

THE SYSTEM OF INEQUALITIES

$$a_{rs} > X_r - X_s$$

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PREFACE

In theory of economic index-numbers, as also in theory of a much wider class of scaling measurement problems, there has to be considered the solution of systems of simultaneous inequalities of the form

$$\lambda_r > 0, \lambda_r D_{rs} > \varphi_r - \varphi_s \quad (r, s = 1, \dots, n),$$

where numbers D_{rs} are supposed measured and given, and all the admissible number pairs (λ_r, φ_r) ($r = 1, \dots, n$) are to be found.

This involves consideration of systems of the form $a_{rs} > X_r - X_s$.

This paper is to develop the theory of such systems, for the independent interest, as well as for these necessary applications.

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TABLE OF CONTENTS

1. Open and closed systems
2. Chain coefficients
3. Minimal chains
4. Derived systems
5. Triangle inequality
6. Extension property of solutions
7. Consistency
8. Cycle reversibility
9. Median solutions
10. Simple systems

1. Open and closed systems

Let $n(n-1)$ numbers a_{rs} ($r \neq s; r, s = 1, \dots, n$) be given; and consider the system of simultaneous inequalities

$$S(a): a_{rs} > X_r - X_s \quad (r \neq s; r, s = 1, \dots, n)$$

defining the open system $S(a)$, of order n , with coefficients a_{rs} . Any set of n numbers X_r ($r = 1, \dots, n$), forming a vector X , which satisfy these inequalities, define a solution X of the system $S(a)$; and the system is said to be consistent if it has solutions.

With the open system $S(a)$, there may also be considered the closed system $\bar{S}(a)$, defined by

$$\bar{S}(a): a_{rs} \geq X_r - X_s \quad (r \neq s; r, s = 1, \dots, n).$$

Obviously, solutions of $S(a)$ are solutions of $\bar{S}(a)$, and the consistency of $S(a)$ implies the consistency of $\bar{S}(a)$, but not conversely.

2. Chain coefficients

Let r, l, m, \dots, p, s denote any chain, that is a sequence of elements taken from $1, \dots, n$ with every successive pair distinct. Now from the coefficients a_{rs} of a system there can be formed the chain coefficient $a_{rlm\dots ps}$, determined on any chain, by the definition

$$a_{rlm\dots ps} = a_{rl} + a_{lm} + \dots + a_{ps}.$$

Obviously

$$a_{r\dots s\dots t} = a_{r\dots s} + a_{s\dots t}.$$

Chains are considered associated with their coefficients, so that by a positive chain is meant one with positive coefficient, and so on similarly. A simple chain is one without loops, that is, one in which no element is repeated. There are $n(n-1) \dots (n-r+1) = \frac{n!}{r!}$ simple chain

of length $r \leq n$, and therefore altogether $n! (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{(n-1)!})$ simple chains.

A chain r, l, m, \dots, p, s whose extremities are the same, that is, with $r = s$, defines a cycle. A simple cycle is one without loops. There are $(n-1) \dots (n-r+1) = \frac{(n-1)!}{r!}$ simple cycles of $r \leq n$ elements, and the total number of simple cycles is made up accordingly. The coefficients $a_{rs} + a_{sr}$ on the cycles of two elements define the intervals of the system.

Any chain can be represented uniquely as a simple chain, with loops at certain of its elements, given by cycles through those elements; and the coefficient on it is then expressed as the sum of coefficients on the simple chain and on the cycles. Also, any cycle can be represented uniquely as a simple cycle, looping in simple cycles at certain of its elements, which loop in simple cycles at certain of their elements, and so forth, with termination in simple cycles. The coefficient on the cycle is then expressed as a sum of coefficients on simple cycles. Thus out of these generating elements of simple chains and cycles, finite in number, is formed the infinite set of all possible chains.

3. Minimal chains

THEOREM 1. For the chains to have a minimum it is necessary and sufficient that the cycles be non-negative.

