

A MODEL OF
BORROWING AND LENDING WITH BANKRUPTCY

Martin F. Hellwig

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Econometric Research Program
PRINCETON UNIVERSITY
207 Dickinson Hall
Princeton, New Jersey

1. Introduction

"Imperfections in the capital markets" prevent us from living in the best of all possible worlds. But while we have a fair idea about what the best of all possible worlds would look like, we know very little about the functioning of capital markets, let alone about capital market imperfections.

Analogies from commodity markets are not very helpful, because they tend to lead into paradoxes: Thus, a borrower who behaves like a price taker in the market for his own personal debt will want to play the Ponzi game of financing the repayment of his debt with the issue of new debt. We know that this is "wrong" in that actual capital markets do not permit it, but we cannot rule it out in the formalism of perfect markets with competition.

We feel safe when we can consider capital market transactions as exchanges of commodities at different times and/or in different states of nature and do not have to worry about agents' ability or willingness to keep their promises. Therefore, it is often assumed that default on a promise is accompanied by such unspeakable penalties that no agent in his right mind would ever take the risk of exposing himself to such an eventuality even with a zero probability.^{1/}

Yet, default and the prospect of default are a central factor in the functioning of capital markets. The penalty on default is not so terrifying that people want to avoid the slightest exposure to it. Therefore, the circumstances of default will enter the considerations of both debtors and

^{1/} See for instance Hart (1974). On the Ponzi paradox see Foley and Hellwig (1974). The original call for a precise analysis of the working of the capital market was made by Stigler (1967).

creditors when they make a contract. When the debtor's future prospects are uncertain and there is a likelihood of default, the creditor is interested in how much he can salvage when the debtor goes bankrupt. Some of the alleged "imperfections in the capital markets" can be directly traced to such considerations.^{2/}

The major difference between exchanges of commodities and exchanges of commodities for a promise to deliver commodities in the future is this: In an exchange of commodities, each partner knows what he gets. He can look at it and if he does not like it cancel the whole exchange. At worst, he can call the police or the Walrasian auctioneer to support him. The person who receives a promise for the future delivery of goods is in a completely different situation. The quality of the promise cannot be determined by looking at the promise. In fact, how much the promise was worth is not completely determined until it comes due and delivery is made or not made. Furthermore, in most cases, the agent making the promise can affect his own ability to pay up by actions taken between the time when the promise is made and the time when it comes due. By the time the debtor defaults on his promise to pay, it is too late for the creditor to cancel the original contract and he may have lost whatever he initially gave to the debtor. Not even a Walrasian auctioneer can restore him to the original position.

This paper studies one particular aspect of this fundamental asymmetry between debtor and creditor. We consider the case where the borrower promises

^{2/}For instance, expected future labour income is not as good a collateral as expected future capital income, because the institution of bankruptcy is arranged so as to prevent involuntary servitude, so that the creditor cannot impound the debtor's future labour income in the event of default. See Stigler (1967).

a stream of interest payments to the creditor rather than a single repayment of debt. In this case, the timing of default becomes important: The later a debtor defaults and goes bankrupt, the more interest does the creditor actually receive before the default. On the other hand, the debtor affects the time to bankruptcy by choosing a more or less cautious financial policy.

In the model presented in this paper, a debtor borrows to finance current consumption in excess of his income because he hopes to earn more in the future. To pose the problem of the asymmetry between borrower and lender in its most extreme form, we assume that the borrower has complete freedom to do as he likes and consume as much as he wants as long as he is not in default on his obligations. In practice, one would expect creditors to limit this freedom of action and make their credits conditional on the borrower's "good behaviour"; to the extent that they do not succeed in completely determining the borrower's behaviour, the problems discussed in this paper will still be present, if in a weaker form.

On the creditor's side, we study the determination of the credit limit given to any single debtor. The creditor who lends on uncertain income, must consider the possibility that the debtor has no luck before he exhausts his credit limit. If the chance of an income increase were still viable for the "next day," should the creditor continue to give the debtor credit? More precisely, we study the point at which the creditor decides not to lend any more on the debtor's uncertain income prospect. In the course of this discussion, we show that the decision to cut off credit is a different decision from the decision to announce a certain credit limit at the outset. As a consequence, we shall see that in normal circumstances, the creditor's decisions are structured in such a way that he is not able to set the actual credit limit at its optimal value. Indeed, there are nonpathological cases, in

which his decision structure leads him to set the credit limit in such a way that he makes a loss on average. If this is the case, we show that there is no consistent method for the creditor to determine the credit limit. The notion of rationality is not well defined.

In the final section, we show that the problems of the interaction between creditor and debtor are attenuated when we consider the creditor determining the credit conditions for a population of debtors rather than a single debtor.

2. A Model of Borrowing

We use a one-asset optimal saving model to discuss borrowing on uncertain income. An agent's opportunity set is given as follows:

Noninterest income $y(t)$ is a random variable which takes on the values 0 and $a > 0$, according to the following rule:

$$y(0) = 0 .$$

(1) For all t , T with $t > T$, $y(T) = a$ implies $y(t) = a$.

For all t , $\text{Prob}\{y(t) = a\} = 1 - e^{-\lambda t}$, where λ is a constant.

We can interpret the situation as one of employment and unemployment. Initially, the agent is unemployed. At a random time, he finds employment, which he then keeps forever. This simple specification allows a clear distinction between times when the agent is poorly off and times when he is well off and enables us to study the notion of borrowing on uncertain income without

too many complications.^{3/} The assumption that the time of the income switch is exponentially distributed is made solely for formal convenience. Most of the results do generalize to the case where λ depends on t .

The agent's intertemporal budget set is characterized by the usual accumulation equation:

$$(2) \quad \dot{k}(t) = R(k(t)) + y(t) - c(t) .$$

Capital accumulation is given as the difference between income and consumption, where interest income $R(k)$ is a continuous concave function of capital, which satisfies:

$$(3) \quad R(0) = 0$$

$R'(k) = r$ for $k \geq 0$, where $r > 0$ is the market rate for riskless securities.

In addition to the accumulation equation, we need a constraint on the admissible values of capital to describe the budget set. Normally, capital is constrained to be nonnegative, but here, we want to allow for the possibility of borrowing. Therefore, we introduce the notion of a credit limit. The credit limit A constrains the agent's holding of capital to be no less than $-A$. Negative holdings of capital are of course, debt, so that we

^{3/} Another interpretation would view the income switch as the inheritance of an annuity.

constrain debt not to exceed the credit limit A .

While the credit limit is not yet reached, the agent has no difficulty in obtaining more credit. He can finance both consumption and interest payments on previously incurred debt by incurring more debt. When he reaches the credit limit, no more debt can be incurred. Given the one-asset structure of the model, his interest income at $k = -A$ is negative. If his noninterest income is zero, the accumulation condition would drive him through the credit limit. In this case, he goes bankrupt. If his noninterest income exceeds the interest owed at $k = -A$, he does not go bankrupt, because he can fulfill his interest obligations. Formally, the agent goes bankrupt whenever his capital holding is $-A$ and his capital accumulation is negative for all nonnegative values of consumption.

To simplify the model further, we assume that

$$(4) \quad R(-A) + a \geq 0 .$$

After the agent has found employment, he can always pay the interest on his debt and does not go bankrupt. In other words: The credit limit is not set so high that in its neighbourhood, the agent cannot avoid bankruptcy whether or not he finds employment.

The agent maximizes the expected present value of utility of consumption over an infinite time horizon:

$$(5) \quad E \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

The agent's utility satisfies the following conditions:

Assumption I: At any moment, when the agent has not yet exhausted the credit limit, his utility of consumption $u(\cdot)$ is a continuously differentiable, strictly increasing, strictly concave function that is bounded above and below. Furthermore, $\lim_{c \rightarrow 0} u'(c) = \infty$ ^{4/}

Assumption II: At the moment when the agent goes bankrupt, the value of the program after bankruptcy, (5), is B , a constant, independent of the time of bankruptcy or of the consumption path before bankruptcy.

The value of the agent's consumption path after he goes bankrupt is given exogenously, e.g., through institutional factors. There may be a penalty: He may be jailed or sold into slavery. He may merely find his credit rating gone. Such sanctions to bankruptcy are subsumed in the parameter B . Assumption II asserts that the value of sanctions on bankruptcy is definite. The agent may dislike the sanctions, but this dislike can be expressed as a definite number. This allows us to discuss the choice of programs that expose the agent to the possibility of bankruptcy. On the other hand, this assumption forces us to rule out utility functions before bankruptcy with $\lim_{c \rightarrow 0} u(c) = -\infty$; for such utility functions, there would exist states from which the definite value B of the program after bankruptcy was a reward rather than a penalty.

Furthermore, Assumption II asserts that the value of bankruptcy, B is independent of the path preceding bankruptcy. B need not be independent of the level of debt at the moment of bankruptcy, but it does not depend on the time of bankruptcy. If one wants to distinguish between fraudulent

^{4/} In Appendix 3, we give an example that is due to Goldman (1974).

and nonfraudulent bankruptcy, this assumption is too extreme, but in this first approach, we neglect this fine point.

We specify the value of bankruptcy sufficiently to make sure that bankruptcy is not a desirable event. When the agent goes bankrupt, all debt is cancelled, but an unspecified penalty p is levied. This penalty is sufficiently harsh to make bankruptcy worse than the alternative of having a moratorium on one's debt until one finds employment with the obligation to resume interest payments thereafter. Anticipating the development of the next section, we express this formally as:

Assumption III: The value of bankruptcy B is given as:

$$B = \frac{u(0) + \lambda V(0)}{\rho + \lambda} - p$$

and it satisfies:

$$B \leq \frac{u(0) + \lambda V(-A)}{\rho + \lambda}$$

$V(\cdot)$ is the value of the optimal program after the agent has found employment. After the cancelling of the debt, the agent is constrained to zero consumption and has the prospect of finding employment when the value of his capital is zero. Assumption III gives B as the expected present value of this program minus the penalty.

3. The Borrower's Optimization Problem

When the agent finds employment, he can no longer go bankrupt, because he can always pay the interest he owes. Then, the maximization of (5) reduces

to the ordinary optimal saving problem:

$$(5') \quad \text{Max} \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

$$\text{subject to: } \dot{k}(t) = R(k(t)) + a - c(t)$$

$$k(t) \geq -A$$

$$k(0) = k_0 .$$

It is well known that under Assumption I and Condition 3, a unique optimal consumption path for (5') exists. Let $V(k)$ be the value of the maximum at the optimal consumption path for initial capital k . Then $V(\cdot)$ is continuous, strictly increasing and strictly concave. Furthermore, $V(\cdot)$ is bounded above and below.

To compare consumption paths before the income switch has occurred, we note that by the principle of optimality, any program that has any claim to being optimal, must be optimal after the income switch. For a given consumption path $c(t)$, consider the following program:

Before the income switch has occurred, consume $c(t)$ at the time t . This consumption path defines a time path for capital, conditional on the income switch not yet having occurred. Let this be $k(t)$.

If the income switch occurs at t , at the capital $k(t)$, pursue the optimal program for (5') with $k(t) = k_0$. This has the present value $V(k(t))$.

If the income switch occurs at t , this program has the present value:

$$\int_0^t e^{-\rho \tau} u(c(\tau)) d\tau + e^{-\rho t} V(k(t)) .$$

The density of an income switch at t is $\lambda e^{-\lambda t}$, from (1).

Let T be the time when the path of capital $k(t)$ that is induced by the consumption path $c(t)$ exhausts the credit limit, so that $K(T) = -A$. T is unique (but not necessarily finite): with $y = 0$, (2) and (3) imply that $\dot{k} < 0$ if $k = -A$. T is the time when the program under consideration leads into bankruptcy, unless the agent has found employment before T . The probability of this is $e^{-\lambda T}$. The path ending in bankruptcy has the present value:

$$\int_0^T e^{-\rho t} u(c(t)) dt + e^{-\rho T} B.$$

Now we take expectations over all the possible contingencies and write the value of the maximand (5) at the program under consideration as:

$$\int_0^T \lambda e^{-\lambda t} \left(\int_0^t e^{-\rho \tau} u(c(\tau)) d\tau + e^{-\rho t} V(k(t)) \right) dt + e^{-\lambda T} \left(\int_0^T e^{-\rho t} u(c(t)) dt + e^{-\rho T} B \right).$$

The first term under the first integral can be integrated by parts. At the program under consideration--optimally after the income switch, $c(t)$ before--the maximand (5) has the value:

$$(6) \quad \int_0^T e^{-(\rho+\lambda)t} (u(c(t)) + \lambda V(k(t))) dt + e^{-(\rho+\lambda)T} B.$$

Thus, we have transformed the stochastic saving problem (5) into a problem that, formally at least, does not differ from a nonstochastic problem. The agent chooses the consumption path that maximizes (6). He follows this

path until he finds employment. Thereafter, he follows the path that solves problem (5') and has the value V .

In the formalism of (6), a consumption path $c(t)$ gives utility directly through the consumption before the income switch and indirectly through the value of the optimal program after the income switch. The latter is evaluated at the flow rate λ at which the income switch actually occurs. The discount rate is augmented by λ , to take account of the improbability of the income switch not having occurred yet.^{5/}

The agent selects a consumption path to maximize (6) subject to the accumulation equation:

$$(2') \quad \dot{k}(t) = R(k(t)) - c(t)$$

the initial condition:

$$(7) \quad k(0) = k_0$$

and the terminal condition:

$$(8) \quad k(T) = -A .$$

A minor additional complication allows us to take account of the possibility that the agent does not want to exhaust the credit limit. If he refuses

^{5/} A similar derivation was used by Cass and Yaari (1967) and Merton (1971) to analyse optimal saving with a random lifetime. In our case, "lifetime" at a low income is random and we have to take account of the value of the "after-life" at a high income.

to incur more debt, this means immediate bankruptcy. We can let the borrower determine the amount of debt at which he wants to go bankrupt, provided it does not exceed the credit limit. In the preceding formulation of the borrower's problem, let A be the amount of debt at which he chooses to go bankrupt, so that condition (8) must still hold. But now, he can choose A subject to the constraint:

$$(9) \quad A \leq \bar{A},$$

where \bar{A} is the credit limit set by the creditor.

The problem of maximizing (6) subject to (2'), (7), (8) and (9) differs from the normal optimal saving problem in that the time of bankruptcy is a decision variable of the agent. This complicates the analysis, because the set of feasible consumption paths is not, in general, convex.

On the other hand, the time to bankruptcy is a convenient parameter of the agent's borrowing policy. T will be finite if and only if the agent does choose to borrow. In lemma 1 of Appendix 2, we show that the maximand (6) is continuous in T even at $T = \infty$, when the agent does not borrow. Thus, we can infer whether the agent wants to borrow from the choice of optimal T to maximize (6). If (6) has a global maximum at finite T , borrowing will be desired, whereas borrowing is not desired if (6) has its global maximum with $T = \infty$.

In the next section, we discuss the agent's policy when he cannot borrow so that $T = \infty$. We then use this as a reference to discuss the desirability of borrowing and the choice of the time to bankruptcy.

