FURTHER NOTES ON THE ALLOCATION OF EFFORT*

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Econometric Research Program
Research Memorandum No. 171
December 1974

*I am grateful to Roy Radner for many discussions of the topics presented here. Some of the most important ideas are due to him. This research was partially supported by the National Science Foundation.

1. SUMMARY AND INTRODUCTION

In a previous paper [3], Roy Radner and I analyzed the following situation: An agent is in charge of I distinct activities. At time t the level of activity i is $U_i(t)$. Although the agent is responsible for all I activities, his ability to control them is severely limited. In the first place, regardless of his actions, the evolution of $U(t) = (U_1(t),\ldots,U_I(t))$ is stochastic. More importantly, he can only devote himself to one activity at a time and while his efforts are generally successful (attended activities tend to improve), their absence is likely to be harmful (neglected activities deteriorate). The agent's problem is to choose a rule for allocating his effort or attention among the various competing activities.

A formal version of this model is as follows: The allocation of the agent's effort at time t is determined by a vector $a(t) = (a_1(t),\ldots a_I(t))$ satisfying $a_i(t) \geq 0$; $\Sigma\, a_i(t) = 1$. In general, we shall consider here only cases where the $a_i(t)$ are equal to 1 or 0 ; fractional allocations, which are discussed in [3], can arise from mixed strategies or because the agent is actually able to divide his attention among several activities. The allocation vector $a(t)$ affects the evolution of $U(\cdot)$ through its effects on the increments of $U(\cdot)$, $Z(t+1) = U(t+1) - U(t)$.

Thus, the distribution of $Z(t+1) = (Z_1(t+1),\dots,Z_I(t+1))$ is determined by $a(t)$ and, possibly, by the past history of the process. In [3] we modeled the influence of the allocation of effort by assuming

$$E\ Z_i(t+1) = a_i(t)\eta_i - (1-a_i(t))\xi_i \tag{M.1}$$

where η_i and ξ_i are given positive parameters. We further assumed that:

The distribution of $Z(t+1)$ is determined solely by $a(t)$; (M.2)

and

Given $a(t)$, the coordinates of $Z(t+1)$ are mutually independent. (M.2)

In the sequel, I shall refer to assumptions (M.1), (M.2) and (M.3) as the Markov assumptions. In addition to these three assumptions we made some regularity assumptions very similar to the following conditions, which I shall adopt in in this paper:

$Z_i(t+1)$ is integer valued, (R.1)

$|Z_i(t+1)| < b$ for some $b > 0$, (R.2)

and

If

$$H = \{h \in R^I \mid |h_i| = 1, \quad i=1,\ldots,I\},$$

then there exists $\gamma > 0$ such that for all values of $a(t)$, all past histories of events up to and including time t, and $h \in H$, (R.3)

$$P\{Z(t+1) = h \mid a(t)\} > \gamma .$$

As usual, the underlying probability structure is represented by the triple $(\Omega, \mathcal{F}, P)$. $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_t \ldots$ is a sequence of increasing sub-sigma fields of $\mathcal{F}$; $\mathcal{F}_t$ represents the events observed up to and including date t. Thus a variable dated t, such as $U(t)$, $a(t)$ or $Z(t)$, is $\mathcal{F}_s$-measureable as long as $t \geq s$.

This model can be used to analyze a number of different questions. One of the most natural is determining whether the manager can control all I activities simultaneously or whether his attempts to look after them all will lead naturally to disaster for one or more of the activities. In [3] we defined survival as keeping all indices above some arbitrary level for all time and asked whether or not it was possible to survive. Formally, if

$$M(t) = \min_i U_i(t) , \tag{1.1}$$

survival is possible if

$$P\{M(t) > 0 , \quad \text{all } t\} > 0 \tag{1.2}$$

The notion of being in control captured somewhat more sharply by a different notion. If

$$\liminf M(t) = \infty , \text{ a.s.,} \tag{1.3}$$

then I shall say the manager eventually succeeds. There are three reasons for believing that (1.3) is at least as interesting a property as (1.2). First it is more pleasing to know that events in which one is interested will occur almost surely than to know there is some positive (but possibly very small) probability that they will occur. Secondly, if the manager eventually succeeds, things get better and better in the sense that for any arbitrary level of performance, L , and any $\varepsilon > 0$, there is a (non-random) time T such that with probability at least $1-\varepsilon$, $M(t) \geq L$ for all $t \geq T$. Finally eventual success implies there is with arbitrarily high probability, a limit to how bad things get in the sense that for every $\varepsilon > 0$ there is a finite $B(\varepsilon)$ such that $M(t) \geq B(\varepsilon)$ for all t with probability at least $1-\varepsilon$. The following Proposition demonstrates that these are the implications of eventual success.

Proposition 1

If $\liminf M(t) = \infty$, a.s. then

(i) For every L, and every $\varepsilon > 0$, there exists $T(L,\varepsilon)$ such that

$$P\{M(t) \geq L, \text{ all } t \geq T(L,\varepsilon)\} \geq 1-\varepsilon .$$

(ii) For every $\varepsilon > 0$, there exists $B(\varepsilon)$ such that

$$P\{M(t) \geq B(\varepsilon) \quad \text{all } t\} \geq 1-\varepsilon$$

PROOF:

(i) Let $Y_t(\omega) = \mathrm{Max}[M(t,\omega), L + \delta]$ for some $\delta > 0$. Then $Y_t \to L + \delta$ a.s. and Egorov's Theorem implies that for every $\varepsilon > 0$ there is a set **F** with $P(F) \geq 1-\varepsilon$ such that $Y_t(\omega) \to L + \delta$ uniformly on F.

