ON THE CONTROL OF NONLINEAR ECONOMETRIC SYSTEMS WITH UNKNOWN PARAMETERS

Gregory C. Chow

ECONOMETRIC RESEARCH PROGRAM
Research Memorandum No. 175
March 1975

The research described in this paper
is supported by NSF Grant GS43747X.
This paper was typed by Alice Furth.

Econometric Research Program
PRINCETON UNIVERSITY
207 Dickinson Hall
Princeton, New Jersey

ON THE CONTROL OF NONLINEAR ECONOMETRIC SYSTEMS WITH UNKNOWN PARAMETERS

Gregory C. Chow*

An approximate solution, based on the method of dynamic programming, is provided for the optimal control of a system of nonlinear structural equations in econometrics with unknown parameters using a quadratic loss function. It generalizes the methods previously proposed by the author for the control of a nonlinear econometric model with constant parameters and of a linear econometric model with uncertain parameters. It is an improvement over the method of certainty equivalence which replaces the unknown parameters by their mathematical expectations and utilizes the solution for the resulting model. Since the solution is given in the form of feedback control equations, many of the useful concepts and techniques developed in the theory of optimal feedback control for linear systems are now applicable to the control of nonlinear systems using the method proposed, including the calculation of the expected loss of the system under control by analytical rather than Monte Carlo techniques.

1. INTRODUCTION

In this paper, I present an approximate solution to the optimal control of a system of nonlinear structural equations using a quadratic welfare loss function when the parameters of the system are unknown. This is a generalization of the solution given in Chapter 12 of Chow [2] for the control of

*The research described in this paper was supported by the National Science Foundation Grant GS43747X and was performed while the author was consultant to the NBER Computer Research Center.

nonlinear econometric systems with known parameters. It is also a generalization of the solution given in Chow [1] for the control of linear econometric systems with unknown parameters. It applies the method of dynamic programming to solve an optimal control problem involving a nonlinear econometric system with unkown parameters.

This paper advances the state of the art in the control of nonlinear econometric systems as it improves upon the certainty-equivalence solution which is obtained by replacing the random parameters in a system by their mathematical expectations. It provides for a set of numerical feedback control equations based on a system of nonlinear structural equations in econometrics. It will show that many useful analytical concepts and tools developed in the theory of control of linear systems are indeed applicable to the control of nonlinear systems. Furthermore, in the derivation of an approximate solution using the method of dynamic programming, it will indicate precisely where the approximation takes place and why an exact solution is difficult to achieve.

In section 2, we set up the control problem and provide an exact solution to the optimal control problem for the last period. In section 3, we give an approximate solution to the multiperiod control problem using dynamic programming. In section 4, the mathematical expectations required in the solution of section 3 will be evaluated approximately to simplify computations. Section 5 contains some concluding remarks.

2. AN EXACT SOLUTION TO A ONE-PERIOD CONTROL PROBLEM

The t-th observation of the i-th structural equation is written as

$$y_{it} = \Phi_i(y_t, y_{t-1}, x_t, \eta_{it}) \qquad (i = 1,\ldots,p) \tag{2.1}$$

where y_{it} is the t-th observation on the i-th dependent variable, y_t is a column vector of p dependent variables, x_t is a vector of q control variables, and η_{it} is a vector of unknown parameters in equation i including the random residual and variables not subject to control. Higher-order lagged dependent variables are eliminated by the introduction of appropriate new dependent variables and identities. Denoting the column vector of functions $\Phi_1,\ldots,\Phi_p$ by Φ, we write the system of structural equations as

$$y_t = \Phi(y_t, y_{t-1}, x_t, \eta_t) \tag{2.2}$$

where η_t consists of η_{it} as elements. In this section and the following we assume that a posterior density function for η_t is available after the system is observed for certain periods. We further assume that, when we look ahead in the calculation of the optimal decision x_1 for the first period in a T-period control problem, we ignore the possibility of learning further about these parameters.

