

MONOPOLISTIC DISCRIMINATION OVER TIME:
REMARKS ON PERIODIZATION IN ECONOMIC MODELS

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I. INTRODUCTION

Suppose the U. S. Treasury wanted to sell its holdings of gold so as to maximize the proceeds. What is the best sales strategy they could pursue? After they will have sold their gold, the equilibrium price of gold will be lower than before. If they have a good knowledge of the demand for gold, they would best like to realize perfect discrimination so as to appropriate the buyers' total consumers' surplus.

But if there is a perfect market for gold, perfect discrimination is ruled out, because at any one time, two units of gold cannot sell for different prices. On the other hand, there is nothing to prevent the ounce of gold to be traded at different prices on subsequent days. This suggests the possibility of a slow sale over a certain length of time. If they can keep a straight face in assuring the market that the current sale of gold will be the last one, they will be able to sell the first units at a higher price. This is traded off against the interest loss from delaying the sale of the last units sold.

The present paper analyses the optimal sales sequence for a monopolist in an asset market, such as the gold market with a large supplier. The problem of finding the optimal time path of sales from a limited supply is similar to the sales problem faced by the owner of a natural resource such as oil. The major difference arises from the fact that the commodity is an asset that is not continuously used up, so that the market price will depend on the total quantity outstanding and sales in any one period affect the price in future periods through their effect on the outstanding quantity of the asset.

In the first part of the paper, we assume that the public has static price expectations. In particular, current sales by the monopolist do not

generate expectations of future sales. In later parts of the paper, we discuss the role of expectations in an asset market and its significance for the model we consider here. The starting point for this discussion is provided by a postulate that Duncan Foley has recently proposed for period models in economic theory, namely that the length of the period be not an important parameter of the properties of a theoretical model.

The model proposed here does not conform to this postulate; we discuss its failure to do so in terms of the role of speculative expectations. In the very short run, we propose, these are analogous to the multilateral communications that prevent price discrimination in the ordinary market model. Thus, the "period" of the basic model incorporates the effects of speculative expectations. Variations in the length of the period then are illegitimate because the period is tied to the expectations process.

In the final section we use this discussion as a starting point for a new assessment of the role of periodization as a tool of economic theory. It is proposed that the concept of a period is an ordinal concept which represents the amount of information about the time ordering of events that the theorist wants to take into account in the analysis of the interaction between events.

II. THE BASIC MODEL

Consider an asset with a demand price p_t , which depends on the total quantity y_t that is held by the market--excluding the monopolist--in the t -th period. The function $p(y)$ is the same in each period. It is continuously differentiable, positive and bounded above. Its derivative is strictly negative and bounded.

In the t -th period, the monopolist sells z_t units of the asset to the public, so that the public holds $y_t = y_{t-1} + z_t$ units of the asset. The equilibrium price in the t -th period is $p(y_t)$, the price at which the public is willing to hold the outstanding amount. All transactions in the t -th period including the sale of z_t units by the monopolist take place at the equilibrium price. Thus, within each period the perfect market assumption that there is a single price for the asset is satisfied.

The monopolist's sales in the t -th period bring a revenue of $z_t p(y_t)$ the initial present value of which is $\alpha^t z_t p(y_t)$, where α is the discount factor. The present value of total sales is:

$$\sum_{t=0}^{\infty} \alpha^t z_t p(y_t) = \sum_{t=0}^{\infty} \alpha^t z_t p(y_0 + \sum_{i=0}^t z_i) , \quad (1)$$

where y_0 are the public's initial holdings before any sale by the monopolist has taken place.

The monopolist's initial holdings are X , so that total sales cannot exceed X .

$$\sum_{t=0}^{\infty} z_t \leq X . \quad (2)$$

We constrain sales in each period to be nonnegative. This has the formal convenience of ruling out a Ponzi game based on short sales covered by purchases in the indefinite future. At the same time, because of the peculiar structure, no loss of generality is involved: The constancy of the function $p(\cdot)$ over time takes away the speculative incentive to buy in the hope of a price appreciation. Therefore, the monopolist will not at any time want to buy the asset, because he wants to reduce his overall holding of it, whereas a strategy of first buying and then selling it incurs the interest cost of holding it and the resale price does not exceed the purchase price. If we allowed the function $p(\cdot)$ to shift over time, this argument would no longer hold as there could be a speculative incentive to buy and hold the asset. In that case, nonnegativity of the z_t would be a bad assumption.

The monopolist selects a nonnegative sequence of sales, z , to maximize the present value of sales, (1), subject to the constraint (2).^{1/}

The optimal policy has to satisfy the conditions:

For all t :

$$\alpha^t p(y_t) + \sum_{j=t}^{\infty} \alpha^j z_j p'(y_j) - \lambda \leq 0, \quad (3)$$

^{1/} Under our assumptions, a maximum clearly exists, for $\alpha < 1$: The value of the maximand is bounded between $Xp(y_0)$ and $Xp(y_0+X)$. Impose on the set of sales sequences z , the metric $\rho(z^1, z^2) = \sum_{t=0}^{\infty} \alpha^t |z_t^1 - z_t^2|$. Then, the set of admissible sales sequences is compact and the monopolist maximizes a continuous bounded function over the set of admissible sales sequences.

with an inequality only if $z_t = 0$.