The conclusion follows.

(ii) Assumption (R.2) implies $|M(t) - M(t-1)| < Ib$. Fix L and ε, let $B(\varepsilon) = L - T(L,\varepsilon)Ib$.

The two concepts (1.2) and (1.3) are, despite their different meanings quite closely related. Whether the manager can survive or will eventually succeed depends on two things: the distribution of the $Z(t)$'s and the rules used to determine the allocation of attention. In [3] we showed for the Markov case, that there was a simple test for determining whether any policy could survive. Specifically, we showed in Theorem I of [3] that if

$$\bar{\zeta} = [1 - \sum_i \frac{\xi_i}{\eta_i + \xi_i}] / \sum_i (\eta_i + \xi_i)^{-1} \qquad (1.4)$$

then $\bar{\zeta} > 0$ was a necessary and sufficient condition for the existence of any policy which has a positive probability of survival. It is trivial to adapt the argument of [ ] to prove that $\bar{\zeta} > 0$ is also a necessary and sufficient condition for the existence of any policy which will eventually succeed. Thus $\bar{\zeta}$, a simple function of the conditional means of the distribution of the $Z_i(t)$, emerges as a natural measure of the difficulty of a task facing a manager. If $\bar{\zeta} \leq 0$ the task is impossible; nothing he does can lead survival. If $\bar{\zeta} > 0$, he can, if he chooses the right policy, eventually succeed.

If survival is possible, relatively simple policies can bring it about. In [3] we desribed two. Both policies also will, if $\bar{\zeta} > 0$, eventually succeed. The first, balanced growth, is a behavior in which the allocation of effort is constant. Let

$$\hat{a}_i = \frac{\bar{\zeta} + \xi_i}{\eta_i + \xi_i} , \tag{1.5}$$

then $\Sigma \hat{a}_i = 1$ and if $\bar{\zeta} > 0$, $\hat{a}_i \geq 0$. Consider the behavior which simply sets $a_i(t) = \hat{a}_i$ for all i and t.

It is easy to see that, for the Markov case,

$$E[Z_i(t+1) | a_i(t) = \hat{a}_i] = \bar{\zeta}$$

so that the law of large numbers implies

$$\lim \frac{U_i(t)}{t} = \bar{\zeta} \qquad \text{a.s.}$$

from which it follows that $\lim \frac{M(t)}{t} = \bar{\zeta}$ a.s.. Clearly this behavior will eventually succeed.

Another simple rule which will eventually succeed when $\bar{\zeta} > 0$, is what we called "putting out fires." This is the policy in which all attention is allocated to the activity which is currently performing worst -- apparently a very common kind of administrative behavior. Formally we may define putting out fires by

(i) if $U_i(t) \neq M(t)$, then $a_i(t) = 0$;

(ii) if $U_i(t) = M(t)$ and $a_i(t-1) = 1$, then $a_i(t) = 1$; (1.6)

(iii) if neither (i) nor (ii) holds, then $a_i(t) = 1$ for $i =$ the smallest j such that $U_j(t) = M(t)$.

A notable feature of putting out fires behavior is that, in contrast to the constant proportions behavior described above, it can be pursued without knowledge of the parameters determining the distribution of $Z(t+1)$.

In [3], we proved, again for the Markov case that, if $\bar{\zeta} > 0$ and putting out fires is followed, then the policy will eventually succeed (and survival is possible).

The purpose of this paper is to extend and qualify these results, which seem at once too weak and too strong. They appear too weak because they are restricted to the rather special Markov case; they are too strong because they seem to imply that putting out fires, which is often criticized as being a poor administrative strategy, is as good a rule as any in the sense that putting out fires will persevere when any thing will; if putting out fires fails, nothing will work. These two defects are related. Putting out fires is, I believe, thought to be a bad strategy because it involves too much changing about and such changes are costly. These costs cannot be modeled easily within the Markov framework of [3].

Suppose the Markov assumptions are replaced by the much weaker assumptions below which state in essence only that allocating effort to an activity leads to expected gains and withholding effort causes expected losses:

$$E[Z_i(t+1) | a_i(t) = 1, \mathcal{F}_t] \geq \eta_i \qquad (G.1)$$

$$-\beta > E[Z_i(t+1) | a_i(t) = 0, \mathcal{F}_t] \geq \xi_i \qquad (G.2)$$

where β , η_i and ξ_i are given positive numbers. Then, if $\bar{\zeta}$ is defined as in (1.4) above, its positivity is still a sufficient condition for putting out fires to eventually succeed.

<u>THEOREM 1:</u>

If there is putting out fires behavior, assumptions (R.1-3) and (G.1-2) are satisfied, and $\bar{\zeta} > 0$, then

$$\liminf \frac{U_i(t)}{t} \geq \bar{\zeta} \qquad \text{a.s.} \quad i=1,\ldots,I ,$$

and

$$\liminf M(t) = \infty \qquad \text{a.s.}$$

The proof of this theorem -- which closely parallels the proof of Theorem 3 in [3] -- is given in section 2 of this paper. In section 3 the problem of introducing costs of switching attention from one activity to another is discussed. One plausible model which captures this effect is introduced. It is shown that in this model there are rewards to staying with one activity and not switching around from activity to activity. An example is given in which putting out fires does not persevere but another policy, which involves less frequent changes of attention, eventually succeeds.

