms would lead you to the same choices as invalid axioms. However, this seems unlikely in the present situation.

CHAPTER V

CONCLUSION

1. The Nature of the Military Worth Problem

Our examination of specific military choice problems is now complete and there remains only the task of summarizing our basic position on the general nature of the military worth problem and its solution. The approach taken in each of the specific situations analyzed in Chapters II-IV was animated by a consistent point of view on these matters. Since military worth research is an almost virgin field and since the general approach taken in this study may prove to be of more interest than any one specific result, it seems desirable to conclude by focusing attention on the nature of the approach advocated.

1. In the first place, the term "military worth" in the sense we have used it means nothing more or less than an index of the desirability of alternative military policies--whether the field of discussion be alternative allocations of given items, alternative stock levels or whatever. By the "desirability" of alternative policies we of course mean their purely military desirability abstracting from money cost considerations, space or other type of constraints. A military worth index is thus a guide to choice, a decision-making convenience. Hence in this sense military worth is exactly analogous to the general concept of "utility" as it is now coming to be understood, an index of choice whose properties may differ greatly in character from one decision-making situation to another.

In economics utility theorists have long since given up the nineteenth century pastime of debating the question of the "real" meaning of utility in quasi-philosophical or quasi-psychological terms. They have instead turned to the vastly more fruitful problem of characterizing different types of choice-making situations or preference structures in mathematical terms. The aim of this "modern" utility theory has of course been to describe and in some cases to prescribe actual choices. It is becoming increasingly realized that as the choice-making environment and the underlying structure of preferences assumed to exist changes, so must the "utility theory" associated with it.[1]

The result of this trend has been the development of a diverse collection of utility theories. The earliest of the "modern" utility theories was directed at the analysis of consumer behavior under certainty.[2] Because the preference axioms adopted for this situation were so limited the result was a purely "ordinal" theory and for a long time many economists argued that "utility" was an intrinsically ordinal, non-numerical concept. It was left to von Neumann and Morgenstern to show that in choice situations involving risk a numerical utility is not only necessary but in fact exists if only somewhat more extensive choice axioms are adopted.[3] More recently Luce and others have developed a still different type of "utility" for situations in which the probability of choosing an alternative x over an alternative

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1. See Armen A. Alchian, "The Meaning of Utility Measurement," American Economic Review, Vol. XLIII, No. 1 (March, 1953), pp. 26-50.

2. See for example Vilfredo Pareto, Manuel d'Economie Politique (Paris: 1907), p. 264.

3. Theory of Games, pp. 15-31.

y is assumed to be, in general, different from zero or one.[1]

The moral of this brief story for the military worth theorist is clearly that it is premature, if not entirely beside the point, to debate whether "military worth" is or is not a particular kind of "quantity". We have seen in Chapters III and IV that in some situations it is both possible and desirable to develop military worth indices which are in fact unique up to a linear transformation (Chapter IV) or even unique up to a multiplicative constant (Chapter III). On the other hand, Chapter II demonstrates that there are many important situations in which the appropriate choice axioms can lead at most to an ordinal index. Actually all of the military worth indices developed in this study have in common the fact that they give a numerical summary of the <u>order</u> of preference among the alternatives, although the numbers involved are in fact "cardinal" in the case of the Aumann and Kruskal index and in our adaptation of the von Neumann-Morgenstern index. The actual nature of the index developed in each case depends entirely upon the "richness" of the choice axioms posited. In both Chapters III and IV, for example, it is meaningful to add military worth numbers under some circumstances because this addition has a "natural" correspondence in the structure of preferences. Where the circumstances of the case do not permit such a rich set of preference axioms no such additivity is possible. In any case we have explored the implications of different possible assumptions about the underlying preference structures in the belief

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1. See R. Duncan Luce, <u>Individual Choice Behavior</u> (New York: John Wiley, 1959), p. 2.

that some practical situations will permit the assumption of a rich set of preference axioms and others will not.[1]

