Optimal Policies for Structural Models

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This paper presents a method for approximating optimal policies for a dynamic system of non-linear structural equations and, using an econometric model estimated by the author, applies the method to study the problem of controlling inflation and unemployment. Discussion of many important issues in deterministic control theory has generally assumed that the model is represented by a set of reduced-form equations, with each of n dependent variables in time t given by a function of the form:

$$x_t^i = f_t^i(x_{t-1}, u_t) \qquad i = 1,\dots, n \tag{1}$$

where u_t is the m-dimensional policy or control vector. See, e.g., Canon, Cullum and Polak (1970) and Athans (1972). Many non-linear models of aggregate economic activity, however, are not available in such a reduced-form. Instead they appear as a set of simultaneous equations, exemplified by the structural model:

$$x_t^i = f_t^i(x_{t-1}, u_t, \hat{x}_t^i) \qquad i = 1,\dots, n \tag{2}$$

where $\hat{x}_t^i$ is a subset of the state vector at time t not including the i^{th} component.

This paper is concerned with the characterization and computation of optimal policies for non-linear structural models. It is divided into two parts, one elaborating methods for dealing with such models and the other reporting on an application of the methods to the problem of choosing aggregate economic policies for the American economy when concern is directed towards reducing both inflation and unemployment. The first section establishes criteria for the existence of a reduced-form model "locally-equivalent" to the original structural model. The existence and properties of this equivalent model underlies much of the subsequent discussion. The second section shows how the Lagrangian method of characterizing an optimal policy for a reduced-form model can be amended for structural models. It is shown how application of the results of Section 1 will preserve the interpretation of the Lagrangian multipliers or costates for the more general case. The third section presents an algorithm for approximating an optimal policy for structural models based on linear/quadratic control theory. Some special difficulties with this algorithm, which do not arise when it is applied to reduced-form models, are noted. These methods of characterization and computation were developed to permit analysis of the inflation/unemployment problem in macroeconomic policy-making. Using a non-linear structural model of the post-war American economy and a loss function to measure economic performance the fourth section discusses the computed optimal policies and implications of those policies for aggregate activity. Computational experience with the model and loss function is reported in Section 5. The results of this paper indicate that in theory as well as in computation, optimal control of structural models can be addressed within the framework developed for reduced-form models. While these results do not extend the theoretical frontier of optimal control, they do show how the existing theory can be applied to a class of model frequently used by economists.

1. The Locally-equivalent Reduced-form Model

In this section we will show that, under a mild assumption on the non-singularity of a Jacobian of the structural model, there exists a set of reduced-form functions which imply the same dynamic behavior as the original simultaneous equations. Consider, in a slightly more general form, the structural functions which define the state at time t given the state in the previous period and the current choice of policy:

$$F_t(x_{t-1}, u_t, x_t) = 0 \tag{3}$$

We assume that the structural model is continuously differentiable. Given $x_{t-1} = \bar{x}_{t-1}$ and $u_t = \bar{u}_t$, state $\bar{x}_t$ is realized at time t if $x_t = \bar{x}_t$ is a solution to equation (3). Suppose $\bar{x}_t$ is such a solution. Let p denote the rank of the n x n Jacobian matrix $d_3F_t(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t)$, i.e., the Jacobian of F_t with respect to the components of the state vector x_t evaluated at $(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t)$. Clearly $p \leq n$. For some sufficiently small open neighborhood of $\bar{x}_t$, $M(\bar{x}_t)$, we have:

$$\operatorname{rank}(d_3F_t(\bar{x}_{t-1}, \bar{u}_t, x)) \geq p \text{ for all } x \in M(\bar{x}_t) \tag{4}$$

As x varies in $M(\bar{x}_t)$ the set of p vectors which were linearly independent at $\bar{x}_t$ must remain linearly independent since the partial derivatives in the entries of the Jacobian are continuous functions of x. Thus the rank cannot decrease in $M(\bar{x}_t)$. It is possible, however, that a vector in the Jacobian which could be represented as a linear combination of the p linearly independent set at $\bar{x}_t$ can not be so represented at some other $x \in M(\bar{x}_t)$, so the rank may increase. The weak inequality of (4) is resolved by the occurance of one of two possibilities. Either the rank remains constant:

1. $\operatorname{rank}(d_3F_t(\bar{x}_{t-1}, \bar{u}_t, x)) = p$ for all $x \in M(\bar{x}_t)$

or else it increases at points arbitrarily close to $\bar{x}_t$:

2. for every open set $S \subset M(\bar{x}_t)$ with $\bar{x}_t \in S$, there exists a state $x_S \in S$ such that $\operatorname{rank}(d_3F_t(\bar{x}_{t-1}, \bar{u}_t, x_S)) > p$.

If $p = n$, i.e., if d_3F_t is non-singular at the point $(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t)$, then possibility 2 is excluded, since $\text{rank}(d_3F_t) \leq n$ everywhere. In this case by the Implicit Function Theorem (Dieudonne (1969, pg.270)), there exists an open neighborhood $N(\bar{x}_{t-1}, \bar{u}_t)$ of $(\bar{x}_{t-1}, \bar{u}_t)$, an open neighborhood $N'(\bar{x}_t)$ of $\bar{x}_t$ and a differentiable function f_t:

$$f_t : N(\bar{x}_{t-1}, \bar{u}_t) \rightarrow N'(\bar{x}_t) \tag{5}$$

such that the state x_t given by:

$$x_t = f_t(x_{t-1}, u_t) \tag{6}$$

is a solution of the structural model for all $(x_{t-1}, u_t) \in N(\bar{x}_{t-1}, \bar{u}_t)$. In particular $f_t(\bar{x}_{t-1}, \bar{u}_t) = \bar{x}_t$. Thus if $d_3F_t(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t)$ is non-singular the structural model has, locally, a reduced-form representation at time t.

While we are typically unable to obtain close-form expressions for the component functions of f_t some of their properties are available. In particular the elements of the partitioned Jacobians of f_t evaluated at $(\bar{x}_{t-1}, \bar{u}_t)$ are given by:

$$d_if_t = -(d_3F_t)^{-1} (d_iF_t) \qquad i = 1, 2 \tag{7}$$

where d_3F_t and d_iF_t are evaluated at $(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t)$.

Suppose we have a policy sequence, $\bar{U} = (\bar{u}_1, \ldots, \bar{u}_T)$, an assigned initial state $\bar{x}_o$, and a state trajectory $\bar{X} = (\bar{x}_o, \bar{x}_1, \ldots, \bar{x}_T)$ such that:

$$F_t(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t) = 0 \qquad t = 1, \ldots, T \tag{8}$$

and such that $d_3F_t(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t)$ is non-singular for $t = 1, \ldots, T$. Then a simple extension of the above shows there is a reduced-form model which is equivalent to the structural model for all policy sequences in a neighborhood of $\bar{U}$ and for all initial states in a neighborhood of $\bar{x}_o$. The state trajectory is, at least locally, uniquely determined by the choice of $\bar{x}_o$ and $\bar{U}$. It is in this case of a local reduced-form representation that the results developed for the optimal control of reduced-form models can be applied to structural models. In an appendix we discuss briefly the consequences of $p < n$.