2. PROOF OF THEOREM I

The proof of Theorem 1 follows from the following proposition which is of some independent interest. Let $D(t) = \max_{i \in I} U_i(t) - \min_{j \in I} U_j(t)$.

<u>Proposition 2.</u> Under the conditions of Theorem I, there is a G such that if $G \leq D(s)$ and T^* is the first integer such that $D(s + T^*) < G$ then there exist H and K such that $P\{T^* > n\} \leq H e^{-nK}$.

<u>Remark 1.</u> This implies that $E(T^*)$ is finite as

$$ET^* = \Sigma_n \Pr\{T^* \geq n\} \leq \Sigma_n H e^{-nK} = \frac{H}{(1-e^{-K})} .$$

<u>PROOF OF THEOREM I:</u>

We may without loss, set $s = 0$.

<u>LEMMA 1:</u> The proposition implies

$$\frac{U_i(t)}{t} - \frac{U_j(t)}{t} \longrightarrow 0 , \text{ a.s.}$$

PROOF: Since $|U_i(t) - U_j(t)| < D(t)$, it will suffice to show $\frac{D(t)}{t} \longrightarrow 0$ a.s. No generality is lost if it is assumed that $D(0) \leq G$. Let $A_o = 0$ and for $n=1,2,\ldots$ define B_n as the first date $t > A_{n-1}$ such that $D(t) > G$ and A_n as the first date $t > B_n$ such that $D(t) \leq G$. Then if, for some n,

$$A_{n-1} \leq t < B_n \quad ,$$

$$D(t) \leq G \; ; \qquad \text{if} \qquad B_n \leq t < A_n$$

$$D(t) \leq G + (A_n - B_n)b$$

and

$$\frac{D(t)}{t} \leq \frac{G}{t} + \frac{A_n - B_n}{t} b \leq \frac{G}{t} + \frac{A_n - B_n}{n} b \quad .$$

Thus $\limsup \frac{D(t)}{t} \leq \limsup \frac{G}{t} + \frac{A_n - B_n}{n} = b \limsup \frac{A_n - B_n}{n}$.

To prove the Lemma, it will suffice to show that

$$\limsup \frac{A_n - B_n}{n} = 0 \qquad \text{a.s.}$$

For any $\varepsilon > 0$, let E_n be the event that $\frac{A_n - B_n}{n} \geq \varepsilon$.

Proposition 2 implies $P\{(A_n - B_n) \geq n\varepsilon\} \leq H e^{-Kn\varepsilon}$.

Let X_n be the indicator of E_n, and define $\mathcal{G}_{n-1} = \mathcal{F}_{B_n}$, then $M_n = E[X_n | \mathcal{G}_{n-1}] \leq H e^{-Kn\varepsilon}$ and $\Sigma M_n < \infty$. We conclude from

Freedman [2 , p. 919, Proposition (32)] that $\Sigma X_n < \infty$ a.s. In other words, almost surely only a finite number of the events E_n occurs or $\limsup \frac{B_n - A_n}{n} < \varepsilon$, a.s. Since this holds for all $\varepsilon > 0$, it follows that $\limsup \frac{B_n - A_n}{n} \longrightarrow 0$ a.s.

LEMMA 2: Let $\bar{U}(t) = \sum_i w_i U_i(t)$ where $w_i = (\eta_i + \xi_i)/\Sigma_j(\eta_j + \xi_j)$. Then $\liminf \frac{\bar{U}(t)}{t} \geq \bar{\zeta}$ a.s.

PROOF: Let $\bar{Z}(t+1) = \bar{U}(t+1) - \bar{U}(t)$; then $\bar{Z}(t+1) = \Sigma w_i Z_i(t+1)$ and

$$E\,\bar{Z}(t+1) = \sum_i w_i\, E\, Z_i(t+1)$$

$$\geq \sum_i w_i[a_i(t)\,\eta_i - (1-a_i(t)\xi_i]$$

$$= \sum_i w_i[a_i(t)(\eta_i + \xi_i) - \xi_i] = \bar{\zeta} .$$

Thus,

$$\bar{M}(t) = E[\bar{Z}(t) \mid \mathcal{F}_{t-1}] \geq \bar{\zeta} .$$

Consider

$$\frac{\bar{U}(t)}{t} = \frac{\sum_1^t \bar{Z}(\tau)}{t} = \frac{\sum_1^t \bar{Z}(\tau)}{\sum_1^t \bar{M}(\tau)} \; \frac{\sum_1^t \bar{M}(\tau)}{t}$$

$$\geq \bar{\zeta} \; \frac{\sum_1^t \bar{Z}(\tau)}{\sum_1^t \bar{M}(\tau)} .$$

Since

$$\lim \frac{\sum_{1}^{t} \bar{Z}(\tau)}{\sum_{1}^{t} \bar{M}(\tau)} = 1 \qquad \text{a.s.} \quad \text{[Freedman, 2 p. 921, Theorem (40)].}$$

it follows that

$$\liminf \frac{\bar{U}(t)}{t} \geq \bar{\zeta} \qquad \text{a.s.} ,$$

which proves Lemma 2.