To apply the method of dynamic programming to solve the T-period control problem having the loss function

(2.3) $$W = \sum_{t=1}^{T} (y_t - a_t)'K_t(y_t - a_t) = \sum_{t=1}^{T} (y_t'K_ty_t - 2y_t'K_ta_t + a_t'K_ta_t)$$

where a_t are given targets, and K_t are known symmetric positive semidefinite matrices, we first consider in this section the decision problem for the last period T, given all information available at the end of period $T-1$. Note that the vector x_t is imbedded in the vector y_t so that the loss function (2.3) has only y_t as arguments. In the last period, the problem is to minimize with respect to x_T the conditional expectation (on all information available up to the end of $T-1$) which is assumed to exist:

(2.4) $$V_T = E_{T-1}(y_T'K_Ty_T - 2y_T'K_Ta_T + a_T'K_Ta_T) = E_{T-1}(y_T'H_Ty_T - 2y_T'h_T + c_T)$$

where, for ease of future generalization, we have defined

(2.5) $$H_T = K_T \; ; \; h_T = K_Ta_T \; ; \; c_T = a_T'K_Ta_T \; .$$

The expectation in (2.4) is taken over the random vector η_T on which y_T depends according to (2.2). The vector y_{T-1} is taken as constant.

The one-period optimization problem just described can be solved exactly, at least in principle. Given any x_T, the probability distribution of y_T is induced by the probability distributions of η_T by the use of (2.2). If the nonlinear function Φ is complicated, the distribution of y_T may be difficult to express explicitly, but it can always be evaluated numerically, at least by Monte Carlo techniques. Similarly, the expectation of the quadratic function (2.4) of y_T can also be evaluated. This expectation can then be minimized with respect to x_T by some numerical method. The solution $\hat{x}_T$ will depend on y_{T-1}. If this dependence can be expressed explicitly, we can eliminate x_T as an unknown in the optimal control problem for the last two periods T and $T-1$, and reduce the two-period problem to one involving only one set of control variables x_{T-1} using the method of dynamic programming. If this dependence is not explicitly expressed, one can hardly solve the two-period optimization problem using a closed-loop strategy. It would be possible to find an optimal strategy among the "open loop" policies which specify both x_{T-1} and x_T simultaneously at the beginning of $T-1$, since the expectation

$$(2.6) \qquad E_{T-2}\left[\sum_{t=T-1}^{T} (y_t'K_t y_t - 2y_t'K_t a_t + a_t'K_t a_t) \right]$$

can in principle be evaluated as a function of x_{T-1} and x_T. However, the truly optimal strategy for the last two periods is of the closed loop form; it allows for the choice of x_T sequentially after the outcome y_{T-1} at the end of period $T-1$ is observed.

Therefore, to solve a multiperiod optimal control problem, it is desirable to express the optimal policies x_t for later periods as functions of the initial conditions y_{t-1}. To do so, some approximation is required. Even for a one-period problem, an approximation would be useful because an exact solution as described in the last paragraph can be very costly. To derive the distribution of y_T from the distributions of η_T using Monte Carlo methods, one may have to sample many times from a distribution involving many random variables, and, for each sample of η_T , some numerical method such as the Gauss-Seidel has to be applied iteratively to obtain a numerical solution for y_T . In this section, we will give an exact solution to the one-period control problem. In the next section, we will introduce approximations to the solution of the multiperiod control problem.

To obtain an exact solution for the minimization of (2.4) we differentiate with respect to x_T and interchange the order of integration (or taking expectation) and differentiation, recalling that the expectation is over the random vector η_T , given y_{T-1} and x_T :

(2.7)
$$\frac{\partial}{\partial x_T} E_{T-1}(y_T'H_Ty_T - 2y_T'h_T + c_T)$$
$$= E_{T-1} \frac{\partial}{\partial x_T} (y_T'H_Ty_T - 2y_T'h_T)$$
$$= 2E_{T-1} \frac{\partial y_T'}{\partial x_T} (H_Ty_T - h_T) = 0$$

where the chain rule of differentiation has been applied, and $\partial y_T'/\partial x_T$ denotes the $q \times p$ matrix of derivatives of the p elements of y_T with respect to the q elements of x_T . The solution x_T satisfies the last equation of (2.7).