4. The Optimal Consumption Path Without Borrowing

If borrowing is ruled out, the agent maximizes (6) for $T = \infty$, subject

to the constraints (2a), (7) and the condition that capital should not be negative,

$$(10) \quad \text{for all } t, \quad k(t) \geq 0.$$

Elsewhere, we have discussed a period version of this problem of saving under income uncertainty.^{6/} The optimal path must satisfy the Euler equation:

$$(11) \quad \dot{u}' = -(r-\rho-\lambda)u'(c(t)) - \lambda V'(k(t))$$

and the transversality condition:

$$(12) \quad \lim_{t \rightarrow \infty} e^{-(\rho+\lambda)t} k(t) u'(c(t)) = 0.$$

In Appendix 1, we prove

Proposition 1: The optimal consumption path without borrowing satisfies:

1. a: For all t , $u'(c(t)) > V'(k(t))$.
- b: For all k_0 , initial consumption $c(0; k_0)$ on the path beginning at initial capital k_0 is increasing in λ .
2. a: $r \geq \rho + \lambda$ implies that $\dot{c} > 0$ and $\dot{k} > 0$.
- b: $\rho + \lambda > r > \rho$ implies that for sufficiently small values of initial capital, $\dot{c} < 0$ and $\dot{k} < 0$. Furthermore, if $\lim_{c \rightarrow \infty} (u'(c+a)/u'(c)) = 1$, then for sufficiently large values of initial capital, $\dot{c} > 0$ and $\dot{k} > 0$.
- c: $r \leq \rho$ implies $\dot{c} < 0$ and $\dot{k} < 0$.

^{6/}Foley and Hellwig (1975).

Proposition 1 describes the impact of income uncertainty on the optimal consumption path. The Euler equation (11) differs from the condition for a normal Ramsey problem, $\dot{u}' = -(r-\rho)u'$, by the term $-\lambda(V'(k(t)) - u'(c(t)))$. From the envelope theorem, $V'(k(t))$ is equal to the marginal utility of initial consumption on the path solving problem (5') where income is with certainty high. Part 1. a of Proposition 1 states that initial consumption with just a chance of a higher income in the future is lower than with certain high income now. This implies that the term $-\lambda(V'-u')$ is positive, decreasing the rate at which consumption increases. As a consequence, initial consumption is monotone increasing in λ : the more likely the income switch is, the more one already wants to consume now. There is the less of a reason to save today, the more one believes that one has a higher income tomorrow.

The second part of Proposition 1 gives a perspective on the importance of this effect, by showing when it actually changes the sign of capital accumulation. If $r \leq \rho$, the trade-off between interest and time preference provides an incentive to decumulate, so the prospect of a higher income tomorrow raises the rate at which capital is depleted. On the other hand, if $r \geq \rho + \lambda$, the interest rate is so advantageous that one always prefers to increase one's capital holdings, even in the presence of the income uncertainty. The prospect of the income increase induces one to save less than one would otherwise, but one still saves a positive amount.

In the intermediate case, where $\rho + \lambda > r > \rho$, the prospect of an income increase does change the sign of capital accumulation for sufficiently small values of initial capital, inducing the agent to dissave, where in the absence of such a prospect he would save. For larger values of capital, the dissaving incentive is relatively less important, so that under the additional assumption

that for large enough c , $u'(c+a)/u'(c)$ approaches one, there will be positive saving at high values of initial capital.

The dissaving incentive due to the prospect of an income increase provides the basis of our discussion of borrowing. Borrowing requires a decumulation of capital through the origin and into negative values. The agent may desire this, because the prospect of an income increase provides a dissaving incentive and makes him wish to consume more than his current income. In the following sections we study the conditions under which the agent actually wants to make use of the chance to borrow.

5. The Optimal Time to Bankruptcy

We now proceed to discuss the solution to (6) with borrowing. Necessary conditions for an optimum are the Euler equation:

$$(13) \quad \dot{u}' = -(R' - \rho - \lambda)u'(c^*(t; k, \bar{A})) - \lambda V'(k(t; k, \bar{A})),$$

the condition for the debt at bankruptcy:

$$(14) \quad e^{-(\rho+\lambda)T} \left(u'(c^*(T; k, \bar{A})) + \frac{\partial B}{\partial A} \right) \geq 0$$

and the transversality condition:

$$(15) \quad e^{-(\rho+\lambda)T} \left\{ u(c^*(T; k, \bar{A})) + \lambda V(-A) - (\rho+\lambda) B \right. \\ \left. - [c^*(T; k, \bar{A}) - R(-A)]u'(c^*(T; k, \bar{A})) \right\} = 0$$

Condition (14) determines the amount of debt at which one goes bankrupt,

if one borrows at all and T is finite. If the left hand side of (14) is positive, one exhausts one's credit limit. If the left hand side of (14) is zero, one may not want to exhaust the credit limit. In this case, one trades off the benefit of being able to raise final consumption before bankruptcy (which by the envelope theorem is the marginal contribution of A to the value of the program before bankruptcy) against the cost of incurring more debt in terms of the bankruptcy penalty. In the following, we shall assume that the penalty on bankruptcy is independent of the amount of debt at the moment of bankruptcy, so that the left hand side of (14) is positive and the credit limit is exhausted, provided that T is finite. If B depends on the debt at bankruptcy, this need not be true.

Of greater interest is the determination of the optimal time to bankruptcy. If the agent makes use of the ability to borrow, T is finite and the term in brackets in (15) must be zero. This represents the following trade-off: Consider a marginal delay of bankruptcy. The benefit of this delay is $u(c^*(T; k, \bar{A}) + \lambda V(-A) - (\rho + \lambda) B$, because the agent is able to consume $c^*(T; k, \bar{A})$ and enjoy another chance of finding employment during this additional instant instead of having the flow equivalent $(\rho + \lambda) B$ of the value of bankruptcy. On the other hand, during this delay, one needs an amount $c^*(T; k, \bar{A}) - R(-A)$ to finance consumption and interest payments. This amount is not available for use during the preceding time. By the envelope theorem, it is valued at the marginal utility of final consumption. Thus, the bracketed term in (15) vanishes if the costs and benefits of a marginal delay of bankruptcy are equal.

As the time to bankruptcy is a variable, the set of feasible consumption paths is not convex, so that the first order conditions may have multiple solutions. In fact any policy with $T = \infty$ (no borrowing) satisfies (14)

and (15) automatically, because it makes the discount factor $e^{-(\rho+\lambda)T}$ vanish. To analyze the borrowing policy, we have to investigate the solutions to (15) and see whether the policy without borrowing gives a minimum, a local maximum or a global maximum. In Appendix 2, we prove:

Proposition 2: There exists at most one program with finite T , which corresponds to a local maximum. The unique optimal time to bankruptcy with active borrowing is strictly decreasing in B , the value of bankruptcy and satisfies:

$$(16) \quad \frac{dT}{dK} = - \frac{1}{R(k) - c^*(0; k, \bar{A})} > 0$$

Along the optimal consumption path for this program,
 $u'(c^*(t; k, \bar{A})) > V'(k(t; k, \bar{A}))$.

There exists at most one optimal borrowing path, so that the behaviour of the borrower is well behaved with respect to the parameters k, \bar{A}, B , provided he does want to borrow. Proposition 2 does not deal with multiple solutions and discontinuities between paths on which the agent borrows and goes bankrupt in finite time and paths on which he does not borrow. We discuss this issue in the following two propositions:

Proposition 3: If there exists a value of initial capital, for which a borrowing program is preferred to the optimal program without borrowing, then a borrowing program is preferred to the optimal program without borrowing when the initial capital is zero.

If borrowing is preferred from an initial capital of zero, it is preferred to the optimal program without borrowing for all values of initial capital, from which the optimal program without borrowing decumulates capital towards zero. In this case, the policy without borrowing is locally minimal.

Proof: The proposition is a straightforward application of the principle of optimality. Any path that leads into debt passes through the origin. The principle of optimality implies that borrowing can only be optimal if it is optimal from zero initial capital.

Conversely, suppose that there exists $k > 0$, from which the policy without borrowing is preferred to all borrowing policies, and from which the policy without borrowing decumulates capital towards zero. Within finite time, the optimal policy without borrowing comes arbitrarily close to the origin. Again, by the principle of optimality, if borrowing is not optimal from k , it must not be optimal from any value of capital that is reached from k in finite time. Therefore, for values of initial capital arbitrarily close to the origin, $T = \infty$ is globally optimal. Since for each T , the value of the maximand (6) is continuous with respect to the value of initial capital, $T = \infty$ must also be globally optimal when the initial capital is zero.

The proof of the last statement of Proposition 3 is given in Appendix 2.

Proposition 3 states that the optimal time to bankruptcy is unique when the initial capital is zero, so that there are no multiple optima; either one wants to borrow or one does not want to borrow, but one cannot be indifferent between an optimal path with borrowing and an optimal path without borrowing.

For positive values of initial capital from which the optimal program without borrowing depletes capital to zero, the desirability of borrowing is solely determined by the desirability of borrowing from the origin. Therefore, from such values of initial capital, the optimal program also is unique. The crucial question is therefore, under what conditions borrowing is desirable when the initial capital is zero.

We defer the discussion of this question to the next section. First, we continue the discussion of multiple solutions to the first order conditions and consider values of initial capital from which the optimal program without borrowing does not deplete capital towards zero.

Proposition 4: Let borrowing be desirable when the initial capital is zero. Define k^* as the smallest value of initial capital for which borrowing is not preferred to the optimal program without borrowing. At k^* , initial consumption $c^*(0; k, \bar{A})$ is discontinuous with respect to k , unless initial consumption at k^* in the absence of borrowing is equal to the interest income at k^* .

Proof: For values of initial capital below k^* , borrowing is desirable. From (16), interest income does not exceed consumption on the optimal path with borrowing. By definition of k^* , the optimal program without borrowing from k^* does not do worse than any borrowing program. By the principle of optimality, the optimal program without borrowing from k^* cannot lead to values of capital below k^* . Therefore, initial consumption on this program cannot exceed the interest income at k^* . From these two considerations, the proposition follows immediately.

The discontinuity described in proposition 4 is easy to understand: Borrowing requires capital decumulation. By definition of k^* , the policy without borrowing does not require decumulation. Unless the transition between the two is smooth because at k^* , there is neither saving nor dissaving, there must be a discontinuity. The question is whether the possibility of borrowing shifts the largest value of initial capital, from which one wants to decumulate, upwards.

I have not been able to find conditions for the occurrence of the discontinuity. The following heuristic argument may give some insight. Suppose

that $\lambda = 0$ so that there is no chance of ever having a higher income. If $r > \rho$, one always wants to accumulate capital. But at an initial capital of zero, one has no income so that one cannot save and has to stay there. A credit limit under sufficiently favourable conditions would be accepted from zero initial capital, even though one would be certain to go bankrupt in finite time. Similarly, for small positive values of initial capital, borrowing with a sufficiently favourable credit limit would be preferred to the normal accumulation program. The reason is that on an accumulation program, the agent's resources are limited to his own initial capital, whereas a decumulation program can use both his own initial capital and the present value of the credit limit. Resources on the path with borrowing towards bankruptcy are higher than on the accumulation path. Therefore, he may prefer to borrow even though he dislikes the time pattern of consumption that is imposed on him and the penalty to bankruptcy. For high enough values of initial capital this difference in resources between a path with borrowing and a path without borrowing becomes less important and does not outweigh the disadvantages of bankruptcy. At the point where he is indifferent between the accumulation program and the decumulation program with borrowing, there is a discontinuity in his consumption behaviour.

With uncertain income, the same principle is at work. In our formulation, the burden of debt on resources after the income switch is subsumed into the function V , so that explicitly, we only need look at resources available before the income switch. Again the principle applies that the path towards bankruptcy has more resources available than the path without borrowing. This may induce the agent to decumulate and borrow where in the absence of borrowing he would consume less than his income. However, since for the

case $\rho + \lambda > r > \rho$, the stationary value of initial capital without borrowing is positive, by Proposition 1, it is not a priori clear under what conditions the discontinuity will occur.

One may note that in the case $\lambda = 0$, creditors face a sure loss if they set the credit limit in such a way that the consumer wants to borrow. This suggests the conjecture that the discontinuity described in Proposition 4 does not occur if creditors impose conditions so that they do not expect to make a loss by lending to the consumer. However, I have not been able to verify this conjecture.

6. To Borrow or Not To Borrow...

We return to the question of whether the agent finds it desirable to borrow when his initial capital is zero. From Propositions 2 and 3, we know that the optimal T is unique. If the optimal T is finite, the bracketed term in (15) must be zero. We note that this term depends on the agent's choice only through the value of terminal consumption $c^*(T; k, \bar{A})$. Furthermore, it is monotone increasing in the value of terminal consumption. Therefore, there exists a unique value of terminal consumption c^{**} , for which this term is zero, which satisfies:

$$(17) \quad u(c^{**}) - (c^{**} - (R(-A))u'(c^{**}) + \lambda V(-A) - (\rho + \lambda) B = 0 .^{7/}$$

7/This follows from Assumptions I and III. The left hand side of (17) is continuous in c^{**} . $c^{**} = 0$ implies that for A positive, the left hand side of (17) is negative, because of Assumption I. On the other hand, Assumption III implies that for large enough c^{**} , the left hand side of (17) is positive, as otherwise bankruptcy would be bliss.

If the optimal time to bankruptcy is finite, final consumption on the optimal path is equal to c^{**} . c^{**} is increasing in A and B and decreasing in λ . The less terrifying bankruptcy is, the less one is inclined to delay it at the expense of consumption before bankruptcy. Similarly, when the credit limit is higher, one is less willing to delay bankruptcy because the interest cost of doing so is higher and in addition, the benefit of an income increase at the last minute is lower. On the other hand, the higher λ is, the better the prospects are for a last minute turn to the better, so that one is more inclined to delay bankruptcy marginally by consuming less before bankruptcy.^{8/}

The most revealing property of c^{**} is its positivity whenever the credit limit is positive. This is entirely due to the fact that under Assumption I, the value of the interest cost of delaying bankruptcy, $R(-A)u'(c^{**})$ would be infinite, if c^{**} were zero. Thus, the agent chooses to consume a positive amount just before going bankrupt, because consuming less and delaying bankruptcy would entail a net cost of $R(-A)$ which he would pay to his creditor during the delay. At the margin, the main reason for not delaying bankruptcy is the fact that the sooner one goes bankrupt, the less interest one has to pay to one's creditor.

To determine the desirability of borrowing from any initial capital k , consider the consumption path $c(t; k, \bar{A}, T)$ that maximizes (6) for any given T , i.e., that satisfies (13) and (14) and ends at T . If $c(T; k, \bar{A}, T)$, terminal consumption along this path exceeds c^{**} , the bracketed term in (15)

^{8/}This last consideration shows why statement 1. b. of Proposition 1 cannot be applied to paths with borrowing.

is positive for the given value of T ; therefore, one can improve oneself by delaying the time to bankruptcy; if $c(T; k, \bar{A}, T)$ falls short of c^{**} , the bracketed term in (15) is negative and one can improve oneself by reducing T . In Appendix 2, we prove

Lemma 5: When the initial capital is zero, it is optimal to set T at a finite value if and only if

$$(18) \quad \lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) < c^{**} .$$

This lemma indicates that borrowing is desired from the origin, if and only if for very large T , it is desirable to reduce the time to bankruptcy, because terminal consumption falls short of c^{**} .

In general, final consumption depends on the interest schedule. If the marginal interest rate becomes very high as the agent approaches the credit limit, the Euler equation (13) will require that in the neighbourhood of the credit limit, the time derivative of consumption is positive. In this case, interest charges on additional credits become so high that the agent wants to consume at a faster and faster rate to avoid spending much of his resources on interest payments. This is due to the same incentive that also causes c^{**} to be positive. As a consequence, it is difficult to find general expressions for $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T)$, and to give general conditions for the desira-

bility of borrowing. However, in the appendix we do prove the following Proposition 5: For an initial capital of zero, the optimal T is finite (borrowing is desirable), if $R(-\bar{A}) > a$ and $R'(-\bar{A}) < \rho + \lambda$.