If sales in any period are zero, the value of the rest of the program must be zero, or else, an improvement could be made by moving the whole tail of the program up by one period. Therefore, all sales thereafter must be zero.

In periods, when sales are still positive, the first order conditions hold with equality. Consider equations (3) for successive periods and subtract the condition for $t+1$ from that for t , to get:

$$\alpha^t [p(y_t) + z_t p'(y_t) - \alpha p(y_{t+1})] = 0.$$

Thus, we have proved the following proposition:

Proposition 1: The maximization of the present value of sales (1) subject to the constraint (2) requires that there exist a number T (possibly infinite), such that sales in all periods up to and including T are strictly positive, and no sales are made after period T . Furthermore, the following conditions hold: a) For all $t < T$,

$$p(y_t) + z_t p'(y_t) = \alpha p(y_{t+1}) \quad (4)$$

b) At T :

$$p(y_T) + z_T p'(y_T) \geq \alpha p(y_{T+1}) = \alpha p(y_T) = \alpha p(y_0 + X) \quad (4')$$

Condition a) is immediate from the preceding discussion. Condition b) follows by the same derivation, where one notes that in period $T + 1$, there

are no more sales, so that (3) holds with inequality only and $y_{T+1} = y_T = y_0 + X$, because sales must have exhausted the monopolist's initial holdings.

From Condition b), one has the following corollary:

Corollary: Suppose that

$$X > (1 - \alpha) \frac{p(y_0 + X)}{[-p'(y_0 + X)]} ;$$

then, the monopolist does not sell all his holdings at once.

The corollary to proposition 1 states that the monopolist's holdings have to be large enough for him to find it worth his while to forego some interest income in order to achieve some degree of discrimination.

In Appendix I, I give an explicit solution for the optimal sales program for the case of a linear demand function. In general, explicit solutions are difficult to obtain, both because of the integer character of T and because the curvature of the demand function in later periods affects the optimal sales in all previous periods. This latter problem also makes it very difficult to find simple conditions on the demand function (beyond linearity) that would guarantee the concavity of the maximand over the set of feasible sales sequences.

The important condition for intertemporal price discrimination is given in equation (4). This equation gives the condition on whether the marginal unit should be sold in the t -th or the $t+1$ -st period. On the left hand side is the marginal revenue in the t -th period, on the right hand side the price in the $t+1$ -st period, discounted back to the t -th period. The left hand side gives the cost, the right hand side the benefit of selling the marginal unit in the $t+1$ -st rather than the t -th period. It should be noted that this marginal change leaves the total quantity outstanding in the $t+1$ -st period

unaffected. Therefore, the price in the $t+1$ -st period is not affected by it and there is no term in $p'(y_{t+1})$. It does however affect the price in the t -th period, so that the left hand side gives marginal revenue rather than price. Thus, one has the fundamental rule of intertemporal price discrimination, that the marginal revenue in the t -th period be equal to the price in the $t+1$ -st period discounted back to the t -th period, for any two successive periods $t, t+1$, in which sales are positive.

One gains some additional insight into the nature of the trade-off between the gains from discrimination and the costs of waiting by rewriting (4) as:

$$-z_t p'(y_t) = p(y_t) - p(y_{t+1}) + (1-\alpha) p(y_{t+1}) . \quad (5)$$

The left hand side gives the benefits for t -th period revenue from achieving a higher price for those units that are sold; the right hand side gives the cost of the delay: The second term is a direct interest cost of waiting. The first term is the cost from encountering the price $p(y_{t+1})$ rather than $p(y_t)$ in the next period. In addition to the benefits from improved discrimination and the direct interest cost, one has to take account of price changes from the current to the next period.

If we drop for a moment the assumption that the interest factor is the same for all periods, then, even if there is currently no interest ($\alpha_t = 1$), there would still be a cost of waiting if next period's sales were strictly positive. This cost of waiting would induce current sales to be strictly positive. Thus, a positive interest cost at any later period requires sales in all preceding periods to be strictly positive. The adverse effect on the ability to discriminate that arises from an interest cost in the t -th period

is distributed over all the preceding periods and not concentrated in the t -th period alone. Thus, we may regard the price change term in the cost of waiting as the indirect effect of later interest costs, so that the monopolist may be said to trade off the gains from discrimination against direct interest costs now and indirect effects of later interest costs.

The indirect effects of later interest costs create a strong tendency to discriminate less precisely early in the optimal sales sequence, so that sales per period decrease over time and a given quantity is sold over a smaller number of periods earlier along the path. In fact, if the demand function is linear with $p' = -b$, we can write (5) as:

$$z_t = z_{t+1} + (1-\alpha) p(y_{t+1})/b . \quad (5')$$

In this case, it is quite clear that sales are decreasing over time.