It will be useful to write the solution in a different form for future use. Define $\tilde{y}_T$ as the solution of

(2.8)
$$\tilde{y}_T = \Phi(\tilde{y}_T, \overset{o}{y}_{T-1}, \tilde{x}_T, \eta_T)$$

for the given y°_{T-1} and some $\tilde{x}_T$; $\tilde{y}_T$ is a function of the random vector η_T as given by (2.8). Perform a first-order Taylor expansion of (2.2) about $\tilde{y}_T$, y°_{T-1} , and $\tilde{x}_T$:

$$(2.9) \qquad y_T \simeq \tilde{y}_T + B_{1T}(y_T - \tilde{y}_T) + B_{2T}(y_{T-1} - y^{\circ}_{T-1}) + B_{3T}(x_T - \tilde{x}_T)$$

where

$$(2.10) \qquad B'_{1T} = \frac{\partial \Phi'}{\partial y_T} = \left(\frac{\partial \Phi_1}{\partial y_T} \cdots \frac{\partial \Phi_p}{\partial y_T}\right) ;$$

$$B'_{2T} = \frac{\partial \Phi'}{\partial y_{T-1}} ; \quad B'_{3T} = \frac{\partial \Phi'}{\partial x_T} ;$$

and all derivatives of Φ are evaluated at $\tilde{y}_T$, y°_{T-1} and $\tilde{x}_T$ and are functions of η_T . The reduced form of the linearized structure (2.9) is

$$(2.11) \qquad y_T = A_T(\eta_T)y_{T-1} + C_T(\eta_T)x_T + b_T(\eta_T)$$

where

$$(2.12) \qquad A_T(\eta_T) = (I - B_{1T})^{-1}B_{2T} ,$$

$$C_T(\eta_T) = (I - B_{1T})^{-1}B_{3T} ,$$

$$b_T(\eta_T) = \tilde{y}_T - A_T y^{\circ}_{T-1} - C_T \tilde{x}_T .$$

Using the linearized model (2.9) or (2.11), we can express the solution of the last equation of (2.7) for x_T in an iterative form.

An iterative solution of (2.7) is as follows. First, start with some $\tilde{x}_T$, and define the random function $\tilde{y}_T$ by (2.8). Second, use the linearized random function (2.9) to replace y_T in (2.7) by the right-hand side of (2.11),

$$(2.13) \qquad E_{T-1}[C_T'H_T(A_Ty_{T-1} + C_Tx_T + b_T - C_T'h_T] = 0$$

and solve the resulting equation for x_T, yielding

$$(2.14) \qquad \hat{x}_T = -(E_{T-1}C_T'H_TC_T)^{-1}[(E_{T-1}C_T'H_TA_T)y_{T-1} + E_{T-1}C_T'(H_Tb_T - h_T)]$$

$$= G_Ty_{T-1} + g_T$$

where

$$(2.15) \qquad G_T = -(E_{T-1}C_T'H_TC_T)^{-1}(E_{T-1}C_T'H_TA_T) ,$$

$$g_T = -(E_{T-1}C_T'H_TC_T)^{-1}(E_{T-1}C_T'H_Tb_T - E_{T-1}C_T'h_T) .$$

Third, evaluate $\hat{x}_T$ by (2.14) at $y_{T-1} = \overset{\circ}{y}_{T-1}$, and use this value of $\hat{x}_T$ as $\tilde{x}_T$ in the first step. Repeat these three steps until $\tilde{x}_T$ converges.

We claim that if the above iterative process converges, the resulting $\tilde{x}_T$ is an exact solution to (2.7) or to our one-period optimal control problem. This claim is justified if, at convergence, the y_T satisfying the linear function (2.9) or (2.11) is identical with the $\tilde{y}_T$ satisfying the original nonlinear structural equation (2.8). When $x_T = \tilde{x}_T$ and $y_{T-1} = y^{\circ}_{T-1}$, the linear equation (2.9) is reduced to

$$(2.16) \qquad y_T = \tilde{y}_T + B_{1T}(y_T - \tilde{y}_T)$$

or

$$(I - B_{1T})y_T = (I - B_{1T})\tilde{y}_T ,$$

implying $y_T = \tilde{y}_T$, provided $I - B_{1T}$ is nonsingular. Thus, at convergence of our iterative procedure, the y_T given by (2.9) or (2.11) is identical with the $\tilde{y}_T$ given by the nonlinear equation (2.8), and our method provides an exact solution to (2.7).