To prove Theorem I it remains only to observe that

$$\bar{\zeta} \leq \liminf \frac{\bar{U}(t)}{t} = \liminf \sum_{i} w_i \frac{U_i(t)}{t}$$

$$= \liminf \sum_{i} w_i \frac{U_k(t)}{t} = \liminf \frac{U_k(t)}{t} , \quad \text{a.s.}$$

The next to the last step follows from Lemma 1.

Proof of Proposition

The proof is by induction on I. Clearly the proposition holds for $I = 1$. The induction step, that if it holds for $I = J-1$ then it is true for $I = J$, closely parallels the proof of Proposition 2 in [3]. As in that paper, the proof is given in three Lemmas.

LEMMA 3: Suppose $\mathcal{K}$ is any proper subset of $\mathcal{J} = \{1,\ldots,J\}$, that $\mathcal{K}'$ is the complement of $\mathcal{K}$ in $\mathcal{J}$ and that putting out fires is practiced on the activities in $\mathcal{K}$ while no effort is allocated to those in $\mathcal{K}'$. Then there is a (non-random) T such that

$$E \min_{k \in \mathcal{K}} U_k(t) \geq \min_{k \in} U_k(0) + 1 \quad \text{for all} \quad t \geq T \tag{2.1.a}$$

$$E \max_{k \in \mathcal{K}'} U_k(t) \leq \max_{k \in} U_k(0) - 1 \quad \text{for all} \quad t \geq T \tag{2.1.b}$$

T can be chosen so that (2.1) holds for any $\mathcal{K}$ properly contained in $\mathcal{J}$.

<u>PROOF:</u> Consider the activities in $\mathcal{K}$ alone. Let

$$\bar{\zeta}_{\mathcal{K}} = \left(1 - \sum_{k \in \mathcal{K}} \frac{\xi_k}{\eta_k + \xi_k}\right) / \sum_{k \in \mathcal{K}} (\eta_k + \xi_k)^{-1} \tag{2.2}$$

Comparing (2.2) and (1.4) we see that $\bar{\zeta}_{\mathcal{K}} > 0$ whenever $\bar{\zeta} > 0$ so that the induction hypothesis implies Proposition 2 and thus Theorem 1 holds when putting out fires is followed on the activities in $\mathcal{K}$ alone. Thus, for any $k \in \mathcal{K}$

$$\liminf \frac{U_k(t)}{t} \geq \bar{\zeta} \quad ; \quad \text{a.s.} \tag{2.3}$$

and

$$\liminf \frac{1}{t} \min_{k \in \mathcal{K}} U_k(t) \geq \zeta \quad \text{a.s.} \tag{2.4}$$

Since the increments of $U_k(t)$ are uniformly bounded by b ,

$$\left| \min_{k \in \mathcal{K}} \frac{U_k(t)}{t} \right| \leq b \quad ;$$

we may apply the Lebesque Monotone convergence theorem and the Fatou-Lebesque theorem to conclude that

$$\liminf E \frac{1}{t} \min_{k \in \mathcal{K}} U_k(t) \geq$$

$$E \liminf \frac{1}{t} \min_{k \in \mathcal{K}} U_k(t) \geq \bar{\zeta} \qquad .$$

It follows that

$$\liminf E \min_{k \in \mathcal{K}} U_k(t) \longrightarrow + \infty ,$$

and there exists $T_{\mathcal{K}}$ such that $t > T_{\mathcal{K}}$ implies

$$E \min_{k \in \mathcal{K}} U_k(t) > \min_{k \in \mathcal{K}} U_k(0) + 1 .$$

If no attention is paid to an activity then (G.2) states that $E\, Z_k(t) < - \beta$. It follows in a straight forward way from Theorem (40) of Freedman [2] that,

$$\limsup \max_{k \in \mathcal{K}} , \frac{U_k(t)}{t} \leq - \beta .$$

Another application of Lebesque and Lebesque-Fatou produces

$$\limsup E \max_{k \in \mathcal{K}'} U_k(t) \longrightarrow -\infty ,$$

so that there is a $T'_{\mathcal{K}}$ such that $t > T'_{\mathcal{K}}$ implies

$$E \max_{k \in \mathcal{K}'} U_k(t) < \max_{k \in \mathcal{K}'} U_k(0) - 1 .$$

Letting $T = \max (\max(T_{\mathcal{K}}, T'_{\mathcal{K}}))$ where $\mathcal{K}$ ranges over all proper subsets of $\mathcal{J}$ completes the proof.

<u>LEMMA 4:</u> Let $D(t) = \operatorname{Max}_i U_i(t) - \min_i U_i(t)$, and $G = 2 J T(b+1)$; if $D(0) > G$, then $E D(T) < D(0) - 2$ where T is as in Lemma 1.

<u>PROOF:</u> This lemma is identical to and has the same proof as Lemma 2 of [3].

<u>LEMMA 5:</u>

Suppose $G < D(0) < 2 J b + G$. Let T^* be the first T such that $D(T^*) \leq G$. There exist H and K such that $P(T^* > T) \leq H e^{-KT}$.