If any cycle total should be negative, then by taking chains which loop repeatedly round that cycle, chains which have increasingly negative coefficients are obtained, without limit; and so no minimum exists. However, should every cycle coefficient be non-negative, then by cancelling the loops on any chain, there can be no increase in the coefficient, so no chain coefficient will be smaller than the coefficient for some simple chain. But there is only a finite number of simple chains on a finite

number of elements, and the coefficients on these have a minimum.

THEOREM 2: For the cycles to be non-negative it is necessary and sufficient that the simple cycles be non-negative.

For the coefficient on any cycle can be expressed as a sum of coefficients on simple cycles.

THEOREM 3: If the cycles are non-negative then a minimal chain with given extremities always exists and can be chosen to be simple.

For any chain is then not less than the chain obtained from it by cancelling loops, since the cancelling is then the subtraction of a sum of non-negative numbers.

4. Derived systems

According to Theorem 3.3, if the cycles of $S(a)$ are non-negative, that is

$$a_{rlm\dots ps} \geq 0$$

for every cycle r,l,m,\dots,p,s , or equivalently for every simple cycle, by Theorem 3.2, then the coefficients $a_{rlm\dots ps}$ on the chains with given extremities r,s have a minimum, and it is possible to define

$$A_{rs} = \min_{l,m,\dots,p} a_{rlm\dots ps} \quad (r,s=1,\dots,n)$$

Then

$$a_{rlm\dots ps} \geq A_{rs}$$

for every chain and, by Theorem 3.3, the equality is attained for some simple chain. In particular,

$$a_{rs} \geq A_{rs}$$

The number A_{rr} is the minimum coefficient for the cycle through r , so that

$$a_{rlm\dots pr} \geq A_{rr}$$

for every cycle, the equality being attained for some simple cycle. In particular, for a chain of two elements,

$$a_{rs} + a_{sr} \geq A_{rr}$$

The hypothesis of non-negative cycles now has the statement

$$A_{rr} \geq 0$$

The numbers A_{rs} ($r \neq s$), thus constructed from the coefficient of $S(a)$ define the coefficients of a system $S(A)$, which will be called the derived system of $S(a)$.

Any two systems will be said to be equivalent if any solution of one is also a solution of the other.

THEOREM 1: Any system and its derived system, when it exists, are equivalent.

Let a system $S(a)$ have a solution X . Then, for any chain of elements r, l, m, \dots, p, s there are the relations

$$a_{rl} > X_r - X_l, a_{lm} > X_l - X_m, \dots, a_{ps} > X_p - X_s$$

from which, by addition, there follows the relation

$$a_{rlm\dots ps} > X_r - X_s$$

This implies that the derived coefficients A_{rs} exist, and

$$A_{rs} > X_r - X_s$$

That is, X is a solution of $S(A)$.

Now suppose the derived coefficients A_{rs} of $S(a)$ are defined, in which case

$$a_{rs} \geq A_{rs},$$

and let X be any solution of $S(A)$, so that

$$A_{rs} > X_r - X_s.$$

Then it follows immediately that

$$a_{rs} > X_r - X_s$$

or that X is a solution of $S(a)$. Thus $S(a)$ and $S(A)$ have the same solutions, and are equivalent.

THEOREM 2: If the cycles of a system are non-negative or positive, then so correspondingly are the intervals of the derived system.

Since A_{rs} is the coefficient of some chain with extremities r, s it appears that $A_{rs} + A_{sr}$ is the coefficient of some cycle through r , and therefore if the cycles of $S(a)$ are non-negative, or positive, so correspondingly are the intervals $A_{rs} + A_{sr}$ of the derived system $S(A)$.

5. Triangle inequality

From the relation

$$a_{r\dots s} + a_{s\dots t} = a_{r\dots t}$$

it follows that the derived coefficients A_{rs} ($r \neq s$) satisfy the triangle inequality

$$A_{rs} + A_{st} \geq A_{rt}$$

the one side being the minimum for chains connecting r, t restricted to include s , and the other side being the minimum without this restriction.