2. Characterization of Optimal Policies

A familiar technique in characterizing an optimal policy sequence is to approach the optimal policy problem as a programming problem with equality constraints, and then to apply the Lagrange Multiplier Theorem (Canon, Cullum and Polak (1970, pg. 51)). We show in this section that, by using the results of Section 1, this technique may be applied to characterizing optimal policies for structural models in a way that preserves the interpretation of the Lagrangian multipliers. For expositional purposes we treat here only the simple additive loss function on the state trajectory :

$$L(X) = \sum_{t=1}^{T} f_t^o(x_t) \tag{9}$$

where f_t^o is a scalar-valued function of the state at time t. More complex single period loss functions, e.g., $f_t^o(x_t, u_t)$, can be treated either directly or by augmenting the state vector (Chow (1970)).

Consider first the selection of a policy sequence $U = (u_1, \ldots, u_T)$ to minimize $L(X)$ subject to a reduced-form model and an assigned initital state:

$$\min_U \sum_{t=1}^{T} f_t^o(x_t) \tag{10a}$$

$$\text{subject to: } x_t = f_t(x_{t-1}, u_t) \qquad t = 1, \ldots, T \tag{10b}$$

$$x_o = \bar{x}_o \tag{10c}$$

The choice of U is assumed unconstrained and allowed to take on any value in a real vector space of dimension Tm. This optimal policy problem may be viewed as a problem in choosing an (X, U) pair to minimize the loss function subject to the equality constraints of (10b,c). Forming the Lagrangian function $\mathcal{L}$ for this programming problem:

$$\mathcal{L}(X,U) = \sum_{t=1}^{T} f_t^o(x_t) + \left\langle p_o \mid x_o - \bar{x}_o \right\rangle + \sum_{t=1}^{T} \left\langle p_t \mid x_t - f_t(x_{t-1}, u_t) \right\rangle \tag{11}$$

where $P = (p_o, p_1, \ldots, p_T)$ are the multipliers or costates, we know that if U^* is a solution to problem (10) and X^* is the associated state trajectory then (X^*, U^*) is a stationary point for $\mathcal{L}$ (Canon, Cullum and Polak (1970, pg. 51)). This implies the gradients of $\mathcal{L}$ with respect to $(x_o, x_1, \ldots, x_T)$ and $(u_1, \ldots, u_T)$ all vanish at the point (X^*, U^*). The condition on the state gradients leads to the linear dynamic system for the costates:

$$p_T = \nabla_X f_T^o(x_T^*) \tag{12a}$$

$$p_t = (d_1 f_{t+1}(x_t^*, u_{t+1}^*))' p_{t+1} + \nabla_X f_t^o(x_t^*) \qquad t = T-1, \ldots, 1 \tag{12b}$$

$$p_o = (d_1 f_1(x_o^*, u_1^*))' p_1 \tag{12c}$$

while the condition on the policy gradients leads to:

$$0 = (d_2 f_t(x_{t-1}^*, u_t^*))' p_t \qquad t = 1, \ldots T \tag{12d}$$

The costate trajectory can be readily computed from (12a,b,c) given the gradients of the loss function and the state Jacobians of the reduced-form model. As is well-known (Zangwill (1969)) the costate p_t has the interesting interpretation as the marginal change in minimum loss-to-go with respect to a change in state at time t.

We now show that equation (12d) has a particularly important and familiar interpretation. If we make repeated substitutions of the reduced-form model we can obtain an expression for the state at time t as a function only of the initial state and the elapsed policy choices:

$$x_t = \tilde{f}_t(U^t, \bar{x}_o) \qquad U^t = (u_1, \ldots, u_t) \tag{13}$$

Substituting this expression for x_t in the definition of loss, equation (9), we obtain the final-form loss function:

$$\tilde{L}(U, \bar{x}_o) = \sum_{t=1}^{T} f_t^o(\tilde{f}_t(U^t, \bar{x}_o)) \tag{14}$$

where loss now depends directly on the given initial state and chosen policy sequence. If U* is the solution to problem (10) the usual first-order necessary conditions imply $\nabla_{u_t} \tilde{L}(U^*, \bar{x}_o) = 0$ for $t = 1, \ldots, T$. Polak (1971, pg. 67) shows that for any choice of policy, not necessarily optimal:

$$\nabla_{u_t} \tilde{L}(U, \bar{x}_o) = -\nabla_{u_t} \mathcal{L}(X, U) \qquad t = 1, \ldots, T \tag{15}$$

$$= -(d_2 f_t(x_{t-1}, u_t))' p_t$$

where X is the state trajectory associated with the pair $(\bar{x}_o, U)$ by model (10b) and the costates are computed by (12a, b, c) at the given (X, U). Equation (12d) is thus the usual first-order condition on an optimal policy. Equation (15) reveals that by using the costates we can obtain the gradient of final-form loss from the Jacobians of the model and gradients of the definitional loss function, and do not require an analytic expression for $\tilde{L}$. This result is important for implementing the computational algorithm of Section 3.

Consider now minimization of the same loss function subject to the constraint of a structural model:

$$\min_{U} \sum_{t=1}^{T} f_t^o(x_t) \tag{16a}$$

$$\text{subject to: } F_t(x_{t-1}, u_t, x_t) = 0 \quad t = 1, \ldots, T \tag{16b}$$

$$x_o = \overline{x}_o \tag{16c}$$

Suppose U* is the optimal policy for this problem, with associated state trajectory X*. One approach to characterizing U* is to view the optimal policy problem as the programming problem of selecting a pair (X, U) to minimize the loss function subject to the equality constraints of (16b,c). However, if we follow the same steps that we followed for the reduced- form problem, setting up the Lagrangian with the explicit equality constraints, taking the gradients, and setting them to zero, we will lose the interpretation we were formerly able to place on the costate trajectory.

If $d_3F_t(x_{t-1}*, u_t*, x_t*)$ is non-singular for all t we can characterize an optimal policy by applying the results of Section 1. Suppose N(U*) is an open neighborhood of U* and $M(\overline{x}_o)$ is an open neighborhood of $\overline{x}_o$ within which some reduced-form model $\{f_t\}$ is locally equivalent to the structural model $\{F_t\}$ in (16b). We may not have analytic expressions for the f_t functions but we do know they exist and can compute their Jacobians. Since U* solves the unrestricted problem (16) it must also solve the restricted problem:

$$\min_{U \in N(U^*)} \sum_{t=1}^{T} f_t^o(x_t) \tag{17a}$$

$$\text{subject to:} \quad x_t = f_t(x_{t-1}, u_t) \qquad t = 1, \ldots, T \tag{17b}$$

$$x_o = \bar{x}_o \tag{17c}$$

where f_t is the locally equivalent model. Were we to restrict the domain of U to N(U*) in problem (16) it would be identical to problem (17) since the model constraints are locally identical in the two problems.