The reader will have noted that the expectations involved in the computation of G_T and g_T by (2.15) can be difficult to evaluate numerically. The matrices A_T and C_T can be complicated functions of the random variables η_T . We have written the solution to the one-period control problem in the above form to facilitate its generalization to the multi-period case by suitable approximations. Leaving aside the problem of evaluating the expectations in (2.15) until section 4, we proceed in section 3 to obtain an approximate solution to the multiperiod control problem by the method of dynamic programming.

3. AN APPROXIMATE SOLUTION TO MULTIPERIOD CONTROL BY DYNAMIC PROGRAMMING

We utilize the feedback control equation (2.14) for $\hat{x}_T$. Note that this equation provides an exact solution only when $y_{T-1} = y^{\circ}_{T-1}$ since it was derived by using the linear approximation (2.9) or (2.10) for (2.8) and the linear approximation is exact only when $y_{T-1} = y^{\circ}_{T-1}$. All the derivatives in the matrices A_T and C_T are evaluated at $y_{T-1} = y^{\circ}_{T-1}$. For other values of y_{T-1} other than y°_{T-1} , the solution (2.14) is only approximate as a consequence of the linear approximation (2.9). However, we need this approximately optimal feedback control equation to eliminate x_T in order to carry out the dynamic programming solution.

Substituting the right-hand side of (2.14) for x_T in (2.11) and the result for y_T in (2.4), we have the minimum expected loss for the last period

$$(3.1)\qquad \hat{V}_T = E_{T-1}[(A_T + C_TG_T)y_{T-1} + b_T + C_Tg_T]'H_T[(A_T + C_TG_T)y_{T-1} + b_T + C_Tg_T]$$

$$- 2E_{T-1}[(A_T + C_TG_T)y_{T-1} + b_T + C_Tg_T]'h_T + c_T$$

$$= y'_{T-1}E_{T-1}(A_T + C_TG_T)'H_T(A_T + C_TG_T)y_{T-1}$$

$$+ 2y'_{T-1}E_{T-1}(A_T + C_TG_T)'(H_Tb_T - h_T)$$

$$+ E_{T-1}(b_T + C_Tg_T)'H_T(b_T + C_Tg_T)$$

$$- 2E_{T-1}(b_T + C_Tg_T)'h_T + c_T .$$

$\hat{V}_T$ is exactly the minimum expected loss for period T only if $y_{T-1} = y^{\circ}_{T-1}$ in which case the linear approximation (2.9) to y_T is exact. We will use (3.1) to approximate the minimum expected loss and treat it as a quadratic function of y_{T-1} . Since y°_{T-1} is unknown before the end of period T-1 , we will have to perform the linearization (2.9) about some guess of y°_{T-1} , realizing that the matrices A_T and C_T of the resulting derivatives will be affected by this guess.

We proceed to include the period T-1 in our optimization problem. By the principle of optimality in dynamic programming, we minimize with respect to x_{T-1} the expression

(3.2) $$V_{T-1} = E_{T-2}(y'_{T-1}K_{T-1}y_{T-1} - 2y'_{T-1}K_{T-1}a_{T-1} + a'_{T-1}K_{T-1}a_{T-1} + \hat{V}_T)$$

since the optimal policy x_T for the last period has been found and incorporated in $\hat{V}_T$. Substituting the quadratic function of y_{T-1} as given by (3.1) for $\hat{V}_T$ in (3.2), we have