III. THE ROLE OF PERIODIZATION IN THE BASIC MODEL

Economists have recently been warned of the hidden assumptions that are often inherent in period models (May 1970, Foley 1974). To avoid this danger, Foley formulates a central postulate:

"No substantive prediction or explanation in a well-defined macroeconomic period model should depend on the real time length of the period. The results of experiments in a period model must be invariant with respect to period length...In particular, any period model which passes this test can be formulated as a continuous model in which the period has been taken to the limit by decreasing its length to zero."

It seems desirable to analyse our model of intertemporal price discrimination with respect to this postulate. Unfortunately, the "length of the period" is not an explicit variable of the model and has to be introduced as such. A simple specification centers on the interest factor: The longer the "length of the period" the longer the absolute waiting time between two sales, and the higher is the interest cost of delaying a sale. This is formalized by writing:

$$\alpha = e^{-r\Delta t} \quad , \quad (6)$$

where Δt is the length of the period and r the interest rate per unit of absolute time. If we use equation (6) to substitute for the interest factor, we can rewrite the conditions for the optimal sales sequence as:

$$- z_t p'(y_t) = p(y_t) - p(y_{t+1}) + (1 - e^{-r\Delta t}) p(y_{t+1}) \quad (7)$$

$$- z_T p'(y_0+X) \leq (1 - e^{-r\Delta t}) p(y_0+X) \quad (7')$$

In this formulation, variations in the length of the period are equivalent to variations in the rate of interest. Therefore, it will not be surprising that the model violates Foley's postulate, for after all the rate of interest is one of the relevant parameters of the model. In the following, we shall discuss precisely how the length of the period affects the optimal sales sequence. In order to avoid unnecessary complication, we consider only the limit as the length of the period goes to zero.

Proposition 2: If the interest factor in problem (1) is given by (6), then the optimal solution to problem (1) depends on the length of the period as follows: Let T be the last period of positive sales, as in proposition 1;

- a) As Δt goes to zero, T grows out of bounds.
- b) As Δt goes to zero, the present value of sales approaches the value given by immediate perfect discrimination: $\int_0^X p(y_0+z) dz$.
- c) As Δt goes to zero, the absolute time taken to sell off everything, $(T+1)\Delta t$, approaches zero.

Proof: a) We show that for any finite number n , z_{T-n} approaches zero.

From (7'), $\lim_{\Delta t \rightarrow 0} z_T = 0$. Suppose that $\lim_{\Delta t \rightarrow 0} z_{T-n+1} = 0$. Then,

$$\lim_{\Delta t \rightarrow 0} y_{T-n} = \lim_{\Delta t \rightarrow 0} (y_{T-n+1} - z_{T-n+1}) = \lim_{\Delta t \rightarrow 0} y_{T-n+1}.$$

From (7) and the boundedness

of p' , it follows that $\lim_{\Delta t \rightarrow 0} z_{T-n} = 0$, completing the induction. It also

follows that for any finite number n ,

$$\lim_{\Delta t \rightarrow 0} \sum_{i=0}^n z_{T-i} = 0 .$$

Since

$$\lim_{\Delta t \rightarrow 0} \sum_{i=0}^T z_{T-i} = X > 0$$

T must grow out of bounds.

b) The present value of sales can never be larger than the value of immediate perfect discrimination. Hence, it is sufficient to show that there exists a sequence of feasible policies (whether optimal or not), the value of which approaches the value of immediate perfect discrimination as Δt goes to zero. For any Δt , consider the policy of selling X in n equal sales of size X/n each, where n is $[k/\sqrt{\Delta t}]$, the largest integer not exceeding $k/\sqrt{\Delta t}$, k a constant. The present value of sales under this policy is, for given Δt :

$$v = \frac{X}{n} \sum_{j=0}^{n-1} e^{-jr\Delta t} p(y_0 + \frac{j+1}{n} X) > \frac{X}{n} e^{-rn\Delta t} \sum_{j=0}^{n-1} p(y_0 + \frac{j+1}{n} X)$$

If we add and subtract from the right hand side the value of discriminating perfectly at time $n\Delta t$, we have:

$$\begin{aligned} v &> e^{-rn\Delta t} \left\{ \int_0^X p(y_0+z) dz - \sum_{j=0}^{n-1} \left[\int_{j\frac{X}{n}}^{(j+1)\frac{X}{n}} p(y_0+z) dz + \frac{X}{n} \sum_{j=0}^{n-1} p(y_0 + \frac{j+1}{n} X) \right] \right\} \\ &= e^{-rn\Delta t} \left\{ \int_0^X p(y_0+z) dz - \sum_{j=0}^{n-1} \int_{j\frac{X}{n}}^{(j+1)\frac{X}{n}} [p(y_0+z) - p(y_0 + \frac{j+1}{n} X)] dz \right\} \quad (8) \end{aligned}$$

Let q be an upper bound on $-p'$ (which we have assumed to exist). For $z \in [jX/n, (j+1)X/n]$, we can then apply the mean value theorem and write:

$$\begin{aligned} p(y_0 + z) &= p(y_0 + \frac{j+1}{n} X - u) \\ &= p(y_0 + \frac{j+1}{n} X) - u p'(y_0 + \frac{j+1}{n} X - v), \text{ for some } v \in [0, u] \\ &\leq p(y_0 + \frac{j+1}{n} X) + uq \end{aligned}$$