Problem (17) may be placed in the Lagrangian structure of (11) exactly as we did with problem (10), yielding equation (12) as the characterization of the optimal policy sequence. To interpret (12) for the given structural model we need only replace the Jacobian of the implicit reduced-form model in (12) with their equivalent expression from the Jacobians of the structural model by using (7). Once this replacement is completed the costates retain the interpretation as the marginal sensitivity of minimum loss-to-go with respect to a change in state. The fact that N(U*) and $M(\bar{x}_o)$ are open neighborhoods gives us the "room" to conduct the variational analysis required by the Lagrange Multiplier Theorem in characterizing the solution to a programming problem with equality constraints.

For arbitrary policy sequences equation (15) generalizes to:

$$\nabla_{u_t} \tilde{L}(U, \bar{x}_o) = ((d_3F_t(x_{t-1}, u_t, x_t))^{-1}(d_2F_t(x_{t-1}, u_t, x_t)))'p_t \tag{18}$$

The gradient of the implicitly defined final-form loss function can thus be recovered for structural models when d_3F_t is non-singular.

3. Computation of Optimal Policies

In this section we consider a method for computing a policy sequence which, together with its state trajectory, will satisfy equation (12). In the linear/quadratic case (12) can be solved algebraically for the optimal policy (Chow (1972a)), but in more general cases an algebraic solution is not available. While there are many algorithms for computing optimal sequences we restrict our attention here to one based on linear/quadratic control theory, noted by Polak (1971, pp. 69-71). After describing the application of the algorithm to the optimal control of reduced-form models we show how, again using the results of Section 1, it can be applied to structural models.

Consider the problem of computing an optimal sequence for problem (10). Given some tentative sequence U and associated state trajectory X, if we alter U by an increment $\Delta U = (\Delta u_1, \ldots, \Delta u_T)$ the altered state trajectory $X + \Delta X$ must still satisfy the model. Hence:

$$x_t + \Delta x_t = f_t(x_{t-1} + \Delta x_{t-1}, u_t + \Delta u_t) \tag{19}$$

$$\text{with } \Delta x_o = 0$$

Expanding the right-hand-side of (19) about (x_{t-1}, u_t) we obtain:

$$x_t + \Delta x_t = f_t(x_{t-1}, u_t) + (d_1 f_t)\Delta x_{t-1} + (d_2 f_t)\,\Delta u_t + o(\Delta x_{t-1}, \Delta u_t) \tag{20}$$

where $o(\Delta x_{t-1}, \Delta u_t)$ are the high order terms in the expansion. The Jacobians are evaluated at (x_{t-1}, u_t). Since $x_t = f_t(x_{t-1}, u_t)$ equation (20) becomes:

$$\Delta x_t = (d_1 f_t)\Delta x_{t-1} + (d_2 f_t)\Delta u_t + o(\Delta x_{t-1}, \Delta u_t) \tag{21}$$

It follows that a first-order approximation, δx_t, to Δx_t obeys the model:

$$\delta x_t = A_t \delta x_{t-1} + B_t \Delta u_t \tag{22a}$$

$$\delta x_o = 0 \tag{22b}$$

where $A_t = d_1 f_t(x_{t-1}, u_t)$ and $B_t = d_2 f_t(x_{t-1}, u_t)$. Equation (22) is a linear model providing a first-order approximation to the change in state consequent upon a change in the tentative policy sequence.

Consider next the effect on the value of loss of changing the state trajectory from X to X + ΔX. The change in loss, ΔL, is:

$$\Delta L = \sum_{t=1}^{T} f_t^o(x_t + \Delta x_t) - \sum_{t=1}^{T} f_t^o(x_t) \tag{23}$$

and a second-order approximation, $\delta_2 L$, to ΔL is given by:

$$\delta_2 L = \sum_{t=1}^{T} (\tfrac{1}{2}\Delta x_t{}' K_t \Delta x_t + k_t{}' \Delta x_t) \tag{24}$$

where k_t is the gradient of f_t^o and K_t the Hessian, both evaluated at x_t. The definiteness of K_t is generally unknown but if f_t^o is a convex function then K_t is positive-semi-definite, and if f_t^o is strictly convex then K_t is positive-definite (Nikido (1968, pg.50)). Equation (24) provides a second-order approximation to the change in loss consequent upon a change in the tentative state trajectory.

Starting from the tentative policy sequence we now ask for that change in policy which will minimize the second-order approximation to the change in loss subject to the constraint of the first-order model of equation (22). That is, we seek to solve the linear/quadratic problem:

$$\min_{\Delta U} \sum_{t=1}^{T} (\tfrac{1}{2}\delta x_t{}' K_t \delta x_t + k_t{}' \delta x_t) \tag{25a}$$

$$\text{subject to: } \delta x_t = A_t \delta x_{t-1} + B_t \Delta u_t \tag{25b}$$

$$\delta x_o = 0 \tag{25c}$$

This well-known problem (Chow(1972a)) has an exact solution given by:

$$\Delta u_t^* = G_t \delta x_{t-1}^* + g_t \qquad t = 1, \ldots, T \tag{26a}$$

with

$$\delta x_t^* = (A_t + B_t G_t) \delta x_{t-1}^* + B_t g_t \qquad t = 1, \ldots, T \tag{26b}$$

$$\delta x_o^* = 0 \tag{26c}$$

with feedback matrix G_t and forcing vector g_t:

$$G_t = - (B_t' H_t B_t)^{-1} B_t' H_t A_t \tag{27a}$$

$$g_t = - (B_t' H_t B_t)^{-1} B_t' h_t \tag{27b}$$

and matrix and vector Ricatti equations:

$$H_{t-1} = K_{t-1} + (A_t + B_t G_t)' H_t (A_t + B_t G_t) \tag{27c}$$

$$H_T = K_T$$

$$h_{t-1} = k_{t-1} + (A_t + B_t G_t)' h_t \tag{27d}$$

$$h_T = k_T$$

If for some tentative policy sequence we find $\Delta u_t^* = 0$ for all t then (26a, b, c,) imply that g_t must be zero for all t. But then, by (27b), we have $B_t' h_t = 0$ as well and, from (27d), $h_{t-1} = k_{t-1} + A_t' h_t$. Comparing this dynamic equation for h_t with that of p_t in (12a, b, c,) we see that $h_t = p_t$ for all t. But then $B_t' h_t = 0$ is just (12d) and we conclude that the tentative sequence solves the first-order necessary conditions for an optimal policy.