If the conditions of Proposition 5 are satisfied, one can show that $\lim_{T \rightarrow \infty} c(T; 0, A, T) = 0$, which we know is less than c^{**} . If the interest

schedule is not very steep, one is willing to borrow and expose oneself to a positive probability of bankruptcy.

In a sense, Proposition 5 is too strong. It asserts that borrowing is desirable no matter how large the penalty to bankruptcy or how small the credit limit is, provided the interest schedule is not excessively steep. The generality of this result rests crucially on the fact that the agent's utility function satisfies an Inada condition. The fact that for zero consumption, the marginal utility of consumption is infinite, makes c^{**} strictly positive. In addition, this assumption plays an important role in showing that $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) = 0$. If the interest schedule is sufficiently flat, one can spread the extra consumption that the credit permits, no matter how small it is, over a very large time and because of the high marginal utility of consumption, this achieves a large effect. This effect outweighs the dislike for bankruptcy, because bankruptcy is removed sufficiently far into the future.

Since the utility function is bounded below, the assumption that $\lim_{c \rightarrow 0} u'(c) = \infty$ is not very appealing. If $u'(c)$ is finite everywhere, one obtains results that are similar to Proposition 5, but less general.

Suppose now that $r > \rho$. For large values of initial capital, the optimal program in the absence of borrowing requires positive saving, so that the time derivative of consumption is positive. Furthermore, for large enough values of initial capital, initial consumption exceeds c^{**} . Therefore consumption at every instant along the optimal path without borrowing exceeds c^{**} . From Lemma 3 in Appendix 2, consumption on a path with borrowing exceeds consumption on a path without borrowing at all instants. Therefore, for all T , $c(T; k, \bar{A}, T)$ exceeds c^{**} for sufficiently large k . Therefore, the

agent can for all T improve himself by increasing T . He finds it optimal to set $T = \infty$ and to plan not to borrow at all.

If at the same time, the conditions of Proposition 5 are satisfied, the discontinuity of behaviour between paths with borrowing and paths without borrowing that we discussed in Proposition 4 can occur.^{9/}

7. The Credit Limit and the Time to Bankruptcy

To analyze the effects of changes in the credit limit on the borrower's behaviour, we consider the following experiment:

Suppose that a creditor wants to give the debtor a credit limit $A_1 + A_2$. He compares two alternatives: On the one hand, he can grant this amount without further ado. On the other hand, he can grant A_1 and then, when the debtor is about to exhaust A_1 and go bankrupt, grant him an additional credit of A_2 . If we assume that the debtor believes the credit limit that is announced and does not realize the possibility of an extension of credit, how does his behaviour differ under the two alternatives?

Let $T(k, A, B)$ be the optimal time to bankruptcy for an initial capital k , a credit limit A and a value of bankruptcy B .

Proposition 6: For all k, A_1, A_2, B ,

$$(19) \quad T(k, A_1 + A_2, B) \leq T(k, A_1, B) + T(-A_1, A_1 + A_2, B).$$

^{9/}More precisely, one can show that under the assumption of Proposition 5, $\lim c(T; k^{**}, A, T) = rk^{**}$, where k^{**} is stationary under the program without borrowing. By analogous application of Lemma 5, borrowing from k^{**} is strictly preferred if and only if $rk^{**} < c^{**}$. If this is the case, $k^{**} < k^*$ and the discontinuity does occur.

The inequality is strict if $T(k, A_1+A_2, B)$ is finite.

Proof: Since B is insensitive to the amount of debt at bankruptcy, an extension of the credit limit makes the debtor better off. Therefore, $T(k, A_1+A_2, B)$ is finite if $T(k, A_1, B)$ is finite. Consider the optimal program corresponding to a credit limit A_1+A_2 . At a time $\tilde{T}(k, A_1, A_1+A_2, B)$, this path reaches the capital holding $-A_1$, where it has just exhausted the credit A_1 . At this point, one is in the same situation as in the case, when one is granted the additional A_2 only after one has exhausted A_1 . One has a capital of $-A_1$ and a credit limit A_1+A_2 . By the principle of optimality, the two programs that we compare are the same between $-A_1$ and bankruptcy. Thus, Proposition 6 is equivalent to the statement that

$$\tilde{T}(k, A_1, A_1+A_2, B) < T(k, A_1, B) .$$

Define $W^*(k, \bar{A})$ as the value of the maximand (6) at the optimal path for an initial capital k and credit limit A . The, $W^*(-A_1, A_1+A_2)$ is the value of the optimal program beginning at a debt A_1 with a total credit limit A_1+A_2 . By the principle of optimality, the optimal program for (6) with initial capital k , and credit limit A_1+A_2 is optimal after the credit A_1 has been exhausted, with the value $W^*(-A_1, A_1+A_2)$ for the tail after exhaustion of A_1 . We can rewrite problem (6) to make the choice of

$\tilde{T}(k, A_1, A_1+A_2, B)$ explicit:

$$(20) \quad \max \int_0^{\tilde{T}} e^{-(\rho+\lambda)t} (u(c(t)) + \lambda V(k(t))) dt + e^{-(\rho+\lambda)\tilde{T}} W^*(-A_1, A_1+A_2)$$

subject to: $\dot{k} = R(k(t)) - c(t)$

$$k(0) = k$$

$$k(T) = -A_1$$

Since $W^*(-A_1, A_1+A_2)$ satisfies Assumption II (if not Assumption III), problem (20) is formally identical to problem (6), for a credit limit A_1 and a value of bankruptcy $W^*(-A_1, A_1+A_2)$. Therefore,

$$\tilde{T}(k, A_1, A_1+A_2, B) = T(k, A_1, W^*(-A_1, A_1+A_2)) .$$

From Proposition 3, this is strictly less than $T(k, A_1, B)$ if and only if $W^*(-A_1, A_1+A_2) > B$. By definition of W^* , $W^*(-A_1, A_1) = B$. Since an extension of credit makes the debtor off, $W^*(-A_1, A_1+A_2) > W^*(-A_1, A_1) = B$. This completes the proof of Proposition 6.

If the value of bankruptcy were sensitive to the amount of debt at the time of bankruptcy, the same argument would show that (19) holds with weak inequality.

Proposition 6 is the first main result of this paper, and our discussion of lending behaviour will concentrate on the consequence of this proposition. The major point is that if the borrower anticipates a higher credit limit, this will affect his whole consumption path from the beginning. He will not just follow the same path as for the lower credit limit until the lower credit limit is exhausted and then enjoy the extra lease on life given by the higher credit limit. Instead, he increases his consumption from the very beginning, exhausting the lower amount A_1 faster than he would if he did not anticipate more credit afterwards.

As a consequence of Proposition 6, the determination of the credit limit

is of some game theoretical interest. The creditor may attempt to influence the debtor's behaviour to his own advantage by announcing a "false" credit limit initially, that is to say a credit limit that differs from the one that he intends to actually enforce. In this situation, the question becomes what is a credible announcement to the debtor, i.e., one that the debtor really believes is going to be enforced. We shall return to this question after a general introduction to the position of the creditor in our model.

The principle behind Proposition 6 is far more general than one might expect from the rather special assumptions that we made. It is based solely on the two facts that the optimal time to bankruptcy is decreasing in the value of bankruptcy and that the value of the optimal path after the exhaustion of A_1 is independent of the path before the exhaustion of A_1 and exceeds B . Consider the first order condition for \tilde{T} in problem (20), the analogue of (15):

$$(21) \quad u(c^*(\tilde{T}; k, A_1)) - (c^*(\tilde{T}; k, A_1) - R(-A_1))u'(c^*(\tilde{T}; k, A_1)) \\ + \lambda V(-A_1) - (\rho + \lambda)W^*(-A_1, A_1 + A_2) = 0$$

An increase in A_2 raises W^* and lowers the left hand side. \tilde{T} is adjusted to counteract this effect. By the second order condition for a maximum, the left hand side of (21) is decreasing in \tilde{T} , so that \tilde{T} is decreased when A_2 is raised. This argument is independent of the various stationarity assumptions contained in our model. While the stationarity properties of λ , B , W^* , etc., were sufficient to make borrowing behaviour well-behaved and to rule out multiple optima among borrowing paths, they

are not needed to establish Proposition 6. Whenever we can disregard the possibility of discontinuities in the borrower's behaviour, the second order condition for the determination of \tilde{T} , together with the independence of W^* from the consumption path before A_1 is reached will ensure that Proposition 6 holds, even if λ, V, W^* are functions of \tilde{T} . Thus, there is a general presumption that consumption before the exhaustion of A_1 is the more lavish, the more additional credit the borrower expects to receive afterwards.

8. The Returns from Lending to the Borrower

We now turn to a discussion of the lenders who do business with the borrower described in previous sections. In any instant, all the creditors together receive the amount $-R(k)$ in interest payments and the amount \dot{k} as a repayment of debt. (If more debt is incurred, \dot{k} is negative). Therefore, in any instant, lenders have a net inflow of resources from the borrower

$$\dot{k} - R(k) = -c + y.$$

When the debtor's noninterest income is zero, creditors have a net inflow of resources $-c(t)$ in any instant t . In other words, they experience a net outflow at the rate needed to finance the debtor's consumption.

After the debtor has found employment, his obligation to the creditors has become a safe asset, paying the interest $-R(k) \geq -rk$ on the nominal value $-k$. If this interest charge is strictly greater than the interest income from a safe asset of the same nominal value at the market rate, then the present value of this asset depends on the time when the debtor repays his debt. For at this time, the creditor exchanges an asset with a higher

return for an asset with just the market rate of return.

To simplify our analysis, we shall assume that the debtor's obligation is a nonredeemable perpetuity, so that its market value after the debtor finds employment is simply $-R(k)/r \geq -k$. This new assumption requires a redefinition of the function V in previous sections, but otherwise leaves the preceding discussion unchanged.

If the assumption that debt is not redeemable is at one extreme, at the other extreme, we could consider the case where immediately after the income increase the debtor was able to float new debt at the safe rate of interest r and use it to redeem his outstanding debt. The present value to creditors of the old debt just before the conversion would just be the face value $-k$.

In the intermediate case, where the debtor could gradually repay the debt out of his own saving, the present value of his obligation would be somewhere between the two values $-R(k)/r$ and $-k$, depending on the debtor's saving behaviour. Most of the subsequent discussion is not affected, if, instead of $-R(k)/r$, we use $-k$ or the intermediate value.

If debt is a nonredeemable perpetuity, then the present value of a path $c(\cdot)$ with an income switch at the time t to the creditors is:

$$- \int_0^t e^{-r\tau} c(\tau) d\tau - e^{-rt} R(k(t))/r .$$

This path has the density $\lambda e^{-\lambda t}$.

We assume that when the debtor goes bankrupt, creditors receive nothing. This assumption appears plausible in the light of actual experiences, but it has to be justified in view of the fact that even at bankruptcy, there may still be this prospect of a higher income in the future. One might argue that bankruptcy destroys this prospect, because nobody is willing to employ

somebody who has gone bankrupt. The assumption can be justified in a more substantial way by considering that the institution of bankruptcy is designed to prevent creditors from impounding the debtor's future labour income to prohibit this form of involuntary servitude.

If T is the time to bankruptcy for the given consumption path $c(\cdot)$ and no income switch occurs before T , the present value of the path to the creditor is:

$$- \int_0^T e^{-rt} c(t) dt .$$

This occurs with probability $e^{-\lambda T}$.

We can take expectations over t , the time of the income switch and use integration by parts as in the derivation of (6) to find the expected present value of returns from lending to the borrower as

$$(22) \quad - \int_0^T e^{-(r+\lambda)t} (c(t) + \lambda R(k(t))/r) dt .$$

The first term under the integral represents the cost to lenders of financing the borrower, the second term the benefit for those cases when the borrower succeeds in finding employment.

Remark 1: For the expected present value of returns to lenders to be non-negative, the marginal rate of interest paid by the borrower must be strictly higher than the safe market rate.^{10/}

^{10/} Suppose that $R' = r$ for all k . One uses (2') to evaluate (22) as:

$-e^{-\lambda T} \int_0^T e^{-rt} c(t) dt$, the expected value of the loss if the borrower fails to find employment.

Remark 2: Suppose that for the time path under consideration, $c(\cdot)$, consumption decreases over time. Then, the expected present value of returns to lenders is nonnegative, only if $c(T) + \lambda R(k(T))/r < 0$.^{11/}

One gains some insight into the structure of (22) by considering, for a given consumption path $c(\cdot)$, the expected present value of lending to the borrower over the last ϵ time units before he goes bankrupt:

$$\begin{aligned}
 (22') \quad & - \int_{T-\epsilon}^T e^{-(r+\lambda)t} (c(t) + \lambda R(k(t))/r) dt \\
 & = - \int_{T-\epsilon}^T e^{-(r+\lambda)t} c(t) dt - \lambda R(k(T-\epsilon)) \int_{T-\epsilon}^T (e^{-(r+\lambda)t}/r) dt \\
 & \quad - \frac{\lambda}{r} \int_{T-\epsilon}^T e^{-(r+\lambda)t} (R(k(t)) - R(k(T-\epsilon))) dt .
 \end{aligned}$$

We have expanded expression (22') by adding and subtracting terms in $R(k(T-\epsilon))$ under the integral. The first term on the right hand side is the cost to lenders of financing the borrower's consumption during the last ϵ time units before bankruptcy. The second term is the expected return from an income switch between $T-\epsilon$ and T on that part of the borrower's debt that was incurred before $T-\epsilon$. The third term on the right hand side of (22') is the expected return from an income switch on those parts of the borrower's debt that are incurred between $T-\epsilon$ and T .

The first two terms are of the order of magnitude of ϵ , while the last term is of the order of magnitude of ϵ^2 . For small ϵ , the first

^{11/} This is obvious, because k is decreasing, by (2'), and $-k$ is increasing over time.

term, the cost of lending exceeds the last term, the returns on the amount that is lent during the last ϵ time units. The total present value of returns from lending during the last ϵ time units can be positive only because there are additional returns on loans that were given earlier.

This is a somewhat special instance of a rather general principle: A loan can be made on easier terms to the borrower, if it improves the quality of loans that the creditor has given to the debtor previously, than if it were the first loan from this creditor to this debtor. This principle has important consequences for the competitive structure of the loan market: The old creditor has a competitive advantage over a potential new creditor, if the new loan improves the quality of the old loans. This suggests that there is a strong tendency for a debtor to take loans from just one creditor. In the rest of this paper, we shall assume that this is in fact the case, so that we can neglect problems of interactions between different creditors.

For a given consumption path, one can also consider the present value of net returns to the creditor during the first ϵ time units after the debtor first goes into debt.

$$(22") \quad - \int_0^{\epsilon} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t)) / r) dt .$$

The same argument as before shows that during the first ϵ time units, the cost of lending is of the order of magnitude ϵ , while the returns that can be earned during this time are of the order of magnitude ϵ^2 . Thus, for small ϵ , the cost of financing the debtor's consumption during the first ϵ time units exceeds the returns to be earned during this time. In other words, the first loans have a positive overall return only because later loans keep the borrower out of bankruptcy for some time and raise the

probability that he finds employment and is able to fulfill his obligations on all his debts. The point is that the costs of the first loans are of an order of magnitude ϵ , while both the returns on the first loans if the agent finds employment and the probability that the agent finds employment during the first ϵ time units are of the order of magnitude ϵ , so that expected returns during the first ϵ time units are of the order of magnitude ϵ^2 .^{12/}

The arguments of the preceding paragraphs show that externalities between the first loans and the last loans are essential to the creditor's evaluation of the profitability of lending. It does not make sense to evaluate the profitability of each loan separately as if the other loans did not exist; the fact that later loans extend the time to bankruptcy for any given consumption path and improve the profitability of earlier loans is essential for considering the profitability of both earlier and later loans. For a correct evaluation of the profitability of lending, the creditor has to consider all the loans that he intends to make to the borrower and look at them as a whole.^{13/}

^{12/} It is wrong to deduce from this that small credit limits cannot be profitable. As one varies the credit limit, the debtor's consumption path changes, so that for small credit limits the time to bankruptcy need not be small. In Appendix 3, I give an example where the expected present value of lending to the borrower is positive for arbitrarily small values of the credit limit. I am grateful to Professor Dreze for pointing out to me this fallacy of deducing the unprofitability of small credit limits from the discussion of (22").