2. Since a military worth index is basically an ordering of alternatives, military worth becomes a "problem" only in situations where ordinary "hand" methods of ranking alternatives prove inadequate and it appears that more systematic methods can offer some chance of improvement. Any military worth problem of the type we have considered can always be "solved" simply by appointing a military authority to order all alternatives of interest in accordance with his best military judgment. Any military worth index does no more than provide such an ordering.[2] Nevertheless, as we have seen throughout the course of this study, there may be many reasons why it is desirable to replace the "hand" solution by a systematically constructed military worth function. (1) It may be desirable to take advantage of a kind of division of labor among experts. Thus different individuals may order the elements of different subsets of the over-all set of alternatives with which they are particularly familiar. In this case a systematic method of amalgamating the component orderings into an over-all ordering is needed.[3] (2) Hand solutions may be inadequate if it is thought that the complexity of the alternatives involved is too great for meaningful judgment, even by experts. In

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1. See the discussion of the principles of measurement in the Theory of Games, particularly pp. 20-21.

2. There is of course nothing paradoxical about the fact that even cardinal military worth functions "merely" order the alternatives. Their cardinality results from the fact that somewhere in the preference structure one has assumed that one can combine preferences in a certain way to get other preferences.

3. See above, pp. 19-20, 87-93

this case it may be possible to systematically combine preferences among small-scale alternatives so as indirectly to obtain preferences for the complex ones. In such situations a true military worth "theory" is both desirable and possible.[1] (3) Finally, numerical military worth techniques may be desirable to facilitate routine computation of optimum alternatives within whatever constraints are present on the part of logistics personnel.[2]

As we have seen repeatedly, even in situations where a systematic "military worth" solution to the problem of ranking alternatives may be desirable, it may be impossible to obtain an adequate solution for any one of several reasons. Chief among these is the inability to find a sufficiently rich and at the same time meaningful set of choice assumptions. No doubt many, though perhaps not the majority of military choice problems will always be solved in the old-fashioned ways--a remark which should not be taken to minimize the potential usefulness of military worth theory. The amounts of money involved in military decisions are so huge that even relatively minor improvements would pay for the cost of military worth research many times over.[3]

3. The only really satisfactory military worth theory starts with the explicit statement of a set of choice axioms applicable to the problem at hand. Ideally the axioms should be sufficiently rich to generate preferences among all alternatives of interest.

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1. See above, pp. 67-71, 97, and 108.

2. See above, p. 97.

3. For a particularly forceful statement of this point of view see O. Morgenstern, The Question of National Defense, pp. 203-205.

As we have seen, in the type of problem in which we are interested the first step in the computation of an actual military worth function is the collection of preferences among some alternatives by direct questioning of a military authority. These preferences are then combined or extended according to certain rules to obtain preferences over other, usually more complex alternatives. Thus our choice axioms do two things: (1) They state the kind of preferences which can be obtained directly. (2) They state the ways in which these preferences can be combined to yield other preference relations. The choice of axioms in a particular situation will of course depend in part upon expediency and in part upon questions of "naturalness" or "intuitive acceptability".[1]

The reason for advocating an axiomatic approach is simple, but basic. Military worth devices serve a normative purpose in the following sense: they tell us in a given situation which of two alternatives is to be preferred. In view of this fact there must be a test as to whether the choices actually ground out by the mechanism are in fact the "correct" ones. It is of course possible to determine the "correctness" of the choices simply by checking the answers ground out by the military worth mechanism against the good judgment of authorized military authorities. But as we have noted above, in the situations in which military worth systems are most likely to be needed this check is either impossible or unreliable. If such a direct test were entirely possible and reliable there would be little need for the construction of a systematic method for computing military worth

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1. See The Theory of Games, p. 25.

in the first place. This dilemma provides the basic justification for the introduction of an axiomatic approach. For if we can show that the preferences ground out are the logical consequence of the "elementary" preferences obtained by direct questioning and the axioms for their manipulation, then to check the "correctness" of the derived preferences we need merely check the acceptability of the axioms which produced them.[1] In the absence of an explicit set of choice axioms it is of course always possible to invent an ad hoc military worth scheme which will "tell" whether one alternative is or is not to be preferred over another. The only trouble is that in this case one has no idea why the given alternative should or should not be preferred.