Therefore,

$$\int_{\frac{jX}{n}}^{\frac{(j+1)X}{n}} [p(y_0 + z) - p(y_0 + \frac{j+1}{n} X)] dz \leq \int_0^{\frac{X}{n}} qu du = q(X/n)^2/2 .$$

Substituting this into (8), one finds that:

$$V > e^{-rn\Delta t} [\int_0^X p(y_0 + z) dz - qX^2/2n]$$

For $n = [k/\sqrt{\Delta t}]$, $\lim_{\Delta t \rightarrow 0} n\Delta t = 0$ and $\lim_{\Delta t \rightarrow 0} (1/n) = 0$, so that the right hand

side approaches $\int_0^X p(y_0 + z) dz$, the value of immediate perfect discrimination.

Since V cannot exceed that value, V must approach it, i.e.

$$\lim_{\Delta t \rightarrow 0} V = \int_0^X p(y_0 + z) dz ,$$

as was to be shown.

c) If any positive amount of the asset is held over a period of positive absolute duration, there is a strictly positive interest cost of waiting and

the value of such a program is strictly less than the value of immediate perfect discrimination. From part b, it follows that such a program cannot be optimal as Δt becomes small.

If the periodization of our model of intertemporal price discrimination is given by equation (6), then changes in the length of the period affect the attainable present value of the monopolist's sales in the following way: If the period becomes shorter, the same amount of discrimination can be achieved at a lower interest cost. Alternately, at the same interest cost, the monopolist can achieve a higher degree of discrimination at no interest cost. To put these properties of the periodization (6) into even sharper perspective, we formulate a corollary to proposition 2:

Corollary: The continuous time version of problem (1) with the periodization (6)

(6)

$$\max \int_0^{\infty} e^{-rt} z(t) p[y(t)] dt \quad (9)$$

$$\text{subject to } z(t) = \dot{y}(t)$$

$$\int_0^x z(t) dt = x$$

does not have a solution.

Proof: Consider the sequence of sales policies for varying Δt , used to prove part b of proposition 2. Each of these policies is feasible under problem (9). The value of these policies approaches the value of immediate perfect discrimination. By the argument used to prove part c of proposition 2, any policy with a strictly positive sales time is overtaken by an element of this sequence for sufficiently small Δt . Along this sequence, the length

of time needed for active sales approaches zero. The limiting policy consists of selling everything immediately. It earns $Xp(y_0+X)$, which is strictly less than $\int_0^X p(y_0+z) dz$, the limiting value of the sequence of policies, so that the supremum of (9) is not a maximum.

The discontinuity that we are dealing with is due to the fact that we impose no constraint on the time derivative of the price, while we require there to be a single price at any one moment. The ability to discriminate depends on the number of prices called within a given length of time, without any constraint on the movement of these prices over time. As the length of the period, Δt , --the waiting time between different prices--becomes small, the number of prices called during any positive time interval becomes large, countably infinite in the limit. Since a countable number of prices can trace out the whole demand curve, the monopolist can over any positive time interval come as close as he likes to perfect discrimination, for sufficiently small Δt . Thus, there is no cost in terms of the precision of discrimination to a shorter time of active sales, while there is a saving on interest costs. He can always do better by selling over a shorter period; but when the length of the sales time is zero, there is no discrimination at all and he earns merely $Xp(y_0+X)$.^{2/}

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This is shown heuristically from condition (7). After division by Δt , one has:

$$-p'_t z_t / \Delta t = (p_t - p_{t+1}) / \Delta t + r p_{t-1} + o(\Delta t)$$

As Δt goes to zero, the left hand side approaches $-p' \dot{y} = -\dot{p}$. The right hand side approaches $-\dot{p} + r p$, where in the limit, the arguments are the same as on the right hand side. In the limit, the right hand side exceeds the left hand side by rp , implying that (7) cannot hold and sales must be zero, i.e., X is already exhausted. Since this is true at any moment, the whole supply must have been sold at the very beginning, contradicting the desirability of discrimination.

IV. EXPECTATIONS AND THE PERIODIZATION

The most obvious objection to the somewhat paradoxical results of the preceding section is that we have assumed demand to be stationary and that we have not considered speculative expectations. One might expect the more extreme puzzles to be resolved, if one assumes that the public takes notice of the monopolist's sales and takes them as an indicator of his further intentions.

It may be reasonable to think that expectations have some role to play. On the other hand, one has to remember that the sale of the asset by the monopolist is a unique historical event, so that it is difficult to form expectations on it, especially if the monopolist manages to disguise his ultimate intentions. In this respect, the U. S. Treasury for instance should not have too many difficulties.