^{13/} This argument invalidates the strict validity of the principle that any loan has to be able to "stand alone," irrespective of future loans. We postulated this principle elsewhere (Foley and Hellwig 1974) to rule out the possibility of Ponzi games. But it is still true that loans made by any single creditor have to be able to stand alone and be profitable independent of loans made by other creditors.

9. The Determination of the Credit Limit: Ex Ante and Ex Post Criteria

If we take the debtor to be small relatively to the creditor, we can, for a first approximation, abstract from problems of risk aversion or liquidity constraints on the side of the creditor. We therefore assume that the creditor attempts to maximize the expected present value of net returns from lending to the borrower.

Remark 3: If it were completely up to the lender to determine the borrower's consumption path so as to maximize (22), he would prescribe a spike of consumption at the very first instant and no consumption thereafter.^{14/}

The creditor likes the debtor to consume much at the beginning so that he has a large claim on the debtor in case he finds employment early. As bankruptcy approaches, additional consumption by the debtor brings bankruptcy closer and decreases the quality of the claim on the debtor that the creditor already holds.

In the subsequent discussion we assume, as we did in the analysis of the borrower's behaviour, that the creditor has no direct control over the debtor's consumption path. He merely determines the credit limit and possibly the interest conditions. This assumption is somewhat extreme in view of the fact that most creditors require their borrowers to satisfy some "good behaviour" conditions. On the other hand, I cannot conceive of any creditor who has perfect control of a debtor's actions. Furthermore, the analysis of the extreme case seems to be warranted because it can reveal the problems that

^{14/} Any path with positive consumption at an instant $t > 0$ can be improved upon by consuming more earlier in such a way that indebtedness after the time t is unchanged: In this case, if income switches before t , the creditor has a higher claim on the debtor.

"good behaviour" conditions are designed to meet.

We shall neglect the determination of the interest schedule because this would require an analysis of the competition between creditors, which we are not equipped to undertake. Instead, we concentrate on how the creditor determines the credit limit for a given interest schedule. One would like to think that when the debtor first comes to borrow, the creditor announces the credit limit that maximizes the expected present value of net returns (22) and the debtor accepts this as a parameter of his behaviour. In this view the credit limit is determined by the first order condition for the maximization of (22).

This view of the credit limit neglects the fact that the announcement of a credit limit is an asymmetrical precommitment. Nothing prevents the creditor from raising the credit limit at some later time, if he finds it convenient to do so. In fact, the debtor will be very happy about this. On the other hand, it would be a breach of contract to lower the credit limit in the absence of contract violations by the debtor. Thus, at any later instant, the creditor is free to raise, but not to lower the credit limit from the value that he announced initially.

The actually enforced credit limit is equal to the announced credit limit only if the creditor does not at some later instant find it to be in his interest to increase the credit limit. At any time τ along the borrower's path, the expected future net returns from additional lending to the borrower are:

$$(23) \quad - \int_{\tau}^T e^{-(r+\lambda)(t-\tau)} (c(t) + \lambda R(k(t)))/r dt$$

This quantity differs from the initial expected net returns (22) by the terms that concern the time path between 0 and τ . From Proposition 6, we know that these terms are affected by a change in the credit limit. Therefore, they play a role in the determination of the credit limit that is originally announced. At the time τ , these terms are historically given. If the creditor considers an increase in the credit limit so as to maximize (23) rather than (22), he need not consider effects of the credit limit on these terms which lie in the past. Therefore, the value of the credit limit that maximizes (23) is not in general the same as the value that maximizes (22).

The one decision that determines the credit limit irrevocably is the decision to cut off credit and let the debtor go bankrupt. Suppose that this decision is taken at T , when the debtor's debt is A . It must be true that at T , any further extension of credit appears unprofitable to the creditor. Let ΔA the amount of additional credit that the creditor considers and $\Theta(A, T, \Delta A)$ the time when the debtor will have exhausted this amount. Credit is cut off at T , if for all $\Delta A > 0$,

$$(24) \quad - \int_T^{\Theta(A, T, \Delta A)} e^{-(r+\lambda)(t-T)} (c(t) + \lambda R(k(t)) / r) dt < 0 .$$

In particular, as we let ΔA go to zero, by (17), $\Theta(A, T, \Delta A)$ approaches T , unless $A = 0$; a necessary condition for the creditor to cut off credit is:

$$(25) \quad c(T) + \lambda R(k(T)) / r \geq 0 .$$

To understand condition (24) and to distinguish it from the maximization

of (22), consider the cut-off credit limit A . The creditor decides whether to give an additional credit ΔA . At a first order of approximation, this additional credit is used up by the debtor's consumption and interest payments at a rate $c(T) - R(-A)$, so that it lasts for a time span of $\Delta A / (c(T) - R(-A))$. During this additional time that is given to the debtor, the creditor experiences a cost of $c(T)$ per time unit, whereas the benefits from the chance that the debtor finds employment are $-\lambda R(k(T))/r$ per time unit. If the additional credit, however small it may be, is unprofitable, the costs per time unit exceed the benefits per time unit during the extension of the debtor's time to bankruptcy.

This consideration completely neglects the effects of ΔA on the debtor's consumption path before he has reached the cut-off credit limit A . From Proposition 6, we know that an increase in ΔA , if it were announced at the beginning, would raise the debtor's consumption level before he even had used up the credit A . Thus, it would shorten the time needed to exhaust A . This effect is taken into account in the determination of the credit limit that maximizes the ex ante present value of returns (22); it is not taken into account in the ex post consideration of whether to cut off credit or not.^{15/}

^{15/}I have not been able to show that the cut-off credit limit satisfying (24) is always higher than the credit limit that maximizes (22). Remark 2 above strongly suggests that this is true for many cases. On the other hand, we know that if the cut-off credit limit is lower than the credit limit that maximizes (22), the creditor can precommit himself to the credit limit that maximizes (22), because he can precommit himself not to give less than he announced.

Condition (25) suggests an objection to the assumption that the creditor has no direct control over the debtor's consumption. If he cuts off credit, because the cost of an additional credit, which is given by the debtor's consumption, exceeds the expected benefits, there would be a possibility for the debtor and the creditor to make a mutually advantageous agreement whereby the creditor gives additional credit under the condition that the debtor limit his consumption. Since the debtor dislikes the alternative of going bankrupt immediately, he would be willing to agree to such an arrangement. If the creditor is in a position to control the rate at which he pays out cash on the additional credit (as opposed to the rate at which he writes up the debt in lieu of interest payments to himself), he is even able to police the agreement. To justify the lack of direct control by the creditor, one would have to introduce a model where the current expenditures are "needed" to keep alive the chance of a gain, as in an oildrilling operation, and where the debtor has more information about what really is needed than the creditor, so that the creditor cannot exert control over the actual use of cash. Such a model would be beyond the scope of this paper.

Another problematic feature of (25) is more formal than real. For many utility functions, there exists no credit limit that satisfies (25). The costs of extending more credit are always below the benefits.^{16/} This is due to the fact that while $c(T)$ increases with the credit limit (see equation (17)), so does the creditor's claim on the debtor, the quality of which is improved by an additional loan. The creditor considers that if he found it worthwhile to give credit yesterday, it is even more worthwhile to give

^{16/} For an example see Appendix 3.

credit today, because he already has a vested interest in this particular debtor. This requires of course that the debtor's lack of success yesterday does not affect the creditor's estimate of the probability of his success today. In the formalism of our model, this problem can be avoided if λ , the parameter of the income switch depends on time and tends to zero. For a full theory of the credit limit, it is therefore necessary to analyze the evolution of the creditor's estimate of the debtor's future changes of success as the debtor has failed to succeed in the past.^{17/}

10. The Determination of the Credit Limit: Sophistication and Super-Sophistication

The debtor may realize that the announced credit limit that maximizes (22) is not the credit limit that will actually be enforced, that the credit limit that is actually enforced, has to satisfy (24). In that case, he will behave as though he expected the actual credit limit from the very beginning, independent of the initially announced credit limit. This change of behaviour on the side of the debtor does not affect the criterion (24) for the cut-off point. The criterion (24) is independent of the debtor's behaviour before the cut-off point was reached. In particular, it is independent of whether the debtor behaved as though he anticipated the announced credit limit or the actual cut-off point.

^{17/} In the formalism of our model, where λ is a constant, the possibility of an infinite credit limit is ruled out by the assumption that creditors obtain nothing when the debtor goes bankrupt. Under this assumption, they are not willing to grant credit beyond the point indicated by condition (4): Any further credit would involve a sure loss as the debtor would be sure to go bankrupt even if he did obtain the higher income, and on bankruptcy, creditors would not be allowed to impound the debtor's income.

On the other hand, the dependence of the debtor's behaviour on the cut-off credit limit rather than the announced credit limit has the consequence that the ex ante expected present value of returns to the creditor (22) falls short of its maximum whenever the cut-off credit limit exceeds the credit limit that maximizes (22). If the original announcement is not credible to the debtor, because he anticipates the change in the creditor's interest between the beginning and the cut-off point, then the creditor's ability to change his mind later makes him worse off, on average than he would otherwise be.

It is essential for an understanding of this problem to see that the creditor's dilemma does not arise out of some misperception of his own interests. The creditor's true interests at the cut-off point dictate that the cut-off point satisfy condition (24). These interests differ from his interests when he just begins to lend to the debtor, because he now has more information, for instance about the fact that up to now the debtor was unsuccessful in finding employment. Thus, it is rational for him to desire to change his original decision.

However, the expectation of this revision by the debtor from the very beginning puts him into a position that is on average worse in the ex ante sense. Before he begins to lend, he would find it advantageous to precommit himself never to raise the credit limit. This precommitment is not credible, because when the time comes to let the debtor go bankrupt, he will not be ready to keep it and the debtor will be quite willing to oblige him. It is the anticipation by the debtor of the creditor's true future interests that puts the creditor in the position where he cannot achieve the ex ante maximum of expected returns.

Indeed, it is quite possible that when the cut-off point satisfies (25) and the debtor anticipates the cut-off point correctly, the ex ante expected present value of returns is negative, and the creditor makes a loss on average. Indeed, from remark 2 above, we know that this is the case whenever the Euler equation (13) prescribes a time path for the debtor such that consumption decreases over time.^{18/} But if the debtor can anticipate the point at which criterion (24) requires credit to be cut off, the creditor should be able to anticipate it too. Thus, he will notice that what he forecasts to be his own behaviour in the future induces him to make a loss, on average. Therefore, he will not be willing to embark on this path at all and he will cut off credit before he even granted it.

This consideration leads to the formulation of a new rule for cutting off credit from the debtor. Suppose that the creditor considers the announcement of an additional credit ΔA . He forecasts that with this announcement, he will eventually give a total additional credit $\Delta A + f(A+\Delta A) \geq \Delta A$, i.e., after the debtor has used the amount ΔA , the creditor expects himself to give further credits adding up to $f(A+\Delta A)$. Furthermore, he expects the debtor to have the same forecast and to behave accordingly, so that the true time to bankruptcy is $\theta(A, T, \Delta A + f(A+\Delta A))$. Now the sophisticated cut-off criterion requires the creditor to cut off credit from the debtor whenever, for all announcements of additional credits $\Delta A > 0$, expected returns are negative, where the expectation is calculated in anticipation of the actual cut-off credit limit $A + \Delta A + f(A + \Delta A)$:

^{18/} For an example see Appendix 3.

$$(26) \quad - \int_0^{\Theta(A, 0, \Delta A + f(A + \Delta A))} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t)) / r) dt < 0 \quad .^{19/}$$

In a sense, this sophisticated criterion is more general than the naive criterion (24): The naive criterion (24) may be considered to be the special case of (26) where $f(A + \Delta A) = 0$ for all $A, \Delta A$, so that one always expects to stick with one's announcement.

It is also easy to see that whenever the naive criterion (24) requires a cutting of additional credit, the sophisticated criterion (26) will require a cutting of credit, independent of the specific function $f(\cdot)$ that is used in the sophisticated criterion.

The sophisticated cut-off criterion contains a logical difficulty: On the one hand, it is a criterion for cutting off credit. On the other hand, it makes use of a forecast about one's own future behaviour in cutting off credit should one decide not to cut credit immediately. This forecast appears to be arbitrary, unless it were to correspond to actual behaviour. Therefore, to accept criterion (26) as a valid description of creditor behaviour, one wants the two notions of cutting off credit to coincide, so that the behaviour that is forecast is also the behaviour that results from applying (26). In a sense, the creditor's forecast of his own behaviour should be based on rational expectations.

In the remainder of this section, it is shown that whenever sophisticated

^{19/} The concept of sophisticated behaviour is developed in the small literature on changing preferences that was pioneered by Strotz (1956). See also Pollak (1968) and Peleg and Yaari (1973).

and naive criteria diverge, it is not possible to find such a consistent behaviour that has the property that if the creditor anticipates it and expects the debtor to anticipate it he himself will want to apply it. Before giving the general theorem, I give an example to illustrate the problem in the consistency requirement.

Assume that interest, time preference and the income switch parameter are such that for any optimal path, the debtor's consumption is a decreasing function of time. If the cut-off point is determined by the naive criterion, ex ante expected returns are negative, by (25) and remark 2. Define sophisticated behaviour of the first degree to be the behaviour that cuts off credit by the sophisticated criterion (26), where the forecast of the actual credit limit $A+\Delta A+f(A+\Delta A)$ is the credit limit at which the naive criterion requires a cut-off. Thus, the forecast credit limit satisfies (25), so that under our assumptions on borrower behaviour first degree sophistication foresees negative expected returns, no matter what the current state and what the additional announced credit should be. Therefore, first degree sophistication requires an immediate cut-off of additional credit, no matter where the borrower is currently.

Define sophisticated behaviour of the second degree or super-sophisticated behaviour of the first degree as the behaviour that cuts off credit by the sophisticated criterion (26), where the forecast actual credit limit $A+\Delta A+\bar{f}(A+\Delta A)$ is the credit limit that first degree sophisticated behaviour will enforce after the initial announcement ΔA . Under our conditions on borrower behaviour, first degree sophisticated behaviour always requires an immediate cut-off of additional credit; therefore, $\bar{f}(A+\Delta A) = 0$ for all $A+\Delta A$. The credit limit that is expected to be enforced is $A+\Delta A$. But this

is the forecast made by the naive creditor! Thus, we have shown:

Proposition 7: Suppose that ex ante expected returns under the application of the naive criterion are always negative. Then first degree super-sophisticated behaviour is the same as naive behaviour.