(3.3) $$V_{T-1} = E_{T-2}(y'_{T-1}H_{T-1}y_{t-1} - 2y'_{T-1}h_{T-1} + c_{T-1}) ,$$

where

(3.4) $$H_{T-1} = K_{T-1} + E_{T-1}(A_T + C_TG_T)'H_T(A_T + C_TG_T)$$

$$= K_{T-1} + E_{T-1}(A'_TH_TA_T) + G'_T(E_{T-1}C'_TH_TA_T) ,$$

(3.5) $$h_{T-1} = K_{T-1}a_{T-1} + E_{T-1}(A_T + C_TG_T)'(h_T - H_Tb_T)$$

$$= K_{T-1}a_{T-1} + E_{T-1}(A_T + C_TG_T)'h_T$$

$$- E_{T-1}(A'_TH_Tb_T) - G'_T(E_{T-1}C'_TH_Tb_T) ,$$

$$(3.6) \qquad c_{T-1} = E_{T-1}(b_T + C_T g_T)'H_T(b_T + C_T g_T) - 2E_{T-1}(b_T + C_T g_T)'h_T$$

$$+ a'_{T-1}K_{T-1}a_{T-1} + c_T \ .$$

Observe that (3.3) has the same form as (2.4) with the subscript T-1 replacing T . The steps following (2.4) can therefore be repeated to yield the solution $\hat{x}_{T-1}$ as given by (2.14) with T-1 replacing T . When this solution is substituted in (3.3) and a similar approximation is used, the minimum expected loss $\hat{V}_{T-1}$ for the last two periods becomes a quadratic function of y_{T-2} as given by (3.1) with T-1 replacing T . The process continues until the approximately optimal policy $\hat{x}_1$ for the first period and the associated expected loss $\hat{V}_1$ for the T-period policy are obtained. Our multiperiod control problem is solved.

We will state our solution in the form of an iterative procedure consisting of the following steps.

1. Choose some initial guess $\tilde{x}_1, \ldots, \tilde{x}_T$ of the vectors of control variables for the T periods. Using the econometric model, with the unknown parameters and disturbances η_t set equal to their expected values, and the above values of control variables, solve for a set of initial values $y_1^\circ, \ldots, y_{T-1}^\circ$ of the dependent variables by the Gauss-Seidel method.

2. For each period t, t=1,..., T, linearize the nonlinear model for y_t about the above values of y_{t-1}° and $\tilde{x}_t$ as is done in (2.9), using a value of η_t drawn at random from the given distribution. In other words, consider the linearized structure

$$(3.7) \qquad y_t \simeq \tilde{y}_t + B_{1t}(y_t - \tilde{y}_t) + B_{2t}(y_{t-1} - y^{\circ}_{t-1}) + B_{3t}(x_t - \tilde{x}_t)$$

where $\tilde{y}_t$ is the solution of

$$(3.8) \qquad \tilde{y}_t = \Phi(\tilde{y}_t, y^{\circ}_{t-1}, \tilde{x}_t, \eta_t)$$

obtained by some iterative method such as the Gauss-Seidel. (The iterations to solve (3.8) could be saved by using y°_t obtained in step 1 for $\tilde{y}_t$, this value being the solution of (3.8) corresponding to the expected value of η_t. But we save the approximations and computational short-cuts for section 4.) Computationally the partial derivatives B_{1t}, B_{2t} and B_{3t} are easy to obtain. Each derivative is the change in the value of Φ from $\Phi(\tilde{y}_t, y^{\circ}_{t-1}, \tilde{x}_t, \eta_t)$ with respect to a small change in one element of its first three (vector) arguments. Having computed B_{1t}, B_{2t} and B_{3t}, we compute $A_t(\eta_t)$, $C_t(\eta_t)$ and $b_t(\eta_t)$ using (2.12).

3. The expectations $E_{T-1}C'_TH_TC_T$, $E_{T-1}C'_TH_TA$, $E_{T-1}C'_TH_Tb_T$ and $E_{T-1}C'_T$ are computed by using numerical integration or Monte Carolo methods, the latter by averaging over repeated random drawings of η_T in step 2. Compute G_T and g_T by (2.15) and obtain the optimal $\hat{x}_T$ associated with y°_{T-1} by (2.14). Replace $\tilde{x}_T$ by this $\hat{x}_T$ and repeat step 2 for period T and step 3 until $\tilde{x}_T$ converges. As we have pointed out in section 2, the solution $\tilde{x}_T$ is optimal for the last period, provided that the initial condition is indeed y°_{T-1}.