<u>PROOF:</u> Let $D = D(0)$ and consider the random variables

$$X_n \equiv D(n T) - D[(n-1)T)] .$$

Let $\mathcal{G}_n = \mathcal{F}_{nT}$; $\mathcal{G}_n$ is an increasing sequence of sigma fields and X_n is $\mathcal{G}_n$-measurable. Furthermore, if $Y_n = \sum_{m=1}^{n} X_m$, then $D(n\,T) = D + Y_n$. Let $C = G - D$. Then $-C \leq 2\,b\,J$. If $Y_n \leq C$ then $D(n\,T) \leq G$. Let N^* be the first N such that $Y_{N^*} \leq C$. It will suffice to show that there exist H' and K' such that for all $n > 2\,J\,B$

$$P\{N^* > n\} \leq H' e^{-K'n} \quad . \tag{2.5}$$

The random variables X_n, Y_n have the following properties

$$|X_n| \leq B \tag{2.6}$$

where $B = 2\,J\,T\,b$ and Lemma 2 implies $E[X_n | Y_{n-1} > C] < -2$.

Let

$$W_n \equiv \frac{X_n + B}{2B}$$

$$S_n = \sum_1^n w_m = \frac{Y_n + nB}{2B}$$

$$R_n = E[w_n | \mathcal{B}_{n-1}] \quad .$$

Suppose

$$2n > 2\,J\,b \geq -C \; ; \tag{2.7}$$

if $N^* > n$, then $Y_n > C$ and $R_m < \frac{-2+B}{2B}$ for $m=1,\ldots,n$

so that

$$S_n > \frac{C + nB}{2B} \equiv a_n \quad \text{and} \quad \sum_1^n R_m < \frac{n(-2+B)}{2B} \equiv b_n \quad .$$

Since (2.7) implies $a_n > b_n$ we may use (4.b) of Freedman [p. 91] to conclude that

$$P[N^* > n] \leq \exp[-\frac{(a_n - b_n)^2}{a_n}] \tag{2.8}$$

Note that

$$\frac{(a_n - b_n)^2}{2\, a_n} = \frac{(C - 2n)^2}{2\, B(C + nB)} \geq \frac{4\, n^2}{2n\, B^2} = \frac{2n}{B^2}$$

because of (2.7). Thus (2.8) may be replaced by

$$P\{N^* > n\} \leq \exp -(\frac{2}{B^2})\, n \quad , \qquad \text{for} \quad n > 2\, J\, b \quad ,$$

which with $H' = 1$, $K' = \frac{2}{B^2}$ is (2.5).

Proposition 2 follows immediately.

Remark 3. The assumption $\bar{\zeta} > 0$ is stronger than is necessary to prove Proposition 2. As the structure of the proof (particularly the proof of Lemma 3) makes clear all that is needed is $\bar{\zeta}_j > 0$, $j=1,\ldots,I$, where

$$\bar{\zeta}_j = [1 - \sum_{i \neq j} \frac{\xi_i}{\eta_i + \xi_i}] \Big/ \sum_{i \neq j} (\eta_i + \xi_i)^{-1} \quad . \tag{2.9}$$

III. COSTS OF PUTTING OUT FIRES

We now consider a model in which following putting out fires behavior may be ill advised as there is a cost involved in switching from one activity to another; in such a situation, wise policies would, as putting out fires does not, economize on the number of such switches. The simplest possible model is the following: Suppose that the allocation of effort is indivisible so that $a_i(t) = 0$ or $a_i(t) = 1$. Let $m(t)$ be the activity to which effort is allocated at time t. Let $s(t)$ be the number of consecutive periods immediately preceeding (but excluding period t) during which attention has been allocated to activity $m(t)$. Thus if $m(t) = a_i(t)$ while $m(t-1) \neq a_i(t)$ $s(t) = 0$; if $m(t) = m(t-1) =, \ldots, m(t-\tau) = a_i(t)$ while $m(t-(\tau+1)) \neq a_i$ then $s(t) = \tau$.

Now suppose,

The distribution of $Z(t-1)$ is determined by $a(t)$ and $s(t)$. (3.1.a)

$$E\, Z_i(t+1) = a_i(t)(\eta_i + \delta_{is(t)}) - (1-a_i(t))\xi_i, \qquad (3.1.b)$$

where for each i, $\{\delta_{is}\}$ is a bounded sequence of non-negative numbers such that

$$\delta_{io} = 0; \qquad (3.2.a)$$

$$\delta_i \leq \delta_{is+1} \qquad (3.2.b)$$

$$\delta_{is} > 0 \quad \text{for some } s \qquad (3.2.c)$$

The specification (3.1.b) and (3.2) captures the notion that there is a cost to switching attention often and that there are increasing returns to continuing to do the same thing. We shall show this formally below. The criterion we shall use to measure quality of performance is the long run rate of growth of the $U_i(t)$'s which we define as

$$R_i = \lim U_i(t)/t \qquad \text{a.s.}$$

when that limit exists. Recall the definition of $\bar{\zeta}_i$ (2.9). The following theorem is our first result about growth rates.

<u>THEOREM 2:</u> Suppose assumptions (3.1) and (3.2) hold and putting out fires is followed. If $\bar{\zeta}_i > 0 \quad i=1,\ldots,I$ then the growth rates R_i exist and are equal to one another. This common growth rate, R^1, exceeds $\bar{\zeta}$.