The logic of Proposition 7 is very simple: Under naive behaviour, one comes out with a loss, on average. First degree sophistication foresees this and gets out immediately. But if one is sophisticated enough to get out now, surely, one can expect to be sophisticated enough to get out tomorrow. Therefore, one can feel justified in applying the naive criterion now. Unfortunately, tomorrow's decision is again going to be supersophisticated rather than just sophisticated, so that, considering one decision after the other, the super-sophisticated creditor makes the same mistakes as the naive creditor.^{20/}

Proposition 7 shows that an "increasing degree" of sophistication (defined inductively) need not lead to the "right" amount of sophistication, but may lead right back to naivete. The problem with this example is that both sophistication of the first degree and super-sophistication use the "wrong" forecasts of future creditor behaviour. They use different forecasts than would be obtained if they themselves were applied. For instance the super-sophisticated behaviour forecasts sophisticated behaviour, which differs from what super-sophisticated behaviour itself would produce. Therefore, it is desirable

^{20/} This problem is similar to some aspects of the decision making that led into the Vietnam War (Halberstam 1973). Instead of a single decision on whether or not to engage in a full scale land war, which might have been answered in the negative (first degree sophistication), there was a series of piecemeal decisions in favour of "escalation," each of which was made under the assumption that in the next decision, one still had the option not to escalate any further. Of course, in this case, misjudgement of the chances of success of each of the piecemeal measures seems to have been more important than the game theoretical aspects of the different parties' behaviours.

to develop some notion of consistency to see whether this kind of schizophrenia can be avoided.

Define the following mapping from the space of nonnegative upper-semicontinuous real valued correspondences on the nonnegative halfline into itself:

$$(27) \quad (Tf)(A) = \{\Delta A^* + \bar{f}(A+\Delta A^*)\},$$

where $\Delta A^* \geq 0$, $\bar{f}(A+\Delta A^*) \in f(A+\Delta A^*)$ satisfy the condition that for all $\Delta A \geq 0$, $\tilde{f}(A+\Delta A) \in f(A+\Delta A)$,

$$(28) \quad - \int_0^{\theta(A,0,\Delta A^*+\bar{f}(A+\Delta A^*))} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t)))/r dt \geq - \int_0^{\theta(A,0,\Delta A+\tilde{f}(A+\Delta A))} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t)))/r dt \quad 21/$$

$(Tf)(A)$ is the total additional credit that the creditor expects to give to a debtor with current debt A , if he announces a current credit extension ΔA^* to maximize expected returns, where he anticipates total future credit extensions to be given by the correspondence $f(\cdot)$. If one is at a point where the criterion (26) is satisfied, the best that one can do is not to make any further losses. Then one sets $\Delta A^* = 0$, so that $(Tf)(A) = \bar{f}(A)$. If in this case, $\bar{f}(A) > 0$, one would be at a point, where the maximization

^{21/} We avoid the Peleg-Yaari existence problem by assuming that f is upper-semicontinuous and that the maximization determining the mapping T need only consider those values in the set $f(A+\Delta A^*)$, which are preferred from its own point of view. For an elaboration of this procedure and a proof that the mapping T preserves upper-semicontinuity, see Hellwig (1973). From the nature of the problem considered here, the Peleg-Yaari game equilibrium concept is not appropriate, because it blurs the ex-ante vs. ex-post distinction on which this discussion is built.

(28) requires not to announce a credit, while the behaviour summarized in $\bar{f}(A)$ allows for a continuation of credit. Criterion (26) in anticipation of the behaviour $f(\cdot)$ requires a cutting of credit while $f(\cdot)$ implies a continuation.

In this case, behaviours $f(\cdot)$ and $Tf(\cdot)$ are inconsistent with each other, because they indicate different decisions on the cutting of additional credit. As a minimal condition on consistency, we want to require that there is no ambiguity about the cutting-off of additional credit. Therefore, we say that the correspondence $f^*(\cdot)$ represents a consistent degree of sophistication if and only if: $0 \in (Tf^*)(A)$ if and only if $0 \in f^*(A)$, so that both $f^*(\cdot)$ and $Tf^*(\cdot)$ indicate a cut-off at the same point.^{22/}

Proposition 8: Suppose that the expected present value of returns to the creditor is negative whenever the cut-off point satisfies condition (25). Then there exists no consistent degree of sophistication, unless the naive criterion (24) rules out all lending, or there exists no upper bound on lending.

Proof: Suppose that a consistent degree of sophistication exists and is represented by $f^*(\cdot)$. Define:

$$A = \{A \mid 0 \in (Tf^*)(A)\},$$

the set of values of the credit limit at which anticipation of the behaviour

^{22/} A more stringent consistency condition would require $f^*(\cdot)$ to be a fixed point of the mapping T , so that for all A , $Tf^*(A) = f^*(A)$. The following results hold for this more stringent condition a fortiori.

$f^*(\cdot)$ makes the immediate cutting off of credit a desirable choice. From (28), it is clear that A is closed. For A in the interior of A , $A + \Delta A \in A$ for all sufficiently small ΔA . Since $f^*(\cdot)$ represents a consistent degree of sophistication, $0 \in f^*(A + \Delta A)$ for all sufficiently small ΔA . At a debt A , consider the announcement ΔA , with $\bar{f}^*(A + \Delta A) = 0 \in f^*(A + \Delta A)$. The expected present value of returns for this announcement is:

$$(29) \quad - \int_0^{\theta(A, 0, \Delta A)} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t))/r) dt$$

From (28) and the fact that $A \in A$, expression (29) must be nonpositive for all sufficiently small ΔA . Hence, (25) is satisfied at A in the interior of A . From continuity considerations,^{23/} (25) must also be satisfied at positive points of the boundary of A , except possibly at an isolated point.

Consider now $A_0 = \sup\{\mathbb{R}_+ - A\}$, the smallest upper bound on the amount of debt that the debtor can have such that the creditor positively rules out the alternative of cutting credit. If this quantity is not finite there is no upper bound on lending. If however, the quantity A_0 is finite, then $A_0 \in A$, because A is closed. Furthermore, it is not an isolated point of A , by definition. By the reasoning given above, it must satisfy condition (25).

Furthermore, all $A > A_0$ are interior points of A and satisfy (25) too. We show that $A_0 = 0$. For suppose that $A_0 > 0$. Then, there exists $\eta > 0$, such that for all $\delta < \eta$, $A_0 - \delta \in \{\mathbb{R}_+ - A\}$. At $A_0 - \eta$, one anticipates credit to be cut off at some $A \geq A_0$, i.e., at a point at which (25)

^{23/} $R(-A)$ is continuous in A , by assumption; terminal consumption for the credit limit A is continuous in A for all $A > 0$, by (17).

holds. But by the premise of Proposition 8, anticipation of a cut-off point that satisfies (25) implies that the expected present value of returns from continued lending is negative. On the other hand, from (28) and the fact that $A_0 - \eta \in \{R_+ - A\}$, i.e., that a cutting of credit is not desirable at $A_0 - \eta$, the expected present value of returns from continued lending is positive. Thus, we have derived a contradiction, so that the assumption that $A_0 > 0$ must be false. Hence, $A_0 = 0$, and $A = R_+$. For all $A, \Delta A, A + \Delta A \in A$, so that (29) must be nonpositive for all A and ΔA . It follows that by the naive criterion (24), lending is not profitable at any point. This completes the proof of Proposition 8.

Proposition 8 shows that one cannot generally expect to obtain consistent rules of creditor behaviour. In fact, a consistent degree of sophistication will not exist, if the naive criterion makes some lending desirable, if there exists an upper bound on the credit limit under sophisticated behaviour, and if credit limits satisfying condition (25) lead to negative expected returns from lending. The first two conditions preclude cases of little economic interest in this context. The crucial condition is the third condition that the expected present value of returns be negative whenever the cut-off credit limit satisfies condition (25). From remark 2, we know that this condition is by no means pathological.

We illustrate the problem of inconsistent creditor behaviour by the following corollary to Proposition 8, which generalizes Proposition 7.

Corollary 1: Suppose that under the application of the naive criterion some lending is profitable, but that there exists a finite upper bound on lending, by the naive criterion. For any sophisticated cut-off criterion that requires a cutting off of additional credits at points where a small additional credit

would be profitable by the naive criterion and (25) is violated, there exists a super-sophisticated criterion such that if A is a point at which (25) is violated and the sophisticated criterion requires a cutting, for sufficiently small ΔA , the super-sophisticated criterion requires a continuation of credit at $A - \Delta A$.

Proof: For sufficiently small ΔA , the super-sophisticated creditor at $A - \Delta A$ can announce the credit ΔA . Since he expects the sophisticated criterion to cut off credit at A , the actual credit limit will be A . By assumption, the small additional credit ΔA is profitable.

Corollary 2: If A and $A + \Delta A$ for sufficiently small ΔA are as defined in Corollary 1, then the super-sophisticated criterion requires a continuation of credit at A .

Thus, one can construct a super-sophisticated behaviour which contradicts the sophisticated behaviour when the sophisticated behaviour differs from naive behaviour except possibly at isolated points.

Proposition 8 and its corollaries indicate that we cannot generally find a consistent behaviour for the creditor to follow. The notion of what is rational for the creditor to do is not generally well defined. This seems to be a fundamental problem for sequential decision problems of the structure that we consider here. In situations where there is a question of "getting out" or not, the ability to take a new decision at a later moment, i.e., the inability to precommit oneself can have the consequence that the concept of rationality or consistency is not well defined. The problem is familiar from Ponzi games or "speculative bubbles," in which one may take part because one hopes to get out before the bubble breaks, but in fact one never does get out, because one always thinks one can defer it for just a moment. Our

model shows that the same type of inconsistency arises in nonpathological situations in which in principle, lender and borrower could find a mutually advantageous contract. Furthermore, we have shown that if the inconsistency problem does arise, there is no way in which the lender can, through the "right" anticipation of his own behaviour avoid it.

It should be noted that at this point, randomization of the creditor's decision does not provide a solution to the problem of inconsistency. For while both creditor and debtor may anticipate the cut-off credit limit to be given randomly, the present model does not provide for any reason why the creditor should in fact randomize his cut-off decision. The time path of net returns after any credit announcement is independent of the probability that one assigns to the announcement; therefore, the profitability of the announcement is independent of the probability that one assigns to it. If the announcement is profitable, one therefore makes it with probability one; if it is unprofitable, one cuts credit with probability one. One does not find it desirable to cut credit with a probability between zero and one.

At this point, our only consolation can be that if a creditor cannot determine consistently what he should do, then surely the debtor cannot predict with certainty what the creditor will do, so that the assumption of perfect foresight by the debtor is inappropriate. While this may indicate that the game theoretical problems underlying the present discussion may be less serious in actual interactions between creditors and debtors, under limited information, it leaves us emptyhanded in our attempt to explain the determination of the credit limit in the interaction between borrower and lender in a full information framework.

11. The Determination of the Credit Limit for a Population of Debtors^{24/}

Up to now, we have considered the interaction of a single creditor and a single debtor. While this was the case for the Fugger and Habsburg (and actually captures some of the difficulties the Fugger had with Habsburg towards the end of their relationship), it is not an accurate description of borrowing and lending in most markets. Especially where we made the assumption that the individual borrower was small relative to the individual lender, it seems appropriate to consider the determination of the credit limit for a population of borrowers all borrowing from the same lender.

In this context, the assumption that the debtor can fully anticipate the point at which the creditor will cut off additional credit even if it differs from the announced credit limit, is more reasonable, because the individual debtor can observe what happens to other debtors who began to borrow earlier than he did. If one assumes that the creditor treats all similar debtors alike, the assumption that the debtor has full information on future creditor behaviour is very natural. On the other hand, the creditor who cuts off credit to some debtors will take account of the announcement effect this has for other debtors who are not yet as advanced on their borrowing path. While the cut-off decision no longer affects the initial stages of the consumption path of the particular debtor whose credit is cut off, it does affect the initial stages of the consumption paths of debtors whose cut-off moments lie in the future.

^{24/} I am grateful to Michael Rothschild for several important suggestions on this section.

To make this precise, let $b(K)$ debtors have a capital K at a given instant. Also, let C be a weight indicating the importance of future debtors who have not yet begun to borrow. If the creditor cuts off additional credit at the credit limit A and is precommitted to always cutting off credit at the same credit limit, so that debtors can anticipate it, the expected present value of net returns from lending to the borrowers is given as:

$$(30) \quad - C \int_0^{T(0,A,B)} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t)))/r dt \\ + \int_0^{-A} b(K) \int_{\tau(K)}^{T(0,A,B)} e^{-(r+\lambda)(t-\tau(K))} (c(t) + \lambda R(k(t)))/r dt dK ,$$

where $\tau(K)$ is the epoch at which the path anticipating the credit limit A is at the capital K .

If he has to precommit himself to always choose the same credit limit, the creditor will select the credit limit that maximizes (30). But again, a consistency problem arises, because as time passes, the expected present value of future net returns from lending can change, if the weights C and $b(\cdot)$ change over time. Only if the stream of new borrowers satisfies strong stationarity conditions will these weights be constant over time, so that the maximand (30) does not change over time. The following example illustrates this problem:

Suppose that currently, x new debtors begin to borrow from the creditor and that the number of new debtors grows geometrically at the rate n . If we take C to represent the number of future debtors, discounted to the present, we can set $C = x/(r-n)$, provided that $r > n$. Furthermore, suppose that current debtors have already anticipated the credit limit A that

the creditor is setting, so that the debtor at capital K began to borrow $\tau(K)$ time units ago. At that time, $x e^{-n\tau(K)}$ new debtors began to borrow from the creditor. Of these, an average $(1 - e^{-\lambda\tau(k)})$ have managed to obtain a higher income. Therefore, we can take the weights to be $x e^{-(n+\lambda)\tau(k)}$ 25/

If we substitute for C and $b(K)$ in (30), the expected present value of returns if A is again chosen is given as:

$$-\frac{x}{r-n} \int_0^{T(0,A,B)} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t)))/r dt$$

$$- \int_0^{T(0,A,B)} x e^{-(n+\lambda)t} \int_{\tau}^{T(0,A,B)} e^{-(r+\lambda)(t-\tau)} (c(t) + \lambda R(k(t)))/r dt d\tau .$$

We can integrate the second term by parts to rewrite the expected present value of returns as:

$$(31) \quad -\frac{x}{r-n} \int_0^{T(0,A,B)} e^{-(n+\lambda)t} (c(t) + \lambda R(k(t)))/r dt .$$

This value is compared to the expected present value of returns from choosing any other credit limit A' , which from (30), is simply:

25/ In reality, the creditor knows the actual number of those who have already obtained a higher income. If he used this number to determine the weights $b(K)$, it would not be possible to generate a stationary decision.

(32)

$$\begin{aligned}
 & - \frac{x}{r-n} \int_0^{T(0,A',B)} e^{-(r+\lambda)t} (c(t) + \lambda R(k(t))/r) dt \\
 & + \int_0^{-A} x e^{-(n+\lambda)\tau(K)} \int_{\sigma(k)}^{T(0,A',B)} e^{-(r+\lambda)(t-\sigma(K))} (c(t) + \lambda R(k(t))/r) dt dK ,
 \end{aligned}$$

where $\sigma(K)$ is the epoch at which the path anticipating A' passes through K , and where consumption paths are determined in anticipation of A' .

The creditor will continue to choose the credit limit A , if the value of (31) exceeds the value of (32) for all $A' \neq A$. This criterion is independent of x , the indicator of the absolute size of the population of debtors. Since the ratios of the weights C , $b(K)$ do not change, if the creditor continues to set the credit limit A , the criterion for setting the credit limit remains the same over time. Thus, we have shown that for a population of debtors growing at the constant rate n , if the creditor has always set the credit limit A and finds it optimal to cut off credit at A , presently, he will always continue to set the credit limit A .

The expected present value of returns from lending to the population of borrowers is then given as the expected present value of returns from a single debtor, discounted at the growth rate rather than the interest rate.