4. Ideally speaking a military worth function is a set of numbers applied to alternatives which have the property that the military worth of one alternative is greater than that of another if and only if the first is preferred to the second by some axiomatic criterion of rational preference. As we have seen, this is often too much to expect. But at least the minimum requirement of a military worth function is that the military worth of one alternative be greater than that of another if it is preferred. In other words, preference must at least be a sufficient condition for one alternative to have a higher indexed value than another. With the military worth index constructed in Chapter IV preference is both a necessary and a sufficient condition for this. For the Aumann and Kruskal index and the index associated with Rule 2, however, it is only

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1. See also above, pp. 67-71.

a sufficient condition.

2. The Place of "Ad Hoc" Solutions to Military Worth Problems

Under ideal circumstances the set of preference axioms should be sufficiently rich so that a preference relation can be obtained over any pair of alternatives which may be of potential interest. And of course under ideal circumstances preference of one alternative over another would be both necessary and sufficient to ensure that its military worth value were higher. As we have seen, however, these ideal circumstances are frequently impossible or at least difficult of attainment. We have repeatedly run into situations where our choice assumptions were insufficient to determine whether some particular x was or was not preferred to some particular y. Among the most important instances of this type of difficulty have been (1) failure to obtain complete orders on the a_{ij} from priority and efficiency information[1], (2) failure of Rule 2 to provide a basis for comparison of all pairs of complex over-all assignment plans, and (3) the similar failure of the Aumann and Kruskal military worth procedure and the Aumann choice axioms.

It is of course true that even in cases where the choice axioms are insufficient to determine preference for all pairs of alternatives a choice must in practice somehow be made. This implies that some kind of "ad hoc" method must be developed for determining whether x or y shall be picked when x and y are incomparable in terms of the given elementary preference information and the combinatorial axioms.

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1. See above, pp. 21-24.

But if there are many situations where it will be necessary to resort to some type of ad hoc procedure it may also be important to find a way of distinguishing good ad hoc procedures from bad.

With the exception of one item to be noted in a moment there has been no discussion of possible ad hoc procedures in this paper or of the relative merits of alternative procedures in different situations. This omission has been deliberate and is primarily explained by two considerations. (1) The aim of the present work has been to show the possibilities and limitations of the axiomatic approach. (2) It is our conviction that discussion of ad hoc procedures in the abstract is a rather pointless activity. Only in the presence of a particular real-life problem would such a discussion be meaningful. The real-life problem may present preference information peculiar to itself which is not conveniently axiomatized and it may present particular requirements of convenience, etc., which are not easily forseen.

Actually, however, both the military worth functions asssociated with Rule 2 and the index of the Aumann and Kruskal method are "ad hoc" in a certain restricted sense, for they can be used to make choices between elements that are incomparable in terms of the underlying preference structures from which they are derived. Since preference is only a sufficient condition for a higher military worth number in these cases, $u(x)$ may be greater than $u(y)$ even though x and y are incomparable. Hence with respect to such a pair of alternatives the rule that you should always maximize military worth represents an ad hoc solution in so far as choices between incomparables are concerned. Nevertheless, in the case of both the aforementioned military worth functions, one can at least be sure that

the alternative which maximizes military worth is not inferior to any other feasible alternative. Of course in any collection of feasible alternatives there may be a large number which are "Pareto optimal" with respect to the remainder. The only justification for picking from among these that alternative which happens to maximize the military worth function is sheer mechanical convenience.
In a particular concrete application there might be additional considerations which would further narrow the choice, but such considerations cannot be anticipated in a paper designed to deal with the general problem.

3. The Future of Military Worth Research

If the general line of approach developed in this study on the nature of military worth problems and their most appropriate means of solution is correct, then it would seem that future research in the military worth field should proceed simultaneously along several fronts. First of all there should be a continuing search for different sets of choice axioms or combinatorial rules, hopefully with richer content than the ones unearthed here. Many possibilities should be explored in view of the fact that a set of assumptions which is not plausible or intuitively acceptable as a criterion of "rational" choice in one situation may turn out to work in another. This is obviously primarily a job for the mathematicians.