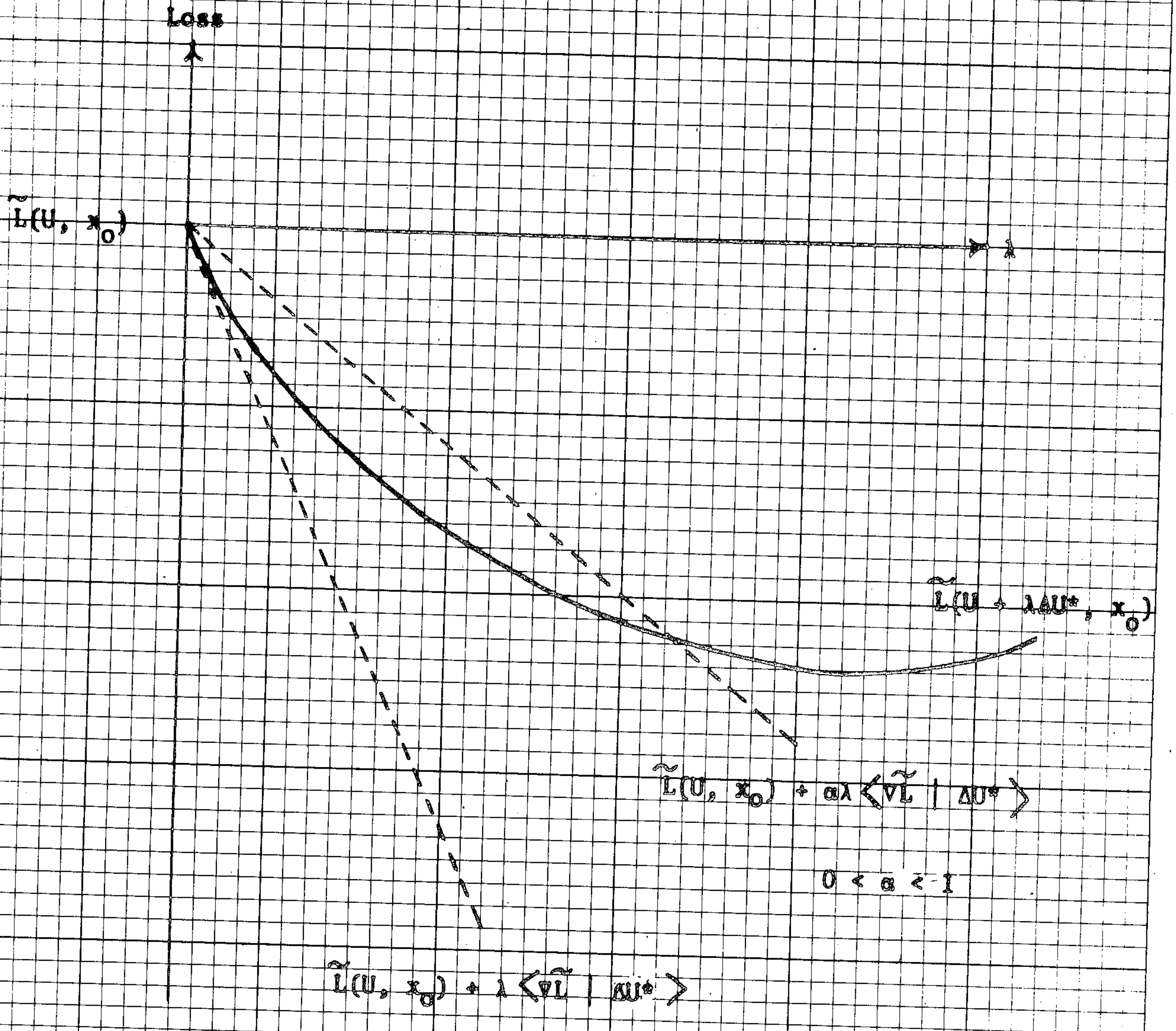
In more general cases, when $\Delta U^* \neq 0$, we wish to increment the tentative sequence U in the direction of ΔU^*. In order to speed convergence we can appeal to a linear search algorithm to obtain the value of λ such that $U + \lambda \Delta U^*$ minimizes the value of the original loss function (9) subject to the non-linear reduced-form model (Polak (1971, chapter 2)). The process of computing the increment and optimizing the step-size is iterated until a test for convergence is successful. As long as the Hessian matrices of L are positive-semi-definite this algorithm will converge to a policy sequence which satisfies the first-order conditions for optimality. If some K_t has negative eigenvalues those eigenvalues can be set to their absolute values to avoid computing a sub-optimal increment to policy in equation (26). (Greenstadt (1967)).

Armijo (1966, noted in Polak (1971, pg. 36)) has suggested a linear search algorithm which we have found both informative and efficient. Choosing an integer N (usually less than zero) and constants $\alpha \in (0,1)$, $\beta \in (0,1)$, the integer n is first set to N and then incremented until the test:

$$\tilde{L}(U + \beta^n \Delta U^*, x_o) \leq \tilde{L}(U, x_o) + \alpha\beta^n \left\langle \nabla\tilde{L}(U, x_o) \mid \Delta U^* \right\rangle \tag{28}$$

is satisfied (cnf. Figure 1). Since $\left\langle \nabla\tilde{L}(U, x_o) \mid \Delta U^* \right\rangle < 0$ when $\Delta U^* \neq 0$ this procedure bounds $\tilde{L}(U + \beta^n \Delta U^*, x_o)$ below $\tilde{L}(U, x_o)$ by some finite amount for all sufficiently large finite n. We denote the first satisfying value of $n \geq N$ by n^s. $\alpha < 1$ guarantees the incrementing of n necessarily terminates for some finite n^s so that $\tilde{L}(U + \beta^{n^s} \Delta U^*, x_o)$ is a finite improvement over $\tilde{L}(U, x_o)$. Note that the test requires knowledge of the gradient of final-form loss, which may be computed from equation (15), demonstrating the usefulness of the costate trajectory in computation as well as in characterization of an optimal policy sequence. The new tentative policy is set to that tested sequence which gave minimum loss, i.e. to $U + \beta^{n^*} \Delta U^*$

Figure 1. Step-size Subprocedure of Armijo



where n* is such that:

$$L(U + \beta^{n^*} \Delta U^*, x_o) = \min_{n \in [N, \ldots, n^s]} L(U + \beta^n \Delta U^*, x_o) \tag{29}$$

If the model is linear and the loss function quadratic one iteration of the algorithm will yield the optimal policy sequence for the original policy problem and n* will equal unity. In more general cases the number of iterations required before attaining convergence gives some idea of how the actual structure differs from the linear/quadratic in a non-local sense, and the value of n* gives an idea of the adequecy of the local linear/quadratic approximations of (22) and (24). If n* is found to be quite small we would conclude that the approximations hold only over a relatively small range of variation in the policy sequence.

This incremental linear/quadratic algorithm can be readily applied to computation of optimal policies for structural models. The only difference is that the matrices A_t and B_t are now defined as the state and policy Jacobians of the locally-equivalent reduced-form model, as obtained from equation (7). The gradient of final-form loss is given by equation (18). Computational experience with the algorithm applied to a structural model is reported in Section 5.