If the growth rate exceeds the discount rate, the weight C will not be finite; (but the weights $b(K)$ are). Future debtors are infinitely more important to the creditor than current debtors, so that the creditor sets the credit limit to maximize the expected present value of net returns from future creditors, disregarding current debtors. This is done--in the overtaking sense--by setting the credit limit to maximize the expected present

value of net returns from any single debtor, discounted at the interest rate. In this case, the creditor is able to set the credit limit so as to maximize the average returns from lending. If on the other hand, the rate of growth falls short of the rate of interest, the creditor does give some weight to current debtors, as expressed in (31), where the later epochs of the typical debtor's consumption path receive a higher weight than in the calculation of the expected present value of returns from any single debtor, (22).^{26/}

Choosing the credit limit A if A has been chosen in the past, and if (31) exceeds (32) for all A' is a consistent strategy, if the creditor is constrained to announce a credit limit under the assumption that he will have to treat all debtors alike. This does not rule out the possibility that the creditor finds it preferable to vary the credit limit over time in order to confuse debtors. Furthermore, we have not discussed the disequilibrium problem of how the credit limit for which (31) exceeds (32) is reached if one has not chosen it in the past. In this case, the weights $b(K)$ are not stationary, and the credit limit does vary over time. To analyze such questions one has to introduce some notion of how debtors form their expectations about the treatment they will receive from what they currently observe. Such an analysis of the strategic interactions between a creditor and a population of debtors goes beyond the scope of this paper, although it is a desideratum for a general theory of creditor behaviour.

Such an analysis seems to be all the more necessary as expressions (31) and (32) indicate that creditor behaviour will not in general be stationary

^{26/} I have not been able to establish an unambiguous relationship between the condition that (31) exceed (32) and the condition that A be the credit limit that maximizes (31).

the limited information of any individual debtor who has to infer the credit limit enforced on him from his observations of currently enforced credit limits. This suggests that the dynamics of a population of debtors are an important element in the determination of the credit limit.

But even if one can resolve the inconsistency of the creditor's behaviour, there seems to be little presumption that the allocation of loans will proceed in accordance with principles that correspond to the intuitive notion of a perfect capital market. The sophisticated creditor who cannot precommit himself may not give a loan, even though a mutually desirable contract could be found, if he could precommit himself. There is no presumption that the debtor for whom the potential ex ante expected present value of returns is highest will be the first to obtain a loan, if the creditor cannot precommit himself.^{27/}

Furthermore, the continuation of credit to previous debtors raises severe problems for the functioning of the loan market. First, a creditor who already has some claims on the debtor has a competitive advantage over other creditors, because the new loan he gives enhances the quality of the loans that he has given previously. Secondly, if this creditor has to decide on whether to allocate funds to continued loans to this old debtor or to some new debtor, there will be a bias to granting loans to the old debtor, because the pay-off to the creditor of lending an additional dollar to a debtor depends on both the interest that this dollar will earn if the debtor makes good and on the improvement in the quality of previously given loans. This creates

^{27/} Thus, there is an imperfection of the loan market in the strict sense defined by Stigler (1967).

an a-priori presumption that the allocation of loans between old and new debtors is not optimal from a social point of view: If the social returns from lending are determined by the prospects of the debtor being successful (e.g., finding oil), the private returns to the creditor will depend on the distribution of these returns between the debtor and himself, which is the more favourable to himself, the more he is already owed by the debtor. This suggests that the allocation of loans through decentralized credit markets is biased towards projects that have been started earlier, as opposed to new projects.

In this paper, we have assumed that creditors and debtors had perfect information about each other. In practice, this is not a good assumption. It has been suggested that an asymmetry between debtors and creditors in the information about the debtor and his project plays an important role in the loan market as it does in the insurance market (Jaffee and Russell 1974). It seems desirable to develop the notion of differences in information in an intertemporal framework of the type proposed in this paper in order to analyze the interaction of the problems of the intertemporal structure of decisions and of the asymmetry of information in order to develop a fully integrated theory of the loan market.

Finally, it has to be stressed again, that the analysis of this paper assumes that the creditor has no control over debtor behaviour aside from determining the credit limit. In practice, creditors do attempt to exert some influence over debtor behaviour, presumably in part because of the problems discussed in this paper. While perfect control by creditors appears unrealistic, it is desirable to investigate potential partial controls and the extent to which they succeed in resolving the creditor's problems.

Appendix 1: The Optimal Program in the Absence of Borrowing

In this appendix I prove Proposition 1.

Statement 1a: For all t , $u'(c(t)) > V'(k(t))$

Proof: Partition the set of admissible values of capital into:

$$A = \{k | u'(c(0; k)) > V'(k)\}$$

$$B = \{k | u'(c(0; k)) \leq V'(k)\}$$

It is sufficient to show that B is empty. Suppose there exists $k \in B$. Then, the optimal path beginning at k will either remain in B forever, or it will leave B at some time. First, suppose that it remains in B forever. Then, for all t ,

$$(A.1) \quad u'(c(t, k)) = u'(c(0, k(t))) \leq V'(k(t)),$$

and from the Euler equation (11),

$$\begin{aligned} \dot{u}'(c(t, k)) &= - (r-\rho)u'(c(t, k)) - \lambda(V'(k(t)) - u'(c(t, k))) \\ &\leq - (r-\rho)u'(c(t, k)). \end{aligned}$$

It follows that for all t ,

$$(A.2) \quad u'(c(t, k)) \leq u'(c(0, k))e^{-(r-\rho)t}.$$

On the other hand, by the envelope theorem, $V'(k) = u'(c^*(0, k))$, where

the path $c^*(t, k)$ is optimal for initial capital k and certain income a and satisfies:

$$(A.3) \quad u'(c^*(t, k)) = u'(c^*(0, k))e^{-(r-\rho)t}$$

and the transversality condition:

$$(A.4) \quad \lim_{t \rightarrow \infty} e^{-\rho t} k(t) u'(c^*(t, k)) = \lim_{t \rightarrow \infty} e^{-rt} k(t) u'(c^*(0, k)) = 0 .$$

Combining (A.1), (A.2) and (A.3), one derives that for all t :

$$u'(c(t, k)) \leq u'(c^*(t, k)) ,$$

or, by the concavity of u ,

$$(A.5) \quad c(t, k) \geq c^*(t, k) .$$

We multiply (A.5) by e^{-rt} and integrate over t , to get:

$$(A.6) \quad \int_0^{\infty} e^{-rt} c^*(t, k) dt \leq \int_0^{\infty} e^{-rt} c(t, k) dt \leq k < k + a/r ,$$

where the second inequality is the budget constraint for $c(t, k)$ and the last inequality is due to the positivity of a . It is easy to verify that (A.6) contradicts the transversality condition for $c^*(t, k)$, (A.4). Hence, there is no optimal path that begins in B and stays in B forever. Therefore an optimal path beginning at $k \in B$ must leave B at some time.

Therefore, there exists a point k^* on the boundary of B , from which the optimal program leads into A , so that:

$$(A.7) \quad u'(c(0, k)) = V'(k^*) = u'(c^*(0, k^*))$$

and along the path $c(t, k^*)$,

$$(A.8) \quad \dot{u}'(c(t, k^*)) \Big|_{t=0} > \dot{V}'(k(t)) \Big|_{t=0} = V''(k^*)(rk^* - c(0, k^*)) .$$

Again, $c^*(t, k^*)$ is the optimal path for initial capital k^* and certain income a . Along this path, $c^*(t, k^*)$, we have,

$$(A.9) \quad \dot{u}'(c^*(t, k^*)) = \dot{V}'(k(t)) = V''(k^*)(rk^* + a - c^*(0, k^*)) .$$

$$(A.10) \quad \dot{u}'(c^*(t, k^*)) = - (r-\rho)u'(c^*(t, k^*))$$

Now, the Euler equation for the path $c(t, k^*)$ requires:

$$\begin{aligned} \dot{u}'(c(t, k^*)) \Big|_{t=0} &= - (r-\rho)u'(c(0, k^*)) - \lambda(V'(k^*) - u'(c(0, k^*))) \\ &= - (r-\rho)u'(c^*(0, k^*)) \\ &= \dot{u}'(c^*(t, k^*)) \Big|_{t=0} \\ &= V''(k^*)(rk^* + a - c^*(0, k^*)) \end{aligned}$$

$$(A.11) \quad < V''(k^*)(rk^* - c^*(0, k^*)) ,$$

where we have successively substituted from (A.7), (A.10) and (A.9). From (A.7), $c^*(0, k^*) = c(0, k^*)$, so (A.11) contradicts (A.8). Therefore, k^* does not exist. It follows that B must be empty.

Statement 1b: For all k , initial consumption $c(0, k, \lambda)$ is increasing in λ .

Proof: For any $\lambda_1, \lambda_2; \lambda_1 > \lambda_2$, partition the set of admissible values of capital into:

$$\tilde{A} = \{k | c(0, k, \lambda_1) > c(0, k, \lambda_2)\}$$

$$\tilde{B} = \{k | c(0, k, \lambda_1) \leq c(0, k, \lambda_2)\}$$

Again, we show that \tilde{B} is empty. Suppose $k \in \tilde{B}$. First, suppose that the optimal path for λ_1 beginning at k stays in \tilde{B} . The Euler equations for $c(t, k, \lambda_1), c(t, k, \lambda_2)$ are, respectively,

$$(A.12) \quad \begin{aligned} \dot{u}'(c(t, k, \lambda_1)) &= - (r-\rho)u'(c(t, k, \lambda_1)) \\ &\quad - \lambda_1 (V'(k(t, k, \lambda_1)) - u'(c(t, k, \lambda_1))) \end{aligned}$$

$$(A.13) \quad \begin{aligned} \dot{u}'(c(t, k, \lambda_2)) &= - (r-\rho)u'(c(t, k, \lambda_2)) \\ &\quad - \lambda_2 (V'(k(t, k, \lambda_2)) - u'(c(t, k, \lambda_2))) \\ &< - (r-\rho)u'(c(t, k, \lambda_2)) \\ &\quad - \lambda_1 (V'(k(t, k, \lambda_2)) - u'(c(t, k, \lambda_2))) \end{aligned}$$

where the inequality in (A.13) follows from statement 1a.

Initial capital is the same along both paths, so (A.12) and (A.13) imply

that for small t , $c(t, k, \lambda_1)$ is strictly less than $c(t, k, \lambda_2)$. Then, for all t , $c(t, k, \lambda_1)$ is less than $c(t, k, \lambda_2)$. For suppose it were not. Then, by the continuity of the consumption path, there must exist t^* such that $c(t^*, k, \lambda_1) = c(t^*, k, \lambda_2)$ and:

$$(A.14) \quad \dot{u}'(c(t, k, \lambda_1)) \Big|_{t=t^*} < \dot{u}'(c(t, k, \lambda_2)) \Big|_{t=t^*} .$$

From (A.12) and (A.13), (A.14) implies that $V'(k(t^*, k, \lambda_1)) > V'(k(t^*, k, \lambda_2))$, thus, $k(t^*, k, \lambda_1) < k(t^*, k, \lambda_2)$. Since $c(0, k, \lambda)$ is increasing in k , we know that

$$(A.15) \quad \begin{aligned} c(0, k(t^*, k, \lambda_1), \lambda_1) &= c(t^*, k, \lambda_1) = c(t^*, k, \lambda_2) \\ &= c(0, k(t^*, k, \lambda_2), \lambda_2) \\ &> c(0, k(t^*, k, \lambda_1), \lambda_2) , \end{aligned}$$

in contradiction to the hypothesis that $k(t^*, k, \lambda_1) \in \tilde{B}$. Therefore, we have for all t , $c(t, k, \lambda_1) \leq c(t, k, \lambda_2)$, with strict inequality for some t . Thus, the consumption path $c(t, k, \lambda_1)$ is not efficient in the budget set defined by $\dot{k} = rk - c$, $k(0) = k_0$ and $k \geq 0$ for all t . It can be shown that this contradicts the transversality condition for the path $c(t, k, \lambda_1)$, so that this path is not optimal. Thus, an optimal path beginning at $k \in \tilde{B}$ cannot stay in \tilde{B} forever.

Suppose therefore, that the optimal path leaves \tilde{B} at some time. Then, there exists k^* on the boundary of \tilde{B} , such that the path $c(t, k, \lambda_1)$ leaves \tilde{B} at k^* . At k^* , we have $c(0, k^*, \lambda_1) = c(0, k^*, \lambda_2)$. Also,

(A.12) and (A.13) imply that:

$$(A.16) \quad \dot{u}'(c(t, k^*, \lambda_1)) > \dot{u}'(c(t, k^*, \lambda_2)) ,$$

and therefore

$$\dot{c}(t, k^*, \lambda_1) < \dot{c}(t, k^*, \lambda_2) ,$$

in contradiction to the assumption that the path leaves \tilde{B} at k^* .^{27/}

Since an optimal λ_1 -path beginning at $k \in \tilde{B}$ can neither stay in \tilde{B} , nor leave \tilde{B} , we conclude that \tilde{B} is empty.

Statement 2: a. If $r \geq \rho + \lambda$, then for all t , $\dot{c} > 0$, $\dot{k} > 0$ along the optimal path.

b. If $\rho + \lambda > r > \rho$, then for small initial capital, $\dot{c} < 0$, $\dot{k} < 0$. If in addition, $\lim_{c \rightarrow \infty} (u'(c+a)/u'(c)) = 1$, then for large

initial capital, $\dot{c} > 0$, $\dot{k} > 0$.

c. If $r \leq \rho$, then for all t , $\dot{c} < 0$, $\dot{k} < 0$.

Proof: Because of the concavity of the maximization, c is increasing in initial capital. Since the problem is stationary over time, this implies that \dot{c} and \dot{k} have the same sign. Parts a and c of the statement are immediate from the Euler equation and from statement 1a. The first

^{27/} Because $c(0, k^*, \lambda_1) = c(0, k^*, \lambda_2)$, at k^* , the time derivative of capital is the same along the λ_1 - and the λ_2 -path, so that the right hand side of (A.16) gives the time change of $u'(c(0, k(t, k^*, \lambda_1), \lambda_2))$ at $t=0$, as well as of $u'(c(0, k(t, k^*, \lambda_2), \lambda_2))$.

half of part b follows from the fact that $\lim_{c \rightarrow 0} u'(c) = \infty$, while $V'(0)$

is finite, because the positive certain income allows positive consumption.

Also, $\lim_{c \rightarrow \infty} (u'(c+a)/u'(c)) = 1$ implies that for large k ,

$$u'(rk+a) \geq u'(rk)(1-\delta),$$

where δ is arbitrarily small. For $\delta < (r-\rho)/\lambda$, this implies:

$$\lambda u'(rk+a) > u'(rk)(r-\rho-\lambda).$$

Since it is also true that $V'(k) > u'(rk+a)$, because the agent does save at the certain income a , the Euler equation cannot be satisfied for consumption in excess of income.

QED

Appendix 2: The Optimal Consumption Path with Borrowing

In this appendix, I discuss the optimal program with borrowing and derive Propositions 2, 3, 4 and 5. First, let T be given and consider the choice of the optimal consumption path and of the amount of debt at which the agent goes bankrupt. The set of feasible consumption paths is convex, because of the concavity of the interest schedule. As the maximand is concave over the set of feasible consumption paths, the optimal path $c(t; k, A, T)$ is unique. It satisfies:

$$(B.1) \quad \dot{u}' = - (R' - \rho - \lambda)u' - \lambda V'$$

$$(B.2) \quad e^{-(\rho+\lambda)T} (u'(c(T; k, \bar{A}, T)) + \partial B/\partial A) \geq 0 ,$$

where we take $\frac{\partial B}{\partial A} = 0$, by assumption.