Secondly, the military logistics personnel themselves should search out situations where the ranking of alternatives by ordinary "hand" solutions seems to be unsatisfactory and where there is some hope that sufficiently "rich" and intuitively acceptable choice rules can be found.

Finally there is a need to test out on real-life problems the workability of the military worth rules developed in this study and the additional rules which we can hopefully expect from the mathematicians in future periods. It would be interesting to see, for example, whether the procedure suggested in Chapter IV or some alternative version of it has practical usefulness. The same applies to uses of the Aumann and Kruskal solution beyond the specific situation for which it was designed.

It is encouraging to note that the Navy is now in the process of developing a "military essentiality coding program" which, roughly speaking and in the language of Chapter II, is an attempt to place in efficiency and priority categories a very large number of "activities" and "models" with which the naval supply system is concerned. However, as the analysis of Chapter II indicates, this is at best a necessary preliminary to obtaining information sufficient really to help solve allocation problems.

In any event, the future of military worth research, if pursued on a large enough scale and with the active cooperation of mathematicians and personnel experienced in concrete logistics problems, seems both intellectually exciting and highly promising from the dollars-and-cents point of view.

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A B S T R A C T

This study seeks to help fill the gap in military decision-making theory created by the absence of a monetary measure of the revenue of alternative "outputs". The study concentrates on situations where no simple physical measure of the value of alternatives --such as "expected kill"--exists, but rather where the aim is to maximize some subjectively determined concept of over-all military "effectiveness".

The basic approach used throughout the discussion begins with the axiomatization of the nature of the elementary preference information that can be obtained from competent military authorities concerning certain simple alternatives. Various other axioms are then considered which permit the manipulation of the elementary preference information to obtain preference relations among more complex alternatives. In each case a numerical "military worth function" is developed which has the property that preference for alternative x over alternative y is a sufficient and, in some cases, both a necessary and a sufficient condition for the military worth of x to be greater than the military worth of y. Maximization of such a military worth function under given constraints would thus assure that the maximizing alternative is at least as good as any other alternative when preferences are "rational"--rationality being defined by consistency with the preference axioms.

In each case the appropriateness of the preference axioms is considered and there is some discussion of the circumstances under which the systematic methods advocated would be superior to more "naive" techniques.

Chapter I defines in detail the nature of the problem to be solved and sets forth the methods to be followed. Chapters II and III deal with the problems of ordering alternative assignments of different types of "models" to different types of "activities"--where these terms are defined in a somewhat special sense. In Chapter II it is assumed that the elementary preference information contains only rankings of the suitability of the models for given activities and of the military importance of the activities. Cases where the technological efficiencies of the models and the military priorities of the activities can be considered separately are distinguished from cases where they cannot. In each case the possibilities and limitations of using such information to obtain rankings of over-all assignment plans are examined. A military worth function which always picks a "Pareto-optimal" alternative from among feasible alternatives is developed. There is some discussion of the appropriate form and interpretation of priority lists as instruments designed to convey elementary preference information.

Chapter III, also dealing with the assignment problem, discusses cases where the elementary preference information contains what is there called a "difference order", in addition to the information considered in Chapter II. A technique devised by Aumann and Kruskal for the development of a numerical military worth function unique up to a factor of proportionality is examined. The nature of the preference structure which would "validate" their procedure is set forth. Finally, a situation is examined where the elementary preference information is derived from different individuals. Some techniques for amalgamating these preferences are considered.

Chapter IV deals with the problem of ranking alternative inventory positions where there are many items to be stocked and many inventory points involved. Each inventory plan is defined in terms of the vector of runout probabilities associated with it and the problem is to rank these vectors. Again the existence of certain elementary preference information is assumed and a technique for manipulating these preferences to obtain rankings of the vectors with the aid of six preference axioms is derived. A military worth function where preference is both a necessary and sufficient condition for one alternative to have a higher computed value than another is developed. The entire procedure in this chapter bears some similarity to the well-known von Neumann-Morgenstern utility theory.

The concluding chapter is devoted to summarizing the author's view of the real nature and appropriate means of solution of the military worth problem. Some parallels with the problem of utility measurement are drawn and suggestions for extension of the research are made.