If the dimension of the state vector is small one may well decide to compute the structural Jacobians of (7) directly by analytic differentiation and evaluation. If, however, numerical approximation is used some special problems arise in approximating A_t and B_t for structural models that do not come up for reduced-form models. Although one could approximate the structural Jacobians numerically and then apply (7), a more direct way would approximate the j^{th} column of A_t as:

$$(A_t)_j = (x_t'' - x_t')/2\delta \qquad j = 1, \ldots, n \tag{30a}$$

where:

$$F_t(x_{t-1} + \delta \ell_j, u_t, x_t'') = 0 \tag{30b}$$

$$F_t(x_{t-1} - \delta \ell_j, u_t, x_t') = 0 \tag{30c}$$

δ is a small real number and ℓ_j is an n-dimensional vector consisting of unity in the j^{th} component and zeros elsewhere. u_t is the tentative policy at time t and x_t is the state at time t from the associated state trajectory. The columns of B_t are similarly approximated by perturbing the m components of policy.

A major distinction now arises between two different classes of structural models. If the state implied by the model can be computed exactly in a finite number of steps, no problem is encountered with applying (30). For example, if the model is recurrsive:

$$x_t^i = f_t^i(x_{t-1}, u_t, \hat{x}_t^i) \qquad i = 1, \ldots, n \tag{31}$$

where $\hat{x}_t^i$ is a subset of the vector $(x_t^1, \ldots, x_t^{i-1})$ for $i > 1$ and is empty for $i = 1$, we can evaluate x_t^1 from x_{t-1} and u_t, then evaluate x_t^2 from x_{t-1}, u_t, and x_t^1, and so on to x_t^n. Recurrsive models can be solved exactly with a single pass throught the structural equations. The non-linear structural model of Fair (1971) appears in a recurrsive form.

For some structural models, however, current state cannot be computed exactly but only approximated by an iterative solution algorithm, Gauss-Siedel for example (Klein and Fromm (1969)). If the structural model is of the form of equation (2) we might begin with a tentative solution state s_o and then iteratively compute successive tentative solutions as:

$$s_i = f_t(x_{t-1}, u_t, s_{i-1}) \qquad i = 1,2, \ldots \qquad (33)$$

When s_K is appropriately close to s_{K-1} for some integer K we would set $x_t = s_K$. If δ in equation (30) is chosen too small the approximation of $(A_t)_j$ may be subject to considerable noise consequent upon the inability to compute the exact values of x_t' and x_t''. At the extreme suppose δ were zero but different tentative solution states were assumed in starting the iterative algorithm used to solve the model in equations (30b) and (30c). The denominator of (30a) is zero, but the numerator may well be non-zero. This problem does not arise if the structural model can be solved exactly in a finite number of steps. Numerical approximation of the reduced-form Jacobians may be more satisfactory if the value of δ is substantially larger than is typically the case for numerical differentiation.

4. An Application to Controlling Unemployment and Inflation

Several authors have investigated the application of optimal control methods to aggregate economic policy in a linear/quadratic framework, including Chow (1972b), Pindyck (1973) and Pindyck and Roberts (1974). However, many macroeconomic models not specifically designed for optimal control purposes exhibit important non-linear specifications, including an interest-elastic proportional demand for money, behavioral explanation of national product account items in real terms and national income account items in nominal terms (necessitating the presence of the implicit price deflator as a state variable) and behavioral explanation of the rate of inflation rather than the level of the deflator. Within the framework of the first three sections questions of optimal policies for these more general models can be addressed.

This section presents quantitative results on applying optimal control to the reduction of inflation and unemployment in the American economy. The model used in this study has been presented in Garbade (1974). Because of its importance to the present question we comment first on the Phillips relation between inflation and unemployment in the model. The section following describes our experience in implementing the algorithm of Section 3.

The model does not have an explicit Phillips curve. Rather it endogenously and independently explains inflation and unemployment, with the opportunities for trading one against the other available to the policy-maker implicit. An important feature of the model is that while trade-offs do exist in the short-run they become increasingly unavailable over longer periods of time. Let Rp be the current rate of inflation, R_p^n a proxy for the expected rate and $\bar{X}$ a standard rate of gross private production assuming 4% unemployment. We postulate that in the long run actual inflation differs from the expected rate as a linear function of the gap between actual and standard production:

$$(Rp - Rp^n)^* = a + b\,(\bar{X}_{-1} - X_{-1}) \tag{34a}$$

with a partial adjustment from quarter to quarter of:

$$\Delta(Rp - Rp^n) = \beta((Rp - Rp^n)^* - (Rp - Rp^n)_{-1}) \tag{34b}$$

Most macro-econometric models, e.g., Fair (1971), assume a long-run relation between inflation and the inflationary gap like $Rp^* = a + b(\bar{X} - X)$, but we felt that incorporation of inflationary expectations as a baseline for price changes was more appropriate in view of recent American experience. The expected rate of inflation is presumed to evolve according to the adaptive model:

$$\Delta Rp^{n} = \alpha(Rp_{-1} - Rp^{n}_{-1}) \tag{35}$$

See Gordon (1973) for a discussion of adaptive expectations in the post-war period. On estimation we identified a = .45 and b = - .0405. With these parameter values it follows from equation (34a) that if it were possible to hold the rate of production above $\overline{X}$ - 11 billion dollars at 1958 prices for a prolonged interval the rate of inflation would approach a positive spread over the expected rate. But (35) implies that R^n_p will rise as long as it is below the actual rate. Both the actual and expected rates will therefore increase indefinitely, which is to say the model has a Phillips curve that approaches, in the long run, a vertical line at an unemployment rate corresponding to a rate of production of $\overline{X}$ - 11. We doubt this interplay between adaptive expectations and equation (34a) would remain valid in hyperinflation, but it seems a reasonable method of modeling the interplay during periods of more moderate price changes.

The model does admit a temporary trade-off between unemployment and inflation. An exogenous increase in production, from a stimulative change in government policy for example, leads to increased demand for labor services and expanded employment as well as a widening of the inflationary gap $(X - \overline{X})$. The reduction in unemployment occurs more rapidly than the increase in the expected rate of inflation, thus yielding a transient trade-off. One of the purposes of the model is to study whether it is optimal to take advantage of such transient characteristics in aggregate economic behavior, as has been suggested by Solow (1969, pg.17).

The loss function used to evaluate economic performance has been described in Garbade (1974). The form of the single-period loss function, the f^o_t of equation (9), is given in Table 1. The principle objectives specified by the function are stabilization of the rate of unemployment (Ru) at 4%

Table 1. Single-period Loss Function

$$f_t^o = (.9925)^t(16.66(Rp)^2 + 33.33(Ru - 4.)^2$$

$$- 20.((Es + En + .2478\ Kd)/Pt) - 10.(Kh/Pt)$$

$$+ ((G - 1.01157\ G_{-1})/2.22)^2 + ((Eg - 1.0093\ Eg_{-1})/.100)^2$$

$$+ ((Rtb - Rtb_{-1})/.372)^2 + ((S - 1.0)/.025)^2$$

$$+ ((FHL - P\ \widehat{FHL})/.882)^2 + 100.((Yg - p\ \widehat{Wg}\ Eg)/.770)^2)$$