Again the proof follows from a proposition of some independent interest.

Let $V_i(t) = U_i(t) - M(t)$, $V(t) = (V_1(t),\ldots,V_I(t))$ and consider the Markov chain

$$C(t) = (m(t), s(t), V(t)) \ . \tag{3.3}$$

Proposition 3: $C(t)$ is positive recurrent.

PROOF of THEOREM 2: Since $C(t)$ is positive recurrent, the long run relative frequency of the events $m(t) = i$ and $s(t) = s$ converges almost surely to the invariant probability of that event, say a_{is}. The a_{is} are strictly positive numbers such that $\sum_{i=1}^{I} \sum_{s=0}^{\infty} a_{is} = 1$.

Let $\bar{a}_i = \sum_{s=0}^{\infty} a_{is}$. Then a straightforward generalization of Theorem 2.b of [3] implies that, R_i exists for all i and that

$$R_i = \bar{a}_i(\eta_i - (1-\bar{a}_i)\xi_i + \sum_{s=1}^{\infty} a_{is}\,\delta_{is} \quad . \tag{3.4}$$

Now suppose $R_i > R_j$ for some i and j. Then

$$V_i(t) = U_i(t) - M(t) \geq U_i(t) - U_j(t) \quad .$$

But this last quantity diverges to $+\infty$ almost surely, which contradicts Proposition 3. Similarly $R_j > R_i$ is impossible and all activities must grow at the same rate.

It only remains to show that this common rate, R^1, exceeds $\bar{\zeta}$. Recall from (1.5) there exists a set of numbers $\hat{a}_i$ such that $\Sigma\, \hat{a}_i = 1$ and

$$\hat{a}_i\eta_i - (1-\hat{a}_i)\xi_i = \bar{\zeta} \,, \qquad i=1,\ldots,I \,. \tag{3.5}$$

Since $\sum_i \bar{a}_i = 1$ it follows that there is an index j such that $\bar{a}_j \geq \hat{a}_j$. Combining (3.4) and (3.5) we get

$$\bar{\zeta} = \hat{a}_j \eta_j - (1-\hat{a}_j)\xi_j \leq \bar{a}_j \eta_j - (1-\bar{a}_j)\xi_j$$

$$< \bar{a}_j \eta_j - (1-\bar{a}_j)\xi_j + \sum_{s=1}^{\infty} \delta_{js} a_{js} = R^1 .$$

The second inequality is strict since (3.2.c) and the fact that all a_{js}'s are strictly positive imply $\sum_{s=1}^{\infty} \delta_{js} a_{js} > 0$.

Proof of Proposition 3. Since $C(t)$ has a single class, it will suffice to show that the expected time to return to a finite set of states is finite. As before, let $D(t) = \max_i U_i(t) - \min_j U_j(t) = \max_i V_i(t)$.

Let G be as in Proposition 2 above, and consider

$$A = \{(M(t), s(t), V(t)) | s(t) = 0, \ D(t) < G\} .$$

For notational ease let $y(t) = (M(t), s(t), V(t))$. Let t_1 be the first date such that $y(t_1) \notin A$ and T be the first date after t_1 such that $y(T) \in A$. I will suffice to show that $E\,T < \infty$.

Consider the set

$$B = \{(m(t), s(t), V(t)) | D(t) < G\} .$$

Suppose
$y(0) \in B$. Let n_1 be the first date $t > 0$ such that
$y(n_1) \notin B$. It follows from (R.3) that

$$\{n_1 \geq k\,G\} < [1-\gamma^{G}]^{k} \qquad \text{for} \quad k=1,2,\ldots$$

so that
$$E\,n_1 \leq G(1 + (1-\gamma^{G})^{-1}) .$$

This bound is independent of $y(0)$. Let N_1 be the first date greater than n_1 such that $y(N_1) \in B$. It follows from Proposition 2 above (which since $\bar{\zeta}_j > 0$ holds for this case) that $E(N_1 - n_1)$ is bounded. Thus there is an L such that $E\,N_1 \leq L$. Now N_1 is the length of the time of the first return to B ; in a similar manner define the random variables n_2 and N_2 as, respectively the length of time it takes $y(t)$ to leave B for the second time and to return to B for the second time. The random variables $n_2, n_3, \ldots, N_3, N_4, \ldots$ are defined similarly. $T_k = \sum_1^k N_j$ is the date of the k^{th} return to B and $E\,T_k \leq kL$.

Consider $y(t)$ from $t=T_j$ to $t=T_j + n_{j+1} - 1$ ($T_j + n_j - 1$ is the date at which $y(t)$ leaves B for the j^{th} time). If for one of those t's, $s(t) = 0$ then we shall say $y(t)$ visits A during its j^{th} visit to B . It follows from (R.3) that the probability of this happening is positive and always exceeds some positive number σ independent of j and of $y(T_j)$.

Thus if the random variable K is defined as the first $k > 1$ such that $y(t)$ visits A during its k^{th} visit to B, $EK < \infty$. Clearly the time of $y(t)$'s first return to A, T, is less than $\sum_{k=j}^{K+1} N_j$. However, it is an immediate consequence of Lemma 6, below that

$$E \sum_{k=j}^{K+1} N_j \leq E[K+1]L < \infty .$$

This completes the proof of Proposition 3.