4. Using the expectations $E_{T-1}C'_TH_TC_T$, $E_{T-1}C'_TH_TA_T$, $E_{T-1}C'_TH_Tb_T$, $E_{T-1}A'_TH_TA_T$ and $E_{T-1}A'_TH_Tb_T$ and the feedback control coefficients G_T in step 3, we compute H_{T-1} and h_{T-1} by (3.4) and (3.5) respectively. H_{T-1} can be applied

to evaluate the expectations $E_{T-2}C'_{T-1}H_{T-1}C_{T-1}$, etc., as in step 3 and to compute $\hat{x}_{T-1}$, G_{T-1} and g_{T-1} by (2.14) and (2.15), with $T-1$ replacing T. This $\hat{x}_{T-1}$ will replace $\tilde{x}_{T-1}$ and the process is repeated until $\tilde{x}_{T-1}$ converges. Essentially, in step 4 so far, we utilize the H_{T-1} and h_{T-1} obtained from the results of step 3 in order to repeat step 3 for $T-1$. The solution will be an optimal $\tilde{x}_{T-1}$ associated with the given y°_{T-2}. Similarly, we can utilize the results of step 4 thus far to obtain H_{T-2} and h_{T-2} in order to repeat step 3 for $T-2$. The process continues backward in time until $\tilde{x}_1$ is obtained.

This solution for $\tilde{x}_1$ would be optimal if the y°_{t-1} in each future period t were the true value to be realized. Insofar as the future y's are not known exactly because of the uncertainties in our model, the solution is only an approximate one. However, this solution improves upon the certainty equivalence solution. One version of the certainty-equivalence solution amounts to replacing $E_{t-1}C'_tH_tC_t$, etc., by $\bar{C}_tH_t\bar{C}_t$, etc., where $\bar{C}_t$ is the expected value of C_t. An even cruder version would replace $E_{t-1}C'_tH_tC_t$ by $C'_t(\bar{\eta}_t)H_tC_t(\bar{\eta}_t)$ where C_t is evaluated at the expected value $\bar{\eta}_t$ of η_t. Since C_t is a nonlinear function of η_t, $C(\bar{\eta}_t)$ is not the same as $\bar{C}_t$. Our solution takes into account the uncertainty in the parameters by evaluating the appropriate expectations $E_{t-1}C'_tH_tC_t$, etc.

5. If our solution deviates from the truly optimal because the initial value y°_{t-1} used in the linear approximation for each future period is not the true one, we can improve upon these values by recomputing them in step 1 using the nearly optimal feedback control equations $\hat{x}_t = G_ty_{t-1} + g_t$ obtained in the above 4 steps for the given y°_{t-1}. Given y°_o, we compute

$\tilde{x}_1$ by its feedback equation. The nonlinear model is solved for y_1°, with $\eta_1 = \bar{\eta}_1$. Given y_1°, we compute $\tilde{x}_2$, and so forth. The steps 1 through 4 can be repeated to yield a new set of more nearly optimal feedback control equations. And another round of computations will generate another set of feedback control equations using the previous set to provide initial values in step 1.

6. The minimum expected loss associated with any set of nearly optimal feedback control equations can be computed by our method. The method as described by equation 3.1 through the paragraph following equation 3.6 carries with it the minimum expected loss $\hat{V}_t$ for all future periods from t onward. Each $\hat{V}_t$ has the same form as (3.1) with t replacing T. The total expected loss for T periods is given by V_1. It can be computed by applying (3.1) and using (3.6) to compute c_{t-1} backward in time until c_1 is acquired.

Our method has been derived and described computationally. In the process of describing it, we have contrasted it with the certainty-equivalence solutions (in step 4). We have also found that the method yields linear feedback control equations which are useful in the analysis of macroeconomic policies using an econometric model as more fully discussed in Chow [2]. The minimum expected loss associated with the approximately optimal policy can be analytically computed.