But even if we do take account of expectations, we are by no means assured of resolving the puzzles of the periodization. To show this, we analyze the problem of the typical holder of the asset in question. Suppose he uses a holding y_t to produce a flow of output $f(y_t) \Delta t$ over a period of length Δt . Also, assume that there is no depreciation, i.e., that the quantity is available for continued production in the next period. Faced with a sequence of prices, $p_t, t = 0, 1, 2, \dots$, he will select a sequence of asset holdings $y_t, t = 0, 1, 2, \dots$ to maximize:

$$\text{Max} \sum_{t=0}^{\infty} \alpha^t [f(y_t) \Delta t - p_t (y_t - y_{t-1})] . \quad (10)$$

From each period's output he must subtract the value of that period's asset purchases. The first order condition for maximization is:

$$\alpha p_{t+1} + f'(y_t) \Delta t - p_t = 0 . \quad (11)$$

Desired holdings in any one period depend only on the price in that and the subsequent period. We write the demand function $y_t = y(p_t, p_{t+1})$. For any given price p_{t+1} in the subsequent period, we obtain the demand price by inverting the demand function

$$p_t = y^{-1}(y_t; p_{t+1})$$

One calculates:

$$p'_t = \frac{\partial p_t}{\partial y_t} = f''(y_t) \Delta t .$$

Although demand is more elastic with respect to current price than in the case where the next period's price is always expected to be the same as this period's, it is not completely elastic, so that there is still some room for the monopolist to maneuver.

The monopolist's problem differs from the one analyzed so far in that demand is no longer stationary. But the model of section II is easily generalized to yield the new first order condition (the analogue to (4)):

$$p_t(y_t) + z_t p'_t(y_t) = \alpha p_{t+1}(y_{t+1}) . \quad (12)$$

Consider now a sequence of sales and prices, such that both the buyer's expectations of future prices and the monopolist's expectations of the buyer's behavior are correct; then (11) and (12) must both hold, and we have:

$$z_t p'_t = - f'(y_t) \Delta t .$$

We can substitute for p'_t from above and get:

$$z_t f''(y_t) \Delta t = - f'(y_t) \Delta t$$

$$z_t = - \frac{f'(y_t)}{f''(y_t)} \tag{13}$$

for periods in which there are positive sales, which do not yet exhaust the monopolist's holdings.

The surprising feature of (13) is that sales are independent of the length of the period and the interest rate. Therefore, as we vary the length of the period, t -th period sales will be unaffected, so the absolute length of time needed by the monopolist to divest himself of his holdings decreases. In fact, it shrinks to zero, as the length of the period vanishes. Then, the same argument as in the corollary to proposition 2 will show that the continuous time problem (9) has no maximum, if we require that the demand price at each instant be derived from the maximization of a problem such as (10), provided that the buyer's price expectations have to be correct.

One may add, that as Δt goes to zero, the value of the optimal program both to the monopolist and to the buyer increases. However, the value to the monopolist does not approach the value of perfect immediate discrimination. The introduction of expectations seems to affect more the distribution between the monopolist and the buyer than the time structure of the problem.

The problem with the preceding discussion is, of course, that whereas

before, the buyer had too little information, now we have given him too much of it. Not only that, we have given him the kind of information that he cannot be expected to have. His current demand depends on the expected price in the subsequent period and how should he know that? Instead of a sequence of actual markets, we have modelled the outcome of simultaneous negotiations with recontracting for a set of futures markets. The length of the period only indicates how fast the actual market can carry out the required transactions for such a sequence of contracts.

An appropriate formulation of the role of expectations in the model of intertemporal monopolistic discrimination would begin with the information the public can be expected to have, namely the history of the asset's prices and possibly the size of sales by the monopolist. From this information, the public presumably tries to guess the monopolist's intentions and thus forms its expectations about the future price of the asset. It is then not difficult to construct models where the current demand for the asset depends through expectations on the flow of current sales by the monopolist.

Such a formulation could overcome the formal puzzles posed in the previous section: one could ensure the existence of an optimal policy in the continuous time model and one could resolve the dependence of the time path of the asset price on the length of the period. But one would find it hard to justify any particular rule for expectations formation beyond its formal convenience. The situation of perfect foresight on both sides of the market leads to unsatisfactory results, and any other expectations model may be artificial.

Therefore, we attack the relationship between expectations and the periodization from another side, namely we discuss the significance of the perfect market constraint that at any one time, any two units of the same commodity are exchanged for the same price. Now we take this condition not

as an assumption, but we analyze the structure of the market that imposes it.

Consider the situation of a monopolist in a single period market who attempts to discriminate between his various customers and to appropriate as much as possible of each customer's surplus. Suppose that he knows each buyer's demand function. Then, he can discriminate perfectly, i.e., appropriate each buyer's total surplus, if the following conditions are satisfied:

- a) he can determine, for each buyer, the maximum quantity to be bought
- b) he can determine, for each buyer and each quantity, the terms of the sale
- c) there is no communication among buyers.