Let $W(k, \bar{A}, T)$ be the value of the maximand (6) at the optimal consumption path, for given k, \bar{A}, T . That is to say:

$$(B.3) \quad W(k, \bar{A}, T) = \text{Max} \int_0^T e^{-(\rho+\lambda)t} (u(c(t)) + \lambda V(k(t))) dt + e^{-(\rho+\lambda)T} B$$

subject to: $k(0) = k, k(T) = A \leq \bar{A}$;

$$\dot{k}(t) = R(k(t)) - c(t)$$

For the policy without borrowing, we write:

$$(B.4) \quad W(k, \bar{A}, \infty) = \text{Max} \int_0^{\infty} e^{-(\rho+\lambda)t} (u(c(t)) + \lambda V(k(t))) dt$$

subject to: $k(0) = k$; for all t , $k(t) \geq 0$;

$$\dot{k}(t) = R(k(t)) - c(t) .$$

Lemma 1: For all $k, A, W(k, A, \cdot)$ is everywhere continuous in T .

Proof: This is elementary for finite T . So we prove: $\lim_{T \rightarrow \infty} W(k, \bar{A}, T)$

$= W(k, \bar{A}, \infty)$. Let $k^*(t, k)$ be the time path of capital for the optimal program in the absence of borrowing. The, for all T ,

$$W(k, \bar{A}, \infty) = W(k, -k^*(T, k), T) + e^{-(\rho+\lambda)T} (W(k^*(T, k), \bar{A}, \infty) - B)$$

Since $-k^*(T, k) \leq \bar{A}$, (B.2) implies that:

$$W(k, \bar{A}, \infty) \leq W(k, \bar{A}, T) + e^{-(\rho+\lambda)T} (W(k^*(T, k), \bar{A}, \infty) - B)$$

Taking limits as T becomes large,

$$(B.5) \quad W(k, \bar{A}, \infty) \leq \lim_{T \rightarrow \infty} W(k, \bar{A}, T),$$

by the boundedness of the utility function.

On the other hand, on the path leading into bankruptcy at T , let $t(\delta, T)$ be the time when indebtedness is δ . Then, since $R'(k) \geq r$, for $k \leq 0$, the budget constraint implies that

$$\delta e^{r(T-t(\delta, T))} \leq \bar{A},$$

or

$$t(\delta, T) \geq T - \frac{1}{r} \ln(\bar{A}/\delta).$$

Thus, $\lim_{T \rightarrow \infty} t(\delta, T) = \infty$.

Also, from the principle of optimality,

$$\begin{aligned} W(k, \bar{A}, T) &= W(k, \delta, t(\delta, T)) \\ &\quad + e^{-(\rho+\lambda)t(\delta, T)} (W(-\delta, \bar{A}, T-t(\delta, T)) - B) \end{aligned}$$

Taking limits as T goes to infinity,

$$\lim_{T \rightarrow \infty} W(k, A, T) = \lim_{T \rightarrow \infty} W(k, \delta, t(\delta, T)) = \lim_{T \rightarrow \infty} W(k, \delta, T)$$

Since this holds for arbitrarily small δ , we have

$$(B.6) \quad \lim_{T \rightarrow \infty} W(k, A, T) = \lim_{T \rightarrow \infty} W(k, 0, T) \leq W(k, 0, \infty),$$

where the latter inequality is true because $\lim_{T \rightarrow \infty} W(k, 0, T)$ is the value of an infinite horizon policy without borrowing. (B.5) and (B.6) together prove the lemma. QED

The same argument as in the proof of this lemma can be used to show that the optimal consumption path $c(t; k, A, T)$ converges pointwise to the path without borrowing, i.e., that for all t , $\lim_{T \rightarrow \infty} c(t; k, \bar{A}, T) = c(t; k, A, \infty)$. By the nature of the problem however, this convergence cannot be uniform. It is therefore not, in general, true that

$$\lim_{t \rightarrow \infty} c(t; k, A, T) = \lim_{t \rightarrow \infty} c(t; k, A, \infty), \text{ even if the latter limit exists.}$$

Lemma 2: For all T , $W(k, \bar{A}, T)$ is strictly concave in k, \bar{A} .

Proof: For any pair of values k_0, A_0, k_1, A_1 , with the optimal consumption paths $c(t; k_0, A_0, T), c(t; k_1, A_1, T)$, let $\lambda k_0 + (1-\lambda)k_1, \lambda A_0 + (1-\lambda)A_1$ be a convex combination. At these new values of the parameters, the convex combination of the optimal consumption paths, $\lambda c(t; k_0, A_0, T) + (1-\lambda)c(t; k_1, A_1, T)$ is feasible because of the concavity of the interest schedule. Since the maximand is strictly concave over the set of consumption paths, this proves the lemma. QED

By the envelope theorem, we can write:

$$(B.7) \quad W_1 = u'(c(0; k, \bar{A}, T))$$

$$(B.8) \quad W_2 = e^{-(\rho+\lambda)T} (u'(c(T; k, \bar{A}, T))) .$$

Lemma 3: For all $t, T, k, A, c(t; k, A, T)$ is increasing in k and \bar{A} and decreasing in T .

Proof: From lemma 2, W_1 is decreasing in k . Therefore, from (B.7), $c(0; k, \bar{A}, T)$ is increasing in k . Let t^* be the smallest value of t , for which, contrary to the lemma, $c(t; k, \bar{A}, T)$ is not increasing in k . For $t < t^*$, $c(t; k, \bar{A}, T)$ is increasing in k ; so the lefthand derivative $\dot{c}(t^*; k, \bar{A}, T)$ must be decreasing in k . Also, the left hand derivative $\dot{u}'(c(t^*; k, \bar{A}, T))$ increases in k . On the other hand, by the principle of optimality, $c(t^*; k, \bar{A}, T) = c(0; k(t^*, k), \bar{A}, T-t^*)$. If $c(t^*; k, \bar{A}, T)$ is not increasing in k , $k(t^*, k)$ is not increasing in k . But then, the right hand side of the Euler equation (B.1) is unchanged, contrary to the proposed increase in \dot{u}' at t^* . Therefore, the assumption that t^* exists implies a contradiction. Hence, $c(t; k, \bar{A}, T)$ is increasing in k for all t . As we increase \bar{A} , lemma 2 and (B.8) imply that $c(T; k, \bar{A}, T)$ is increased. Then, the same argument as before shows that the increase affects the whole path. As we increase T , say from T to $T + \Delta T$, we note that with respect to the original time interval up to T , this is equivalent to a decrease in the credit limit; by the end of T , the credit limit \bar{A} must not yet be exhausted. Then, the previous argument applies. QED

Lemma 4: The dependence of terminal consumption $c(T; k, \bar{A}, T)$ on T is given by:

$$(B.9) \quad \frac{dc(T; k, \bar{A}, T)}{dT} = \frac{dc(T; k, \bar{A}, T)}{dk} (R(k) - c(0; k, \bar{A}, T))$$

Proof: Consider the effect on terminal consumption of increasing the time to bankruptcy from T to $T + \Delta T$. By the principle of optimality, the optimal path for the length $T + \Delta T$ is optimal over the last T time units before bankruptcy. Over these last T time units of the path on the $T + \Delta T$ program, the agent may be viewed as solving a T -program for the same credit limit \bar{A} and the initial capital $k(\Delta T; k, \bar{A}, T + \Delta T)$. Hence, the effect on terminal consumption of increasing the time to bankruptcy from T to $T + \Delta T$ is the same as that of changing the initial capital from k to $k(\Delta T; k, \bar{A}, T + \Delta T)$. As we let ΔT go to zero,

$$\lim_{\Delta T \rightarrow 0} [(K(\Delta T; k, \bar{A}, T + \Delta T) - k) / \Delta T] = R(k) - c(0; k, \bar{A}, T)$$

This completes the proof of the lemma.

QED

Lemma 4 allows us to sign the effect of a change in the time to bankruptcy on terminal consumption. The time to bankruptcy affects terminal consumption both, in the usual way as a parameter to the optimal consumption path, and also, because it determines the time when terminal consumption takes place. The first effect is always negative, because as the time to bankruptcy is increased, the available financial means are spread out over a longer period. The second effect depends on the dynamics of the consumption path and can have either sign. Lemma 4 provides for a criterion as to which effect dominates. In fact, combining lemmas 3 and 4, we see that $dc(T; k, \bar{A}, T)/dT$ changes its sign at most once. For small T , $R(k) - c(0; k, \bar{A}, T)$ is negative; there is dissaving, for otherwise one does not have time to reach the

credit limit. For large T , there may be positive saving. Thus, $c(T; k, \bar{A}, T)$ is decreasing in T , when T is small and possibly increasing in T , when T is large.

To select the optimal time to bankruptcy, the agent sets:

$$(B.10) \quad W_3 = e^{-(\rho+\lambda)T} \{u(c(T; k, \bar{A}, T)) + \lambda V(-A) - (\rho+\lambda)B \\ - (c(T; k, \bar{A}, T) - R(-\bar{A}))u'(c(T; k, A, T))\} = 0 ,$$

if the optimal time to bankruptcy is finite. If it is optimal not to borrow, so that one does not go bankrupt in finite time, it must be the case that for all large T , W_3 is strictly positive.

The second order condition requires that at the optimum, W_3 is not increasing in T , so that $W_{33} < 0$. We notice that the sign of W_3 depends only on the term in brackets. This term depends on T only through terminal consumption $c(T; k, \bar{A}, T)$. At the point where $W_3 = 0$, we have:

(B.11)

$$W_{33} = - e^{-(\rho+\lambda)T} [C(T; k, \bar{A}, T) - R(-\bar{A})] u''(c(T; k, \bar{A}, T)) \frac{dc(T; k, A, T)}{dT}$$

Before proceeding to analyze (B.11), we note that the term in brackets is strictly increasing in $c(T; k, \bar{A}, T)$, so that there exists at most one value of terminal consumption, for which $W_3 = 0$. (Under our assumptions about u , one easily checks that there exists exactly one value of terminal consumption for which $W_3 = 0$). For future reference, we call this unique

value of terminal consumption at which $W_3 = 0$, c^{**} .

We combine (B.11) and (B.9) to see that at the point where $W_3 = 0$, W_{33} has the same sign as $R(k) - c(0; k, \bar{A}, T)$. Since $c(0, k, \bar{A}, T)$ is monotone decreasing in T , by lemma 3, it follows that there exist at most two finite solutions to equation (B.10), one for small T with $W_{33} < 0$, and one for large T with $W_{33} > 0$. Of these, the first one corresponds to a local maximum.

One also notes that as T approaches zero, $c(T; k, \bar{A}, T)$ grows out of bounds, so that for small T , one always has $W_3 > 0$.

The following classification exhausts all the possibilities for the dependence of W on T .

Case 1: There exists no finite solution to equation (B.10). W is strictly increasing in T . It approaches its global maximum as T grows out of bounds, (Figure 1).

Case 2: There exists one finite solution to equation (B.10).

a) The solution occurs where $R(k) - c(0; k, \bar{A}, T) = 0$; it is a saddle point. W is strictly increasing in T , except at the saddle point and approaches the global maximum as T grows out of bounds, (Figure 2a).

b) The solution occurs where $R(k) - c(0; k, \bar{A}, T) < 0$. It is the local and global maximum. As T grows out of bounds, W declines to some limiting value, (Figure 2b).

Case 3: There exist two finite solutions to equation (B.10). For small T , W increases to a local maximum, then decreases to a local minimum and finally increases to the value of the program without borrowing. The global maximum can be either at the local maximum at finite T (Figure 3a) or at the value of the program without borrowing (Figure 3b).

Figure 1

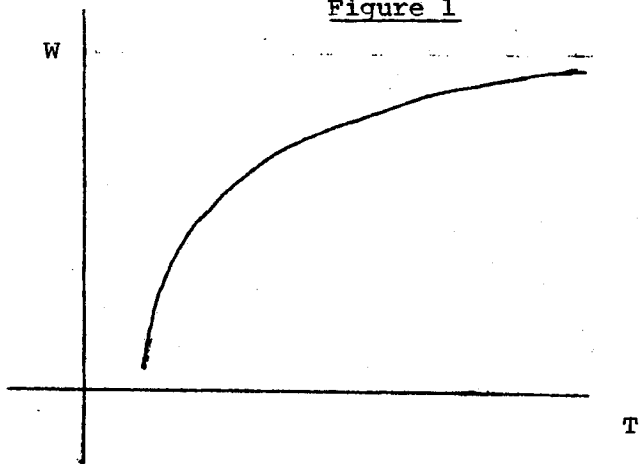


Figure 4

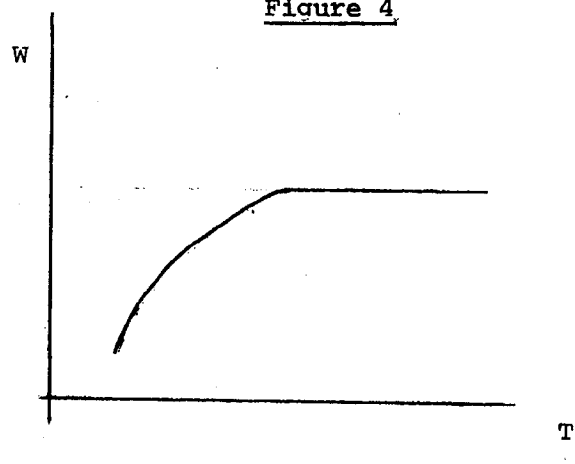


Figure 2a

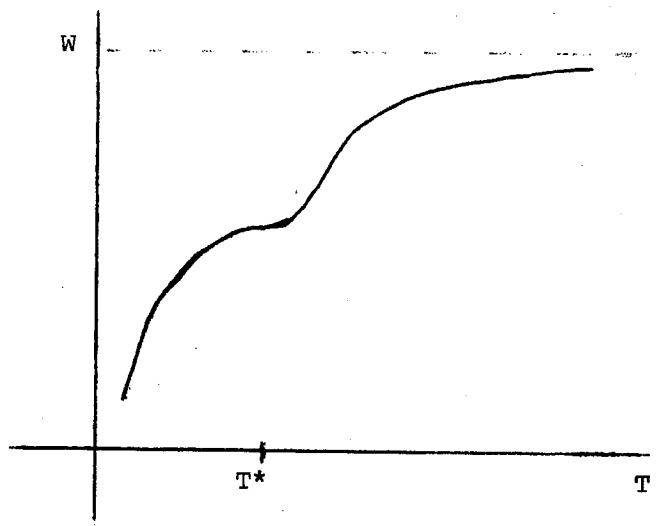


Figure 2b

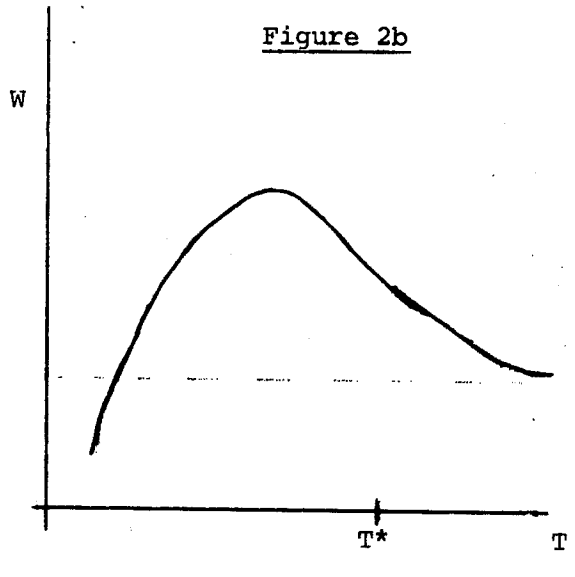


Figure 3a

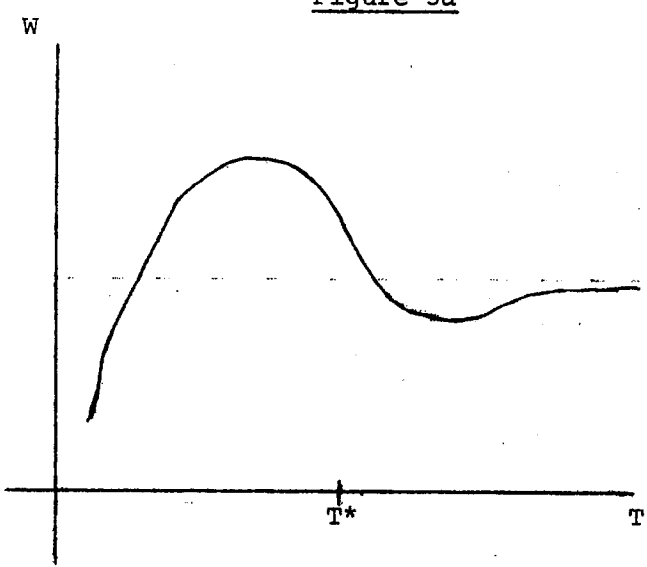
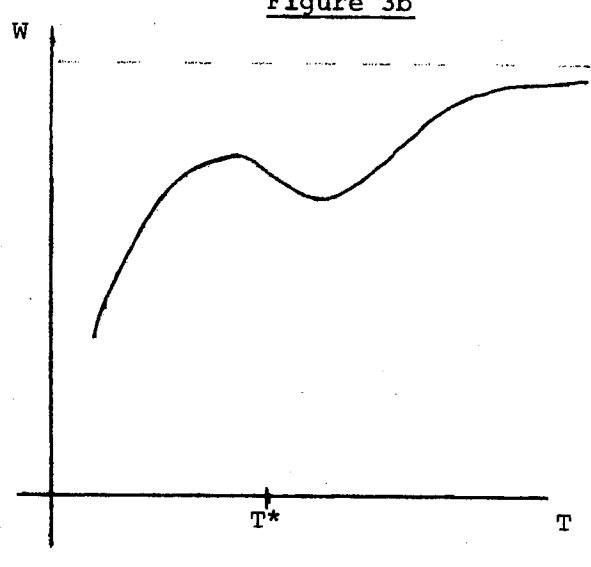