4. APPROXIMATE EVALUATION OF REQUIRED EXPECTATIONS

In section 3, and step 3 in particular, the expectations $E_{t-1}C_t'H_tC_t$, etc., are evaluated by Monte Carlo techniques using random drawings of η_t. This approach can be very costly, and the gain in accuracy may not be worth the cost. We stated that approach in section 3 partly to single out the only source of approximation errors in our method, namely, that of using an inaccurate value of y°_{t-1} in the linear approximation of the nonlinear structure at each stage of the dynamic programming solution. Otherwise, the method described in section 3 would be exact. In this section we introduce a second source of approximation errors which may be tolerable for the sake of economy.

Let us rewrite the required expectations in a more streamlined notation. Denote by Π_t the p by s matrix

$$\Pi_t = (A_t \quad C_t \quad b_t) \tag{4.1}$$

so that the required expectations $E_{t-1}C_t'H_tC_t$, etc., in step 3 of section 3 are submatrices of

$$E_{T-1}(\Pi_t'H_t\Pi_t) \ . \tag{4.2}$$

Denote the s columns of Π by $\pi_1,\ldots,\pi_s$ and the column vector consisting of these columns by π. We suppress the subscript t when understood. If Q is the covariance matrix of π, and $\bar{\pi}$ is the mean of π, we have

$$E\pi\pi' = \bar{\pi}\bar{\pi}' + Q = \begin{bmatrix} \bar{\pi}_1\bar{\pi}_1' & \cdots & \bar{\pi}_1\bar{\pi}_2' \\ & \cdots & \\ \bar{\pi}_s\bar{\pi}_1' & \cdots & \bar{\pi}_s\bar{\pi}_s' \end{bmatrix} + \begin{bmatrix} Q_{11} & \cdots & Q_{1s} \\ & \cdots & \\ Q_{s1} & \cdots & Q_{ss} \end{bmatrix} . \tag{4.3}$$

The i-j element of (4.2) is, with t suppressed,

$$(4.4) \qquad (E\Pi'H\Pi)_{ij} = E\pi_i'H\pi_j = E \text{ tr } (H\pi_j\pi_i')$$

$$= \text{tr } HE\pi_j\pi_i' = \bar{\pi}_i H\bar{\pi}_j + \text{tr } HQ_{ji} .$$

Therefore, once the mean $\bar{\pi}$ and the covariance matrix Q of π are known, the required expectations in (4.2) can be computed by (4.4).

We will now provide an approximation to $\bar{\pi}$ and Q assuming that the econometric model consists of a set of constant, but unknown parameters θ and a vector of random residuals ε_t . Thus η_t consists of θ and ε_t ; η_{it} in equation (2.1) consists of θ_i and ε_{it} . This assumption applies to most econometric models encountered in practice. We also assume that a point estimate $\hat{\theta}$ of θ, a covariance matrix V of the estimator $\hat{\theta}$, and a covariance matrix S of the residual vector ε_t are all available by applying classical estimation methods. Here a change in philosophy from the Bayesian to the classical point of view may be adopted. Rather than treating the unknown parameter θ as random, and consider y_t in (2.2) as induced by the random θ , we replace θ (part of η_t) in (2.2) by $\hat{\theta}$ and consider the resulting estimated model which generates a random vector $\hat{y}_t$ from the random estimator $\hat{\theta}$ and the residuals ε_t . The control problem is to minimize the expectation of a quadratic function in $\hat{y}_t$ thus defined. This viewpoint is explained in Chow [2, p. 243]. Without adopting the above classical viewpoint, one may consider the classical point estimate $\hat{\theta}$ and its covariance matrix V as approximations to the mean and covariance matrix respectively of the Bayesian posterior density of θ . These approximations are valid when the classical estimation method employed is maximum likelihood and the Bayesian posterior density is derived from a diffuse prior density.

In this case, the likelihood function is proportional to the posterior density and its maximum may be close to the mean value of the latter density.