These three conditions together are in fact more than is needed to obtain perfect discrimination. For if only (b) and (c) hold, the monopolist can set the terms of the sale in such a way as to make sure that the buyer will never want more than a certain amount. Similarly, if only (a) and (b) held, he could fix the maximum quantity to be bought by any buyer and the terms of purchase of any quantity in such a way as to make sure that there would be no incentive for buyers to trade among each other, so that communication among buyers would not affect him. His own profit maximization would require that the marginal price charged to each buyer is the same and then there would be no incentive for additional trade among buyers.

If however only (b) holds, i.e., if he can only determine the terms of sale for each quantity and buyer, then communication among buyers as well as the lack of constraints on the quantity purchased by any one of them would make them take advantage of all the available opportunities for arbitrage. The monopolist would be forced to charge the same constant price to every buyer, so that the perfect market condition would be satisfied. Otherwise,

customers who are charged a lower price could buy more than their needs and resell at a profit to those whom the monopolist would charge a higher price.

The structure of the market that guarantees that there is a single price for any one commodity involves a fairly complicated network of multi-lateral negotiations. In principle, the monopolist posts schedules of sales terms to all his potential customers. Then the customers negotiate among each other to determine the cheapest way to buy from the monopolist and to determine the pattern of secondary exchanges among each other. Thereafter they announce to the monopolist the desired quantities. If the monopolist conforms to the perfect market condition and posts the same constant price to every customer, then no secondary exchanges are needed and negotiations are much simpler. But it is their potential presence which forces the monopolist to post a single price to everyone. The perfect market assumption has to be viewed as a consequence of multilateral communication in the market.

In the intertemporal setting, the attempt to anticipate the monopolist's future actions has to be viewed as the analogue of communication and arbitrage between different buyers in a single market. Indeed, in many asset markets with almost continuous trading, in which not every participant is constantly participating, the two forms of arbitrage are quite often the same: Current market participants buy or sell with a view to trade again later, with other participants who are not currently present at the price expected for that time. The major difference between such intertemporal arbitrage based on future price expectations and the secondary exchanges that rule out monopolistic discrimination is the uncertainty about one half of the trade. Since the monopolist's future plans are not known with certainty, intertemporal arbitrage will only work imperfectly. How well it works, will depend on the quality of the forecasts derived from the public's current information. Without

specifying the particular form of expectations formation, we may assume that expectations for the sufficiently near future are quite precise and correct. On the other hand we may consider expectations for the sufficiently distant future to be so imprecise that the public does not consider them but acts myopically. Given the role of expectations as a surrogate for the multilateral communication that prevents discrimination in a single period market, we can analyze markets in the sufficiently near future, where expectations are precise as though they were a single period market with multilateral communication.^{3/} If we then cut the Gordian Knot of expectations and consider only the myopic future apart from this extended present, we obtain the structure analyzed in section II. The "period," over which there is a single price in the market, is not a single market date, but rather a sequence of market dates that are so intimately connected by the participants' expectations as to be a single market for all practical purposes.

Under this interpretation, Foley's postulate no longer is acceptable. The length of the "period" so understood is not an arbitrary parameter of the model, but is intimately connected to the expectations process. Thus, one is not free to vary the length of the period at will. A given length of the period corresponds to the degree of myopia among the public. If the public is completely myopic, the monopolist may in fact initiate a price change from one transaction to the other. If the public perfectly realizes the monopolist's intentions from the start, the price may immediately be driven to its final equilibrium without any chance for the monopolist to

^{3/}This neglects problems of false trading, or transactions costs although they are very important in this context.

discriminate. In the first case, the natural length of the period is zero, in the second case, infinite. Any attempt to vary the length of the period would amount to a substantive change of the model. Of course, if the process under consideration is such that the public's awareness of the monopolist's intentions changes, it may be necessary to have different lengths of the period at different stages of the process.

V. WHAT IS A "PERIOD?"

Since the foregoing discussion is rather directly opposed to Foley's proposals on period analysis, it is appropriate to discuss the concept of a period in greater detail and contrast it with Foley's treatment.

Periodization is a tool which simplifies the analysis of economic processes by treating certain events as synchronous and others sequentially. Insofar as the synchronization is imposed and does not stem from the nature of the process, it represents a restrictive assumption and one would not want the results of a theory to depend on it. Foley argues that most processes do not have natural periods, so that the results of period analysis should not depend on the particular periodization chosen. Therefore, the length of the period should not matter.

While one easily accepts the premise of this postulate, the postulate itself does not necessarily follow. If one finds the synchronization of agents' actions unduly restrictive, the remedy is to consider models without that synchronization. It is not a remedy to postulate that the period over which the synchronization is obtained should not matter. Consider the time path of the equilibria of an economy: For any periodization, this represents a given pattern of synchronization of the actions of all the agents in the

economy. If the length of the period is halved, there is a new time path with twice as many equilibria, i.e., twice as many synchronizations of all agents' actions. To argue that this should not make a difference to the time path may be completely unrelated to the point that in reality there is no complete synchronization at all.