<u>LEMMA 6:</u> Suppose

$$S_n = X_1 + X_2 + \ldots + X_n$$

where

$$E\{X_n | \mathcal{F}_{n-1}\} = M_n \leq \mu .$$

If n^* is a stopping time such that $E\, n^* < \mu$, then

$$E\, S_{n^*} \leq \mu\, E\, n^* .$$

<u>PROOF:</u> Let $Y_n = X_n - M_n$, then

$$E[Y_n | \mathcal{F}_{n-1}] = Y_{n-1}$$

and the sequence

$$T_n = Y_1 + Y_2 + \dots + Y_n$$

is a martingale.

Since

$$E[|T_{n+1} - T_n| \mid \mathcal{F}_n]$$

$$= E[|Y_{n+1}| \mid \mathcal{F}_n] = E[|X_{n+1} - M_{n+1}| \mid \mathcal{F}_n]$$

$$\leq E[|X_{n+1}| \mid \mathcal{F}_n] + E[|M_{n+1}| \mid \mathcal{F}_n]$$

$$= 2 M_{n+1} \leq 2 \mu ,$$

it follows from Proposition 5.33 of Breiman [1, pp. 99] that $E\, T_{n^*} = E\, T_1 = 0$. However,

$$T_{n^*} = \sum_1^{n^*} Y_n = \sum_1^{n^*} X_n - \sum_1^{n^*} M_n \geq S_{n^*} - \mu\, n^*$$

thus

$$0 = E\, T_{n^*} \geq E\, S_{n^*} - \mu\, E\, n^*$$

or

$$\mu\, E\, n^* \geq E\, S_{n^*} ,$$

completing the proof.

It is appropriate to examine the costs of putting-out-fires behavior as contrasted with behaviors which involve less frequent reallocations of attention. Consider in particular the following class of behaviors which I shall call putting out fires with delay d behaviors or simply d - delay behaviors:

i) If $s(t-1) < d-1$ then $a_i(t) = a_i(t-1)$. (3.5.a)

ii) If $s(t-1) \geq d$ then putting-out-fires behavior is followed. (3.5.b)

If $d=1$ then (3.5) is simply putting out fires. If d exceeds unity then the agent is required to attend an activity d consecutive times before switching to a new activity. The consequence of following a d-delay behavior is very similar to that of putting out fires. If $\bar{\zeta}_i > 0$ for all i , the proof of Proposition 2 can be trivially adapted so that its conclusion holds under the assumptions (3.1) and (3.2) when a policy of putting out fires with delay d is followed. Proposition 2 is essentially all that was needed to prove Proposition 3 above. The rest of the proof goes through unchanged for this case ; this proves

<u>Proposition 4:</u> If a d-delay behavior is followed and if $\bar{\zeta}_j > 0$, $j=1,\ldots,I$ then the Markov chain $C(t)$ defined in (3. 3) is positive recurrent.

<u>THEOREM 3:</u> If a d-delay behavior is followed and if $\bar{\zeta}_j > 0$, $j=1,\ldots,I$ then all activities grow at a common rate $R^d > \bar{\zeta}$.

To show that putting out fires, or more generally unnecessary switching of attention is costly, it would be nice to show that R^d is monotone increasing in d . I have not been able to do this; however, a somewhat weaker result in the same spirit can be proved.

<u>THEOREM 4:</u> There exists R^{∞} such that $R^{\infty} > R^d$ for all d and $\lim_{d \to \infty} R^d = R^{\infty}$.

<u>PROOF:</u> The proof is given first under the assumption that for all i

$$\delta_{io} = 0, \quad \delta_{is} = \delta_i > 0, \quad s > 0 \quad . \tag{3.6}$$

Afterwards I discuss how to extend the argument to the more general specification (3.2).

Fix d . Proposition 4 implies the long run frequency of occurrence of the event $M(t) = i$ and $s(t) = s$ converges to the invariant probability a^d_{is} . Thus,

$$R^d = \bar{a}^d_i \eta_i - (1-\bar{a}^d_i)\xi_i + \delta_i \sum_{s=1}^{\infty} a^d_{is} \tag{3.7}$$

where

$$\bar{a}_i^d = \sum_{s=0}^{\infty} a_{is}^d \quad .$$

Let $\bar{b}_i^d = \sum_{s=1}^{\infty} a_{is}^d$, the probability that $m(t) = i$ and $s(t) > 0$. Then (3.7) can be written as

$$R^d = \bar{a}_i^d(\eta_i + \delta_i) - (1 - \bar{a}_i^d)\xi_i - (\bar{a}_i^d - \bar{b}_i^d)\delta_i \qquad (3.8)$$

Note that $(\bar{a}_i^d - \bar{b}_i^d) = a_{io}^d > 0$ the probability of the event $m(t) = i$ and $s(t) = 0$. Let $\hat{b}_i$ be the set of weights such that $\hat{b}_i > 0$; $\Sigma \hat{b}_i = 1$ and, for all i,k .