State Variables

- Rp rate of inflation, percent, annual rate
- Ru rate of unemployment, percent
- Es household expenditures on services, billions 1958 $
- En household expenditures on non-durable goods, billions 1958 $
- Kd stock of consumer durable goods, billions 1958 $
- Pt population, millions
- Kh stock of residential structures, billions 1958 $
- P implicit gross private product deflator, 1958 = 1.00

Policy Instruments

- G government purchases of privately produce goods and services, billions 1958 $
- Eg government employment, millions
- Yg government compensation of its employees, billions $
- Rtb Treasury bill rate, percent
- S Federal personal tax scaling factor, S = 1. implies no surcharge or reduction

FHL Federal Home Loan advances to savings and loan associations billions $

<u>Target Parameters</u>

$\widehat{FHL}$ target level of deflated FHL advances

$\widehat{Wg}$ target per capita real wage index for government employees and stabilization of the rate of inflation (Rp) at 0%. Increases in unemployment above 4% are penalized twice as heavily as increases in inflation above 0%. The loss function also penalizes large departures of the instruments from their trend behavior. The term $((G - 1.01157\ G_{-1})/2.22)$, for example, is a measure of the departure of current government purchases (G) from the target value of $1.01157\ G_{-1}$ in standardized units of 2.22 billions dollars at 1958 prices. The construction of the other policy stabilization terms is similar. With the value of the stabilization terms normalized by the size of the denominators the weights on each were set to unity. The absolute weights on inflation, unemployment, total consumption and the resident-housing stock were arbitrary, although the two former items received the major emphasis.

The initial quarter for the optimal policy problem was taken as 1960/I with an eleven quarter planning interval (T = 11). As noted in Garbade (1974) the optimal policies in the last four quarters of the planning interval diverge significantly from those which would be optimal for an infinite horizon problem. This myopic behavior stems from the proximity of the planning horizon, beyond which the loss function implies the policymaker is indifferent among alternative states of economic activity. The initial state was set at its historical value. Residuals of estimated equations were set at their expected values, yielding a certainty-equivalent version of the original stochastic model.

Table 2 records the optimal and historic policies over the 11 quarter planning interval, 1960/II to 1962/IV. Perhaps the most striking feature of the optimal policy sequence is the secular increase in government employment and compensation compared to the cyclic behavior of government purchases.

Table 2. Optimal Policies

	Rtb	FHL	S	G	Yg	Eg
1960/II	3.92	1.91	1.000	51.7	48.7	10.18
III	3.87	1.88	.999	53.1	51.2	10.57
IV	3.82	2.11	.997	54.2	53.8	10.99
1961/I	3.79	2.16	.998	54.3	56.4	11.39
II	3.77	2.25	.999	53.6	59.0	11.79
III	3.77	2.54	1.000	52.2	61.6	12.19
IV	3.80	2.79	1.001	51.0	64.3	12.56
1962/I	3.83	2.91	1.003	50.5	66.7	12.90
II	3.83	2.03	1.003	51.4	69.1	13.21
III	3.75	2.32	1.000	53.4	71.3	13.48
IV	3.69	3.26	.996	55.6	73.5	13.73

Historic Policies

	Rtb	FHL	S	G	Yg	Eg
1960/I	3.94	1.52	1.000	50.6	46.0	9.75
II	3.09	1.77	1.000	50.9	47.0	9.81
III	2.39	1.74	1.000	51.6	48.1	9.80
IV	2.36	1.98	1.000	52.2	48.8	9.85
1961/I	2.38	1.48	1.000	53.4	49.5	9.92
II	2.33	1.87	1.000	55.2	50.3	9.96
III	2.33	2.12	1.000	57.3	51.2	10.07
IV	2.48	2.16	1.000	57.2	52.6	10.31
1962/I	2.74	2.15	1.000	59.0	53.8	10.52
II	2.72	2.76	1.000	60.8	54.4	10.60
III	2.86	3.04	1.000	60.6	54.8	10.64
IV	2.80	3.48	1.000	61.6	55.7	10.68

Rtb Treasury bill rate, percent

FHL Federal Home Loan advances, billions $

S Federal personal tax scaling factor

G Government purchases of privately produced goods and services, billions 1958 $

Y_g Government compensation of employees, billions $

E_g Government employment, millions

The quarterly growth rates of optimal government employment decline monotonically from 4.4% per quarter in 1960/II to 1.9% per quarter in 1962/IV, but are all well above the target rate of .93% per quarter (cnf. Table 1). The level of optimal employment exceeds the target level of 1.0093 Eg_{-1} by at least one standard unit (.100 million workers) in every quarter and by at least two standard units in eight of the eleven quarters. Government purchases, on the other hand, always lay within a one standard unit band, $\pm$ 2.22 billion dollars at 1958 prices, about the target level of purchases, 1.01157 G_{-1}. This seems to imply, at least for the model and loss function we are considering, that government expenditures directed towards public employment programs are more efficient in achieving preferred configurations of unemployment and inflation than stimulation of aggregate activity by expanded purchases of privately produced goods and services. It is not true, however, that arbitrarily large increases in government employment can be undertaken without any effect on inflation. Increases in compensation must accompany increases in public employment, adding to demand, and inflationary pressure, through the consumption functions. Furthermore, expanded government employment limits the labor force available to the private sector, lowering the standard rate of private production, $\bar{X}$, and adding to inflationary pressure from the supply of product side. The analysis indicates that after these effects, and the corresponding effects from expanded government purchases, have been compared direct employment is somewhat more efficient, albeit not without costs. The initial expansion, and subsequent contraction, of purchases indicates that this policy instrument is more efficient as a stimulus to private employment at times when further expansion of government employment might be too costly. Once total employment is on a satisfactory trend purchases were scaled back to relieve inflationary pressure on the economy.

Table 2 shows the expenditure stimulus actually applied in 1961 and early 1962 came in the form of expanded purchases rather then employment.

These results on the efficiency of the two government expenditure instruments depend heavily on the weights placed on inflation and unemployment relative to the weights on the instruments stabilization terms. Greater emphasis on instrument stability would no doubt reduce the rapid growth in government employment. Optimality of the policies is defined with respect to the criterion function, and there is no reason why a policy-maker may not wish to use a different function than that employed in the present study, perhaps to the end of keeping the public sector more modest.

The optimal Federal personal tax scaling factor, S, does not vary substantially over the planning interval, turning from mildly stimulative in late 1960 to mildly restrictive in early 1961. The fiscal stimulus to economic activity in this study comes entirely from government expenditure rather than revenue policies.

Table 2 indicates that optimal monetary policy is accomodating, with the Treasury bill rate steady at about 3.8% in 1961 and early 1962. This contrasts with the historic behavior of the bill rate, which showed a steep decline in 1960 to about 2.4% and a mild increase through 1962 to 2.8%. The quarter to quarter variation in the optimal bill rate is well within the standard unit of 37 basis points, suggesting that monetary policy is not especially efficient for deterministic policy planning. Optimal Federal Home Loan advances to savings and loan associations grow much more smoothly than historic advances, without the decline and subsequent increase in 1961 evident in the historic choices. Support of residential construction through increases in advances appears more important in the environment of relatively higher yields on direct short-term investments that characterizes the optimal choice of policies than in the lower yield environment of the historic choices.