This moral is very strongly brought home by the models considered here, especially the model with expectations where everybody makes the right forecasts. The point about these models is that as we change the periodization, we change the time pattern of the optimization problems that we let agents solve. Thus, different behaviors may correspond to different periodizations.^{4/}

Foley's postulate seems to be motivated by an essentially cardinal understanding of periodization: The unit period has a given, constant length in terms of absolute time as measured by beats of a clock or the number of sunsets. His postulate then derives from the arbitrariness of the assignment of an absolute time period to a particular synchronization of agents' actions. The concept of periodization underlying our discussion is ordinal and not necessarily related to sunsets or oscillations of a pendulum.

Given a set of events and a preordering ("not later than") on this set, we define a "period" as any equivalence class of this preordering. Two events are synchronic, if we cannot tell whether one occurred before or after the other. Thus, synchronization consists both of true simultaneity and of simple ignorance about the precise ordering of events in time. Note, that in such a framework, there is no need for a period to be of constant absolute length, if indeed one wishes to assign absolute lengths to periods at all.

^{4/}This distinguishes the "period" of economic theory from the observation period in empirical work. The latter is determined by the rhythm of data collection and does not affect the underlying economic process.

The set of all events is completely partitioned by the periodization. We may translate the "variation" in the length of the period into the generation of a new partition of the set of events, which is strictly finer or strictly coarser than the first partition. For instance, if the new periodization has the property that all events A, B with A later than B under the first ordering still have A later than B, and in addition, there are some pairs of events C, D with C not later than D and D not later than C under the first ordering, but C later than D under the new ordering, then the new periodization is strictly finer than the old one and we may define this to be the meaning of "shortening the length of the period."

Thus, different periodizations correspond to different degrees of preciseness of the time ordering of events. In general, one would not expect a model with a lot of information about the time ordering of events to give the same results as one with little information. The information lost through a lengthening of the period may be crucial to a proposition derived for the shorter period, so that with the longer period, the proposition no longer need hold.

Furthermore, if one desires to shorten the period, i.e., to make the time ordering of events more precise, one must be aware of the fact that there are many ways in which this can be done: The less precise model lacks information about the time ordering of "synchronous" events precisely because there is more than one way of specifying this information.

In particular, one cannot automatically make the periods under the finer ordering replicas of the periods under the coarser ordering. For instance, in analyzing a period model of income, saving and portfolio choice, it is not appropriate in halving the length of the period, to assume that

the income in each half period is half the income of the whole period: As one goes to a continuous time model, this method would lead to a model of income accruing as a continuous stream. But that would contradict any notion that income is paid out discretely and generates a transactions demand for cash.

In the present context, if we assume that the market finds its equilibrium inside the period, upon halving the period, we need not obtain an equilibrium for each of the half periods. We saw that the concept of the market underlying the perfect market assumption gave an essential role to multilateral negotiations. As one shortens the length of the period and imposes a finer time structure, one comes to analyzing the negotiations preceding the exchange rather than considering more and more exchanges. The latter however was what we did in varying the length of the period in sections III and IV.

To summarize, we consider Foley's postulate inapplicable to our model, because we consider periodization as an ordinal concept where the length of a period has informational implications, which make variations of the length of the period a nontrivial matter.

On the other hand, we can now notice a peculiar feature of our interpretation of the basic model in terms of expectations. For where the actual market process has a sequence of exchanges in very short succession, so that negotiations for the later exchanges come after the earlier exchanges have been executed, we assumed that the nature of short run expectations was such that we could represent this sequence of exchanges by a single "larger" exchange preceded by the whole multilateral negotiation. Thus, we have in fact reversed the time ordering of certain pairs of events from what it is in reality.

While this assumption is critical, we do not think that it affects the conclusions of the basic model. It does confirm however the lack of a good theory of the interaction of individuals with the market in the very short run.

VI. CONCLUSION

In this paper, we have developed the fundamental rule of intertemporal price discrimination, namely that the current marginal revenue be equal to the present value of the future price. It is easy to generalize the basic model by allowing for production costs. This would apply for instance to the situation of South Africa in the gold market. The new rule then requires that the excess of marginal revenue over marginal cost be equal to the present value of the excess of the future price over the future marginal cost. In general, the introduction of production costs will make it desirable to spread sales over a longer overall period of time.

Other applications of the fundamental result can be made in financial markets. The problem of a large shareholder such as the original owner of a company shortly after the company went public is similar to that discussed here. For such practical applications it is necessary though to specify the process of expectations formation, which we avoided here in order to give the general principle. This is the more important, the easier it is for the public to guess the monopolist's intentions. This point is very well illustrated by the problem of a large market participant acting like a monopsonist who tries to gain control of a company. As soon as the public can guess his intentions, it will move to the new equilibrium price without giving him more room for discrimination.

At the methodological level, the ordinal interpretation of the concept of a period that we propose seems able to solve various formal puzzles of economic dynamics. For instance, the choice between stock and flow equilibrium models which Foley discusses can be seen as a judgment of relative adjustment speeds for asset and commodity markets without any presumption about absolute adjustment speeds in either market.