$$\hat{b}_i(\eta_i + \delta_i) - (1 - \hat{b}_i)\xi_i$$

$$= \hat{b}_k(\eta_k + \delta_k) - (1 - \hat{b}_k)\xi_k \quad .$$

Let R^∞ denote the value of this common sum. I will show that

$$R^\infty > R^d \quad . \qquad (3.9)$$

Since both $\sum_i \bar{a}_i^d = \sum_i \hat{b}_i = 1$, there is a k such that $\bar{a}_k^d \leq \hat{b}_k$

so that

$$\begin{aligned} R^d &= \bar{a}_k^d (\eta_k + \delta_k) - (1 - \bar{a}_k^d)\xi_k - a_{ok}^d\delta_k \\ &< \bar{a}_k^d (\eta_k + \delta_k) - (1 - \bar{a}_k^d)\xi_k \\ &\leq \hat{b}_k (\eta_k + \delta_k) - (1 - \hat{b}_k)\xi_k = R^\infty \quad . \end{aligned}$$

This proves (3.9).

To complete the proof we only need show

$$\lim_{d \to \infty} a_{io}^{d} = 0 \tag{3.10}$$

since (3.8), (3.9) and (3.10) imply that $\lim_{d \to \infty} \bar{b}_i^d = \hat{b}_i$ and $\lim_{d \to \infty} R^d = R^\infty$.

Recall that a_{io}^d is just the invariant probability of the event that $M(t) = i$ and attention has just begun to be allocated to i at time t. Now it is well known that the invariant or stationary probability of an event A is just equal to $[E\, T_A]^{-1}$ where T_A is the time to return to A. (See, for example, Breiman [1, Proposition 6.38, p. 123].)

When A^d is the event that $M(t) = i$, $s(t) = 0$, and a d-behavior is followed, then T^d, the time to return to A^d is always greater than d, thus $\lim_{d \to \infty} E(T^d) = \infty$ from which (3.10) follows.

This argument can be adapted to deal with the more general specification (3.2) in an obvious way. By assumption the sequences $\{\delta_{is}\}$ are monotone increasing and bounded.

Let $\bar{\delta}_i = \lim_{s \to \infty} \delta_{is}$. Then $R^\infty = b_i^*(\eta_i + \bar{\delta}_i) - (1-b_i^*)\xi_i$

where the b_i^* are chosen so that $\sum_i b_i^* = 1$ and

$$b_k^*(\eta_k + \bar{\delta}_k) - (1 - b_k^*)\xi_k = b_i^*(\eta_i + \bar{\delta}_i) - (1 - b_i^*)\xi_i .$$

Choose any set of $\tilde{\delta}_i$ such that $\tilde{\delta}_i < \bar{\delta}_i$. Let $\tilde{b}_i$ be a set of weights such that $\Sigma \tilde{b}_i = 1$ and $\tilde{b}_k(\eta_k + \tilde{\delta}_k) - (1-\tilde{b}_k)\xi_k = \tilde{b}_i(\eta_i + \tilde{\delta}_i) - (1 - \tilde{b}_i)\xi_i$. Call this common sum $\tilde{R}$. Then it is straightforward although tedious to show that $\liminf_{d\to\infty} R^d > \tilde{R}$. This completes the proof.

Theorem 5 shows that if (3.2) holds then putting out fires is a policy which leads to a lower growth rate than other policies which change allocations less frequently. This paper closes with a demonstration that the costs of too frequent switching may be more dramatic; specifically, I give an example where putting out fires cannot survive but for large enough d, putting out fires with a delay of d will eventually succeeed. Let $\xi > 0$ and $\eta > 0$ be such that

$$(I - 1)\frac{\xi}{(\eta + \xi)} < 1$$

while

$$I \cdot \frac{\xi}{(\eta + \xi)} > 1$$

or

$$I^{-1} \eta - (1 - I^{-1}) \xi < 0 \tag{3.11}$$

If for all i, $\eta_i = \eta$ and $\xi_i = \xi$, then $\bar{\zeta}_j > 0$, j=1,...,I and $\bar{\zeta} < 0$.

Choose δ so that $I \dfrac{\xi}{(\eta+\delta+\xi)} < 1$, or

or

$$I^{-1}(\eta + \delta) - (1 - I^{-1}) \xi > 0 . \tag{3.12}$$

Suppose that for all i

$$\delta_{is} = \begin{cases} 0 & \text{for } s < S \\ \delta & \text{for } s \geq S \end{cases}$$

Theorem 4 implies that if a d-delay policy is adopted, all activities grow at the rate R^d .

The symmetry of the example implies that

$$R^d = I^{-1}\eta - (1 - I^{-1})\xi_i + a^d_S \delta \tag{3.13}$$

where a^d_S is the relative frequency of the event $M(t) = i$ and $s(t) \geq S$. For fixed d a^d_S is a function of S , in fact $\lim_{S \to \infty} a^d_S = 0$, (3.11) implies it is possible to pick S so that $R^1 < 0$. If the common rate of growth of all activities when putting out fires is followed, R^1, is negative then $\lim U_i(t) \longrightarrow -\infty$ a.s. and survival is not possible.

However, for S fixed, $\lim_{d \to \infty} a_S^d \longrightarrow 1$ so that (3.12) implies $R^\infty > 0$; with S fixed there is a $\hat{d}$ such that $R^d > 0$ for all $d > \hat{d}$. $R^d > 0$ implies for all i $\lim U_i(t) = +\infty$ which in turn implies the d-delay policy eventually succeeds.

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