Table 3 displays selected components of the optimal and historic state trajectories. The average optimal rate of inflation over the planning interval is 1.9%, while the average optimal unemployment rate is 4.7%. These compare to the historic averages of 1.0% and 5.8% respectively, and the target values of 0% and 4%. Although we were unable to hit the target rates, either on average or in any single quarter, the results indicate that, judged by the loss function of Table 1, the historically lower average rate of inflation was purchased at too high a cost in terms of unemployment.

A notable feature of the optimal state trajectory is the constancy of the rates of inflation and unemployment. Some authors (Samuelson (1967, pg. 163)) have suggested that an optimal policy might drive the economy up and down along the short-run Phillips curve, first stimulating activity to reduce unemployment and then, as inflation begins to emerge, switching to a restrictive policy to forestall the development of inflationary expectations. Until 1968 this was quite similar to the American experience. Since the present model has a transient Phillips trade-off between inflation and unemployment such a stragtegy could be implemented as long as it didn't impart a secular bias to the expected rate of inflation. The optimal state trajectory, however, offers no justification for assuming the strategy is desirable from the point of view of aggregate activity.

Since the long-run Phillips curve of the model is vertical it is conceivable that the optimal choice of policies would have reduced the rate of inflation to the target rate of zero and settled for the natural rate of unemployment (Samuelson (1967, pg. 65) has noted this possibility).

Table 3. Components of State Associated with Optimal Policies

	X	Rp	Ru	Ed	Ip	Ih	R_p^n
1960/II	450.1	1.86	4.86	46.6	48.0	22.5	1.819
III	451.5	1.88	4.68	47.4	48.3	22.2	1.823
IV	451.0	1.87	4.71	47.9	47.8	20.9	1.829
1961/I	453.9	1.80	4.71	48.8	47.5	22.5	1.834
II	456.4	1.86	4.69	49.6	47.3	23.1	1.831
III	457.0	1.92	4.69	50.1	47.0	22.0	1.834
IV	457.9	1.90	4.72	50.6	46.6	21.4	1.842
1962/I	461.3	1.89	4.70	51.0	46.6	22.8	1.848
II	464.5	1.97	4.66	51.8	46.8	22.1	1.852
III	470.3	2.04	4.61	52.9	47.5	21.2	1.864
IV	482.6	2.17	4.36	54.9	49.4	22.8	1.881

Historic States

	X	Rp	Ru	Ed	Ip	Ih	R_p^n
1960/I	447.0	1.80	4.98	45.4	46.6	23.7	1.821
II	445.8	1.70	5.05	45.6	47.6	22.0	1.819
III	443.5	.67	5.37	45.0	47.0	21.0	1.807
IV	439.8	1.96	6.04	43.5	47.0	20.7	1.694
1961/I	438.4	.93	6.54	41.7	44.9	20.9	1.721
II	448.4	.12	6.75	43.2	44.6	21.1	1.641
III	456.6	-.08	6.52	44.5	45.7	21.6	1.490
IV	466.0	1.97	5.95	46.3	46.6	22.6	1.332
1962/I	473.0	1.32	5.40	48.1	47.6	23.1	1.396
II	480.8	.52	5.28	48.1	49.3	23.8	1.388
III	486.3	.83	5.33	49.7	51.1	24.2	1.302
IV	491.3	1.14	5.31	50.8	50.7	23.8	1.254

X Gross private product, billions 1958 $

R_p Rate of inflation, percent

Ru Rate of unemployment, percent

Ed Expenditures on consumer durables, billions 1958 $

Ip Business investment in plant and equipment, billions 1958 $

Ih Investment in residential structures, billions 1958 $

R_p^n Proxy for expected rate of inflation, percent

This can be achieved with little long-run cost if R_p^n, the proxy for the expected rate of inflation, were driven down close to zero, thereby centering the transient Phillips curve at zero inflation. Evidently the costs of unemployment required to drive down the expected rate are too great to justify such a strategy given the value of R_p^n in the initial quarter. Looking at Table 3 we see that the optimal expected rate changed only slightly over the planning interval, leaving the transient Phillips curve unchanged. One of the beneficial effects of the relatively lower historic rate of inflation was to reduce the expected rate by more than half a percentage point by the end of the planning interval (Table 3). This reduction permitted a better transient Phillips curve in years after 1962 than that which results from our study. As a matter of conjecture it seems reasonable to suppose that were the terminal planning horizon sufficiently far away the optimal expected rate of inflation would exhibit a secular drift towards zero, at least in the early and intermediate years of the planning interval. The present study indicates, however, that elimination of inflationary expectations will be optimal only for very long horizons, probably approaching six or eight years. Over any shorter horizon such a policy objective appears too costly in terms of unemployment. This result might be modified, however, in a case where inflationary expectations were high in the initial quarter and where a rapid downward shift in the transient Phillips curve might be worth the burden of substantial unemployment.

5. Computational Experience

The optimal policies exhibited in the preceeding section were computed by the algorithm of Section 3. To start the algorithm we projected the instruments over the planning interval at their no-loss growth rates,e.g.,1.157%

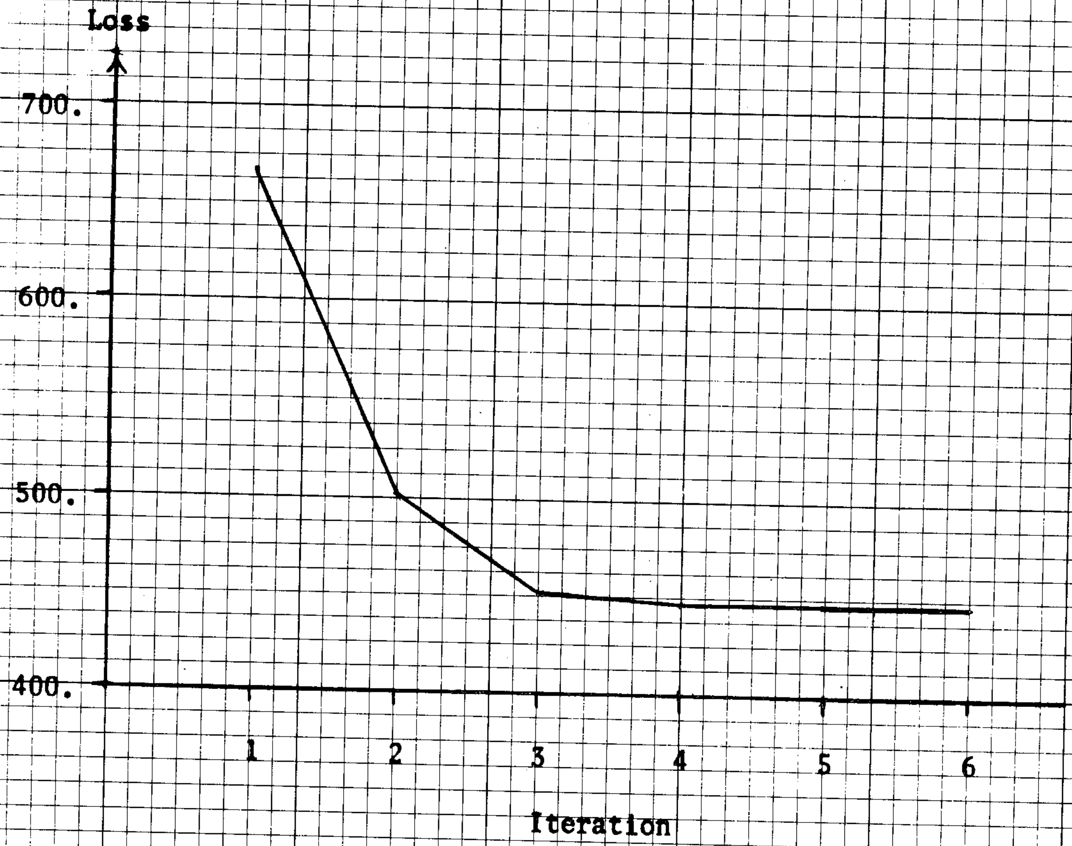
per quarter for government purchases, using the historical policies in the initial quarter as a base, and solved the model for the associated tentative state trajectory. The Jacobians of the implicitly defined reduced-form model were approximated as in equation (30a) and the gradients and Hessians of the definitional loss function directly approximated by numerical differentiation. The optimal policy increment with respect to the approximating linear/quadratic structure was then computed by (26) and the step-size sub-procedure of (28, 29) applied to yield a new tentative policy sequence. This process was repeated until convergence occurred in the sixth iteration.