APPENDIX: THE OPTIMAL POLICY FOR A LINEAR DEMAND FUNCTION

Given a linear demand function $p(z) = a - bz$, the optimal policy satisfies the conditions

$$p_t - bz_t = \alpha p_{t+1} \quad \text{for } t = 0, 1, 2, \dots, T-1 \quad (1)$$

$$p_T - bz_T \geq \alpha p_{T+1} = \alpha p_T \quad (1')$$

Since the demand function is linear, we have $bz_t = p_{t-1} - p_t$. Substituting this into (1), we have a second order difference equation for prices:

$$\alpha p_t - 2 p_{t-1} + p_{t-2} = 0 ; \text{ with the initial conditions:} \quad (2)$$

$$p_{-1} = a - by \quad (2a)$$

$$p_T = a - by - bX \quad (2b)$$

The solution of the difference equation is:

$$p_t = [k_1 (1 + \sqrt{1-\alpha})^{t+1} + k_2 (1 - \sqrt{1-\alpha})^{t+1}] / \alpha^{t+1} \quad (3)$$

with

$$k_1 = \frac{\alpha^{T+1} (a-by-bX) - (1 - \sqrt{1-\alpha})^{T+1} (a-by)}{(1 + \sqrt{1-\alpha})^{T+1} - (1 - \sqrt{1-\alpha})^{T+1}} \quad (3a)$$

$$k_2 = \frac{(1 + \sqrt{1-\alpha})^{T+1}(a-by) - \alpha^{T+1}(a-by-bX)}{(1 + \sqrt{1-\alpha})^{T+1} - (1 - \sqrt{1-\alpha})^{T+1}} \quad (3b)$$

To this price path corresponds the sales path:

$$z_t = (p_{t-1} - p_t)/b = \sqrt{1-\alpha} [k_2(1 - \sqrt{1-\alpha})^{t+1} - k_1(1 + \sqrt{1-\alpha})^{t+1}]/b\alpha^{t+1} \quad (4)$$

Since T is the last sales period, z_t is strictly positive for all $t \leq T$. Hence, the term in brackets on the right hand side of (4) is strictly positive for all $t \leq T$. This will be satisfied, if:

$$k_2(1 - \sqrt{1-\alpha})^{T+1} > k_1(1 + \sqrt{1-\alpha})^{T+1} \quad (5a)$$

We can also substitute for $-bz_T$ in condition (1'). If we use equation (3), we can then write (1') as:

$$k_2(1 - \sqrt{1-\alpha})^{T+2} \leq k_1(1 + \sqrt{1-\alpha})^{T+2} \quad (5b)$$

We can substitute in (5a) and (5b) for k_1, k_2 from equations (3a), (3b).

Remembering that for all x

$$(1 + \sqrt{1-\alpha})^x (1 - \sqrt{1-\alpha})^x = \alpha^x,$$

we have, after rearranging terms:

$$2(a-by) > (a-by-bX) [(1 + \sqrt{1-\alpha})^{T+1} + (1 - \sqrt{1-\alpha})^{T+1}] \quad (6a)$$

and

$$2(a-by) \leq (a-by-bX) [(1 + \sqrt{1-\alpha})^{T+2} + (1 - \sqrt{1-\alpha})^{T+2}] \quad (6b)$$

Since the right hand side of (6a), resp. (6b) is monotone increasing in T , there is a unique value of T , that satisfies both inequalities. Thus we have shown that the first order conditions have a unique solution. Since we also know that a maximum exists and satisfies the first order conditions, the unique path described by (3), (3a), (3b), (6a), (b) is the solution to the maximization problem under a linear demand curve.

From conditions (6a) and (6b), one immediately deduces that the optimal number of sales periods, $T+1$, does not decrease as the rate of interest falls. In particular, if we neglect higher order terms and write the interest factor,

$$\alpha = 1 - r\Delta t,$$

where Δt is the length of the period, the optimal number of sales periods does not fall as Δt becomes smaller and it grows out of bounds as Δt approaches zero. In addition, consider the solution $S(\Delta t)$ of the equation:

$$2 \frac{a-by}{a-by-bX} = (1 + \sqrt{r\Delta t})^{S+2} + (1 - \sqrt{r\Delta t})^{S+2}$$

We note that both $(1 + \sqrt{r\Delta t})^{1/\sqrt{r\Delta t}}$ and $(1 - \sqrt{r\Delta t})^{1/\sqrt{r\Delta t}}$ are decreasing in Δt . Therefore, the quantity $S(\Delta t)\sqrt{\Delta t}$ is decreasing in Δt . As Δt goes to zero, we have:

$$\lim_{\Delta t \rightarrow 0} \{S(\Delta t)\sqrt{\Delta t}\} = \cosh^{-1}\left(\frac{a-by}{a-by-bX}\right)$$

By (6a), $T(\Delta t) = [S(\Delta t)]$ and therefore, $\lim_{\Delta t \rightarrow 0} \{T(\Delta t)\sqrt{\Delta t}\} = \lim_{\Delta t \rightarrow 0} \{S(\Delta t)\sqrt{\Delta t}\}$.

As this quantity is finite, we have:

$$\lim_{\Delta t \rightarrow 0} T(\Delta t)\Delta t = 0$$

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