Figure 3b



It should be noted that W cannot be flat over some interval of values of T . From (B.11) and (B.9), this would require that $R(k) = c(0; k, \bar{A}, T)$ over some interval of values of T , in contradiction to lemma 3. This rules out the curve shown in Figure 4.

Proposition 2: If borrowing is desirable, the optimal time to bankruptcy is unique. It is strictly decreasing in the value of bankruptcy B and strictly increasing in the value of initial capital, k , and satisfies:

$$\frac{dT}{dk} = - \frac{1}{R(k) - c^*(0; k, \bar{A})} > 0$$

The optimal consumption path satisfies statement 1a of Proposition 1.

Proof: The uniqueness of the optimum among finite T is immediate from the preceding discussion. As B increases, W_3 decreases. At the optimum $W_{33} < 0$, so that a decrease in T is required to keep W_3 at zero. Terminal consumption has to be c^{**} , independent of initial capital. Therefore,

$$0 = dc(T; k, \bar{A}, T) = \frac{\partial c(T; k, \bar{A}, T)}{\partial T} dT + \frac{\partial c(T; k, \bar{A}, T)}{\partial k} dk$$

After substitution from (9), this gives

$$(R(k) - c(0; k, \bar{A}, T)) dT + dk = 0,$$

as was to be shown. From (B.11), (B.9) and the fact that W_{33} is negative, T is increasing in k .

Finally, we note that the argument in the proof of Proposition 1 that the optimal path can never leave the set B is true without change if T is

finite. It is therefore sufficient to prove that the path is in the set A , just before bankruptcy.

We have to show that c^{**} is less than consumption at $-\bar{A}$ with certain income a , $\bar{c}(0, -\bar{A})$. It is straightforward to show that the latter must satisfy:

$$(B12) \quad u(\bar{c}(0, -\bar{A})) - (\bar{c}(0, -\bar{A}) - R(-\bar{A}))u'(\bar{c}(0, -\bar{A})) = \rho V(-\bar{A}) .$$

By assumption III, the right hand side is greater than $(\rho+\lambda)B - \lambda V(-\bar{A})$. Comparing (B.12) and (B.10), it immediately follows that $\bar{c}(0, -\bar{A})$ is greater than c^{**} , so that statement 1a of Proposition 1 holds at $-\bar{A}$. QED

Proposition 3, Statement 3: If a program with borrowing is preferred to the optimal program without borrowing from a value of initial capital from which the program without borrowing prescribes a decumulation of capital towards zero, then the policy without borrowing is locally minimal.

Proof: By lemma 3, $R(k) - c(0; k, \bar{A}, T) \leq 0$ for all T . Therefore, we are in case 1 or case 2b of the above classification. In case 1, borrowing is not desired. In case 2b, borrowing is desired and the policy without borrowing is locally minimal. QED

Lemma 5: From an initial capital holding of zero, the agent plans to borrow and sets T at a finite value if and only if:

$$\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) < c^{**} .$$

Proof: $R(0) = 0$. Therefore, for all T , $R(0) - c(0; 0, \bar{A}, T) \leq 0$, with strict inequality for finite T , by lemma 3. From (B.9) and (B.11), there

is at most one finite solution to (B.10). We have either case 1 or case 2b of our classification. If $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) < c^{**}$, there exists $T^* < \infty$ such that $c(T^*, 0, \bar{A}, T^*) = c^{**}$ and T^* is a solution to (B.10). This is case 2b with the local and global maximum at T^* . On the other hand, if $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) \geq c^{**}$, then for all large but finite T , $c(T; 0, \bar{A}, T) > c^{**}$, so that for all large but finite T , $W_3 > 0$.

Proposition 5: Suppose that $R(-\bar{A}) > a$ and $R'(-\bar{A}) < \rho + \lambda$. Then, borrowing is desired from an initial capital of zero.

Proof: By Lemma 5, we have to show that $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) < c^{**}$.

From assumption I, c^{**} is strictly positive. We shall show that

$\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) = 0$. From the Euler equation (B.1), we have:

$$\dot{u}' < (\rho + \lambda)u'$$

and therefore,

$$(B.13) \quad \dot{u}(c(t; 0, \bar{A}, T)) < u'(c(0; 0, \bar{A}, T))e^{(\rho + \lambda)t}$$

Furthermore, we note that for all $k \geq -\bar{A}$, $R'(k) \leq R'(-\bar{A})$ and $V'(k) \leq V'(-\bar{A}) < \infty$. From the Euler equation (B.1), we also have:

$$\dot{u}' \geq (\rho + \lambda - R'(-\bar{A}))u' - \lambda V'(-\bar{A})$$

and therefore,

$$(B.14) \quad u'(c(t; 0, \bar{A}, T)) \geq u'(c(0; 0, \bar{A}, T)) e^{(\rho+\lambda-R'(-\bar{A}))t} - \frac{V'(-\bar{A})\lambda}{\rho+\lambda-R'(-\bar{A})} (e^{(\rho+\lambda-R'(-\bar{A}))t} - 1)$$

Suppose now that $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) > 0$. Then, $\lim_{T \rightarrow \infty} u'(c(T; 0, \bar{A}, T)) < \infty$.

From (B.14), and the fact that $R'(-\bar{A}) < \rho+\lambda$, this implies that

$$\lim_{T \rightarrow \infty} u'(c(0; 0, \bar{A}, T)) < \lambda V'(-\bar{A}) / (\rho+\lambda-R'(-\bar{A})),$$

so that $\lim_{T \rightarrow \infty} c(0; 0, \bar{A}, T) = v > 0$. From (B.13) one can then find a number

and a finite time s , such that for arbitrarily large T , indebtedness at the time s exceeds . This contradicts the argument in the proof of lemma 1. Hence, the assumption that $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) > 0$ leads to a

contradiction. Therefore, $\lim_{T \rightarrow \infty} c(T; 0, \bar{A}, T) = 0 < c^{**}$. QED

Appendix 3: The Returns to Lending in an Example of Borrower Behaviour

In this appendix, we formulate an example of borrower behaviour to illustrate the main points of our discussion of the returns from lending. We use the following generalization of a utility function introduced by Goldman (1974).

$$(C.1) \quad u(c) = \begin{cases} c^\beta / \beta & \text{for } c \leq \bar{c}, \text{ where } 0 < \beta < 1; \\ k_1 c^\gamma / \gamma + k_2 & \text{for } c \geq \bar{c}, \text{ where } \gamma < 0. \end{cases}$$

$$k_1 = \bar{c}^{\beta-\gamma}$$

$$k_2 = \bar{c}^{\beta} \left(\frac{1}{\beta} - \frac{1}{\gamma} \right)$$

Furthermore, we assume that $a = \infty$, so that $V'(k) = 0$ for all finite k . Also, for all $k < 0$, $R'(k) = z$, and:

$$(C.2) \quad \rho < r < z < \rho + \lambda .$$

The Euler equation is:

$$(C.3) \quad \begin{aligned} \dot{c}(t) &= -bc(t) && \text{for } c(t) \leq \bar{c} \\ \dot{c}(t) &= -gc(t) && \text{for } c(t) \geq \bar{c} \end{aligned}$$

where

$$b = \frac{\rho + \lambda - z}{1 - \beta} > 0$$

$$g = \frac{\rho + \lambda - z}{1 - \gamma} > 0 .$$

The value of bankruptcy is independent of the amount of debt at bankruptcy and satisfies assumption III. The first order condition for the time to bankruptcy is:

$$(C.4) \quad u(c^{**}) - (c^{**} + zA)u'(c^{**}) = u(0) - (\rho + \lambda)p ,$$

where we take $V(0) = V(-A)$ for all finite A .

Remark A1: If the credit limit is sufficiently small, the expected present value of returns to the creditor is positive.

Proof: The left hand side of (C.4) is not less than $u(0) - zAu'(c^{**})$, by concavity of $u(\cdot)$. As A becomes small, $zAu'(c^{**})$ cannot fall short of $(\rho + \lambda)p > 0$. (Proposition 5 ensures that for all A , borrowing is desirable.) Hence, $u'(c^{**})$ grows out of bounds, i.e., c^{**} goes to zero. For sufficiently small A , $c^{**} \leq \bar{c}$, and therefore:

$$\lim_{A \rightarrow 0} [Au'(c^{**})] = \lim_{A \rightarrow 0} [Ac^{**\beta-1}] = \lim_{A \rightarrow 0} [c^{**\beta} A/c^{**}] = (\rho + \lambda)p/z > 0$$

Therefore, $\lim_{A \rightarrow 0} c^{**\beta} = 0$ implies that A/c^{**} grows out of bounds as A approaches zero. Because of (C.3) it immediately follows that the time to bankruptcy goes to infinity as A goes to zero.

On the other hand, if the time to bankruptcy grows out of bounds, initial consumption $c(0)$ must go to zero, as A becomes small. Hence, for small enough A , $c(0) \leq \bar{c}$. Then, one has for small enough A ,

$$c(t) = c(0)e^{-bt}$$

and

$$k(t) = k(0)e^{zt} - \int_0^t c(\tau)e^{z(t-\tau)} d\tau$$

$$= -c(0) \frac{e^{zt} - e^{-bt}}{z+b}, \text{ where we take } k(0) = 0.$$

The expected present value of returns for the lender is:

$$EPV = - \int_0^T e^{-(r+\lambda)t} (c(t) + \lambda zk(t)/r) dt$$

$$= c(0) \left\{ \frac{\lambda z}{r(z+b)} \left[\frac{e^{(z-r-\lambda)T} - 1}{z-r-\lambda} - \frac{1 - e^{-(r+b+\lambda)T}}{r+b+\lambda} \right] - \frac{1 - e^{-(r+b+\lambda)T}}{r+b+\lambda} \right\}$$

Under condition (C.2), all the terms in T vanish as T grows out of bounds.

In the limit, we then have:

$$\begin{aligned} \lim_{A \rightarrow 0} \frac{EPV}{c(0)} &= \frac{\lambda z}{r(z+b)(r+\lambda-z)} - \frac{\lambda z}{r(z+b)(r+b+\lambda)} - \frac{1}{r+b+\lambda} \\ &= \frac{1}{r+b+\lambda} \frac{\lambda z}{r(r+\lambda-z)} - 1 \\ &= \frac{1}{r+b+\lambda} \frac{(z-r)(\lambda+r)}{r(r+\lambda-z)} > 0 \end{aligned}$$

It follows that for sufficiently small A , the expected present value of returns from lending is positive.

Remark A2: For given \bar{c} , if the credit limit that is initially announced is sufficiently large, then the naive cut-off criterion does not provide a bound on the total credit that is given.

Proof: If the credit limit that was originally announced is large enough, terminal consumption for this credit limit exceeds \bar{c} and satisfies:

$$(C.5) \quad \frac{1-\gamma}{\gamma} c^{**\gamma} - zAc^{**\gamma-1} = - (k_2 + (\rho+\lambda)p)k_1.$$

Under the naive criterion, credit is cut off only if $c^{**} \geq \lambda zA/r$ (condition (25) in the text). We show that the equation $c^{**} = \lambda zA/r$ has a unique solution and that for higher values of the credit limit, $c^{**} < \lambda zA/r$. Substituting $c^{**} = \lambda zA/r$ into the condition for c^{**} , one has:

$$c^{**\gamma} = \frac{k_2 + (\rho+\lambda)p}{k_1 [r/\lambda - (1-\gamma)/\gamma]}$$

and at this point, one has

$$\frac{dc^{**}}{dA} = \frac{1}{1-\gamma} \frac{z\lambda}{\lambda+r} < \frac{z\lambda}{r} ,$$

from differentiating (C.5) totally and substituting for $c^{**} = \lambda z A / r$. Hence, for A sufficiently large, $c^{**} < \lambda z A / r$, and the naive criterion does not provide for a finite credit limit.

Remark A3: Suppose that $\beta > \lambda / (\lambda + r)$. If \bar{c} is large enough, the naive criterion rules out small additional credits over a whole range of finite credit levels.

Proof: Suppose that $c^{**} < \bar{c}$. c^{**} satisfies:

$$(C.6) \quad \frac{1-\beta}{\beta} c^{**\beta} - z A c^{**\beta-1} = -(\rho+\lambda)p$$

For $\beta > \lambda / (\lambda + r)$, there exists a positive $c^{**} = \lambda z A / r$, satisfying (C.6), which is given as:

$$(C.7) \quad c^{**\beta} = \frac{(\rho+\lambda)p}{r/\lambda - (1-\beta)/\beta} .$$

At this point, we have, from total differentiation of (C.6),

$$\frac{dc^{**}}{dA} = \frac{\lambda z}{(1-\beta)(\beta+r)} > \frac{\lambda z}{r} ,$$

whenever $\beta > \lambda / (\lambda + r)$. Thus, there is a finite credit limit at which small additional credits are unprofitable by the naive criterion, provided that \bar{c} is large enough.

The general statement that at this credit limit, all additional credits are ruled out by the naive criterion is not true, because as the additional credit considered goes to infinity, c^{**} exceeds \bar{c} and (C.5) implies that c^{**} grows out of bounds, as well as A/c^{**} , so that the time to bankruptcy goes out of bounds and remark A1 implies that the expected present value of net returns must become large and positive. However, this will no longer be true, when one considers utility functions that differ in the region above \bar{c} .

From (C.3), we know that consumption is always decreasing over time. Therefore, if the naive criterion cuts credit off at the point indicated by (C.7), the expected present value of net returns from lending is negative, by remark 2 in the text.

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