Table 4 records the reduction in loss obtained at each iteration of the algorithm. The loss associated with the initial tentative choice of policies was 1891.8. Since the initial choice was arbitrary we do not ascribe any significance to the large decrease in loss at the first iteration. It is interesting to note, however, that the second and third iterations also lead to substantial reduction in loss. Were the model linear and the loss function quadratic a single iteration would have been sufficient to gain convergence. Even the mild non-linearities in the present model, which include especially the mixture of real product and nominal income account items, are sufficient to preclude the assumption that a local approximation is an adequate global description of the model and loss function. Inspecting the values of n^* in Table 4 we see they are either zero or unity, implying the linear/quadratic approximation is satisfactory on a local scale.

6. Conclusions

This paper had two objectives, investigation of the problem of controlling inflation and unemployment in the context of a non-linear structural

Table 4. Iterations of the Linear/Quadratic Algorithm



Initial Tentative Loss = 1891.8

Iteration	Loss at Completion of Iteration	Scaling Factors for the Iteration n^s	n^*
1	673.4	3	1
2	502.98	3	0
3	455.48	3	1
4	449.25	3	0
5	449.12	3	0
6	449.12	3	0

$\alpha = .9$

$\beta = .7$

model and the development of methodologies to deal with such models. As to the first it appears that stable economic activity is preferred to cyclic behavior. Furthermore, even in a model which exhibits a vertical long-run Phillips curve there is little evidence to suggest that elimination of inflationary expectations is a desirable goal over even the moderately long planning interval of 11 quarters. On the other hand there is none of the upward bias in the expected rate that might have characterized the optimal policies when significant weight is placed on unemployment. When initial inflationary expectations are low the best policy appears to be one of simple stabilization.

An optimal choice of policies places more emphasis on fiscal over monetary instruments, although it must be kept in mind that coordination of the whole set of available instruments is one of the important benefits of policy analysis by optimal control techniques. Nothing in the present analysis says that monetary policy either doesn't matter or matters but slightly. The results indicate only that stimulation of aggregate activity is best accomplished by an expansionary fiscal rather than monetary policy. Within the set of fiscal instruments considered, increased expenditures seem more important than decreased revenues for achieving a desired expansion, and within total expenditures expanded government employment and compensation is preferred to larger government purchases of private product.

With respect to the second objective of developing methodologies for structural models we showed that many results from the optimal control of reduced-form models can be translated to the more general case by appealing to the local equivalence argument of Section 1. The reason this is possible is that analytic methods for a non-linear model typically use the Jacobians of the model to describe local behavior instead of addressing global behavior with the full model. The Jacobians of a reduced-form model locally equivalent to a given structural model can be recovered even when the reduced-form model itself is unavailable, and hence the analytic methods go through to the more general case.

Appendix: rank $(d_3F_t) = p < n$.

It may be that a particular structural model will exhibit multiple solutions, so that:

$$F_t(\bar{x}_{t-1}, \bar{u}_t, s^i) = 0 \qquad i = 1, 2, \ldots$$

Given assignments $\bar{x}_{t-1}$ and $\bar{u}_t$, the solution set $S = \{s^1, s^2, \ldots\}$ may be finite, countable, or uncountable. In the case where S is finite we are likely to find that only one of the solutions is reasonable, i.e., can be construed as the consequence of current policy $\bar{u}_t$ and previous period state $\bar{x}_{t-1}$. The other elements of S would be dismissed as mathematical artifacts of the model, and we would set $\bar{x}_t$ to the reasonable solution. It is possible that rank $(d_3F_t(\bar{x}_{t-1}, \bar{u}_t, s^i)) = n$ for all i when S if finite or countable, so that each solution is locally unique to neighboring choices of x_{t-1} and u_t. These are cases of isolated solutions. The cases of S finite or countable do not arise with linear models, since there either the solution is unique or, if a solution exists at all, there is an uncountable set of solutions lying on a hyperplane in state space.

Suppose for some solution state $\bar{x}_t$ rank $(d_3F_t(\bar{x}_{t-1}, \bar{u}_t, \bar{x}_t)) = p < n$. Here either possibility 1 or possibility 2 may hold. If rank $(d_3F_t(\bar{x}_{t-1}\bar{u}_t, x)) = p$ for all $x \in M(\bar{x}_t)$, i.e., possibility 1, then, by the Rank Theorem (Dieudonne (1969) pg. 277) there is a connected manifold, C, of dimension n - p contained in $M(\overline{x}_t)$ with $\overline{x}_t \in C$ such that $F_t(\overline{x}_{t-1}, \overline{u}_t, x) = 0$ for all $x \in C$. In this case the solution of the structural model is not even locally unique, and the solution set is uncountable. There are a whole sequence of solutions, arbitrarily near each other. It is impossible to choose one of these as the "true" state of the system, and we would be led to believe that the system has been inaccurately represented by the mathematics of the structural model. If possibility 2 holds no general statements appear possible. The solution $\overline{x}_t$ may or may not be locally unique. Even if it is unique there will be policies arbitrarily close to $\overline{u}_t$ and states arbitrarily close to $\overline{x}_{t-1}$ for which no solution state exists or for which the solution states are not near $\overline{x}_t$. In either case the opportunity of conducting the perturbation analysis required by the Lagrange Multiplier Theorem is precluded. Hence if $p < n$ extension of the results from optimal control of reduced-form models to structural models fails.

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