To derive the mean vector $\bar{\pi}$ and the covariance matrix Q of π required in (4.4) from the mean vector and the covariance matrix of η_t (which includes θ and ε_t) we use a first-order approximation of the function π_t of η_t. Let the partial derivatives of π_t with respect to η_t be represented by the matrix

$$(4.5) \qquad D_t = \left(\frac{\partial \pi_{it}}{\partial \eta_{jt}} \right) .$$

Each element of this matrix, evaluated at $\eta_t = \hat{\eta}_t$, can be computed numerically as the rate of change in π_{it}, an element in $\Pi_t = (A_t \quad C_t \quad b_t)$, with respect to a small change in η_{jt}. Once D_t is found, the covariance matrix Q_t of π_t can be approximated by

$$(4.6) \qquad Q_t = D_t W_t D_t'$$

where W_t is the covariance matrix of η_t having as submatrices the given covariance matrices of $\hat{\theta}$ and ε_t. The mean vector $\bar{\pi}$ can be approximated by the value of π associated with $\hat{\eta}_t$; this approximation can be improved by averaging a sample of π's computed from random drawings from the distribution of η_t. This completes our description of the approximate evaluation of the required expectations.

Before closing this section, it is useful to point out that if there exist exogenous variables z_t not subject to control, they can be treated as a subvector in η_t. Our treatment of η_t allows for the possibility of treating z_t as random but having a given distribution. The randomness in z_t generates uncertainty in the dynamic system in the same way as the randomness in the other parameters. If z_t were regarded as fixed, it is a degenerate random vector; it only affects the mean vector of π_t in our model without contributing to its variances and covariances.

5. CONCLUDING REMARKS

A method is proposed for obtaining an approximate solution to the optimal control of a nonlinear econometric system with uncertain parameters. It results from applying the method of dynamic programming. It provides a set of approximately optimal linear feedback control equations. These equations can then be used to study the dynamic properties of the system under control. Insofar as the method is a generalization of the theory of optimal control for linear systems under uncertainty, many of the useful results and concepts from the linear theory can be applied to the nonlinear case. For example, the comparison in Chow [1] of the optimal feedback control equations and the associated expected welfare loss under uncertainty with the corresponding results under the assumption of constant parameters is valid for nonlinear systems.

As a generalization of the method of Chow [2, Chapter 12] for dealing with nonlinear econometric models with given parameters, the method of this paper is computationally not much more difficult. The main complication lies in the computation of $EC_t'HC_t$, etc. in place of $C(\bar{\eta}_t)'H_tC(\bar{\eta}_t)$, etc. in the certainty case. As we have pointed out in section 4, this amounts to calculating the derivatives of the elements of $\Pi_t = (A_t \;\; C_t \;\; b_t)$ with respect to the elements of η_t , and applying the matrix of these derivatives to form an approximate covariance matrix of the elements of Π_t . These calculations are by no means difficult using the computers available today.

The method of Chow [2, Chapter 12] for the control of nonlinear systems with known parameters, which is identical with the method of this paper except for the use of $C(\bar{\eta}_t)'H_tC(\bar{\eta}_t)$ etc., has been programmed, the Fortran code being available at the Econometric Research Program of Princeton University. The limited experience available indicates that the method is not expensive to use. For example, controlling the Klein-Goldberger model with 23 structural equations for 5 periods with 4 targets and 4 instruments using the Fortran source deck takes 32.7 seconds on the IBM 360-91 computer at Princeton University (costing the user $14.35). The program provides not only the linear feedback control equations for each period but all the matrices A_t , C_t and b_t of the linearized reduced form at each time period and for each iteration until convergence, the expected welfare loss, and the graph of the expected time path of each of the 27 (23 plus 4 control) variables. It took three interactions in the sense of three rounds of the initial values of the control variables $\tilde{x}_t$ as described in section 3. Thus the incorporation of uncertainty by evaluating the required expectations should not be computationally prohibitive. If one does not treat all the parameters in a very large econometric model as random, the method of this paper can be applied to incorporate uncertainty in a subset of parameters (the remaining ones being treated as fixed), and to study the effect of uncertainty on the optimal control policies.

REFERENCES

[1] Chow, G. C., "Effect of Uncertainty on Optimal Control Policies," International Economic Review, Vol. 14, No. 3 (October, 1973), 632-645.

[2] ——————, Analysis and Control of Dynamic Economic Systems. New York: John Wiley & Sons, Inc., 1975.