THEOREM 1: Any system with non-negative cycles is equivalent to a system which satisfies the triangle inequality, given by its derived system.

This is in view of Theorem 3.1 and 4.1,2.

THEOREM 2: Any system which satisfies the triangle inequality has all its intervals non-negative.

Thus, from the triangle inequalities applied to any system $S(a)$,

$$a_{tr} + a_{rs} \geq a_{ts}, \quad a_{ts} + a_{sr} \geq a_{tr}$$

there follows, by addition, the relation

$$a_{rs} + a_{sr} \geq 0$$

THEOREM 3: If a system satisfies the triangle inequality, then its derived system exists, and, moreover, the two systems are identical.

From the triangle inequality, it follows by induction that

$$a_{r1} + a_{1m} + \dots + a_{ps} \geq a_{rs}$$

That is,

$$a_{rlm\dots ps} \geq a_{rs} ,$$

from which it appears that the derived system exists, with coefficients

$$A_{rs} \geq a_{rs} ,$$

so that now

$$A_{rs} = a_{rs}$$

This shows, what is otherwise evident, that no new system is obtained by repeating the operation of derivation, since the first derived system satisfies the triangle inequality.

6. Extension property of solutions

A subsystem $S_m(a)$ of order $m \leq n$ of a system $S(a)$ of order n is defined by

$$S_m(a): a_{rs} > X_r - X_s \quad (r,s=1,\dots,m) .$$

Then the systems $S_m(a)$ ($m = 2,3,\dots,n$) form a nested sequence of subsystems of $S(a)$, each being a subsystem of its successor; and

$$S_n(a) = S(a) .$$

Any solution (X_1, \dots, X_n) of $S(a)$ reduces to a solution (X_1, \dots, X_m) of the subsystem $S_m(a)$. But it is not generally true that any solution of a subsystem of $S(a)$ can be extended to a solution of $S(a)$. However, should this be the case, then the system $S(a)$ will be said to have the extension property.

THEOREM 4. Any closed system which satisfies the closed triangle

inequality has the extension property.

Let X_1, \dots, X_{m-1} be a solution of $\bar{S}_{m-1}(a)$,
so that

$$a_{rs} \geq X_r - X_s \quad (r, s=1, \dots, m-1) .$$

It will be shown that, under the hypothesis of the triangle inequality, it can be extended by an element X_m to a solution of $\bar{S}_m(a)$.

Thus, there is to be found a number X_m such that

$$a_{rm} \geq X_r - X_m, \quad a_{ms} \geq X_m - X_s \quad (r, s=1, \dots, m-1) ,$$

that is

$$a_{ms} + X_s \geq X_m \geq X_r - a_{rm} .$$

So the condition that such an X_m can be found is

$$a_{mq} + X_q \geq X_p - a_{pm} ,$$

where

$$X_p - a_{pm} = \max_r \{X_r - a_{rm}\}, \quad a_{mq} + X_q = \min_r \{a_{mq} + X_q\} .$$

But if $p = q$, this is equivalent to

$$a_{mq} + a_{qm} \geq 0 ,$$

which is verified, by Theorem 5.2, and if $p \neq q$, it is equivalent to

$$a_{pm} + a_{mq} \geq X_p - X_q$$

which is verified, since, by hypothesis

$$a_{pm} + a_{mq} \geq a_{pq}, \quad a_{pq} \geq X_p - X_q .$$

Therefore, under the hypothesis, the considered extension is always possible. It follows now by induction that any solution of $S_m(a)$ can be extended to a solution of $S_n(a) = S(a)$.

This theorem shows how solutions of any system can be practically constructed, step-by-step, by extending the solutions of subsystems of its derived system.

THEOREM 2: Any closed system which satisfies the closed triangle inequality is consistent.

For, by Theorem 5.2, $a_{12} + a_{21} \geq 0$; and this implies that the system

$$\bar{S}_2(a): a_{12} \geq X_1 - X_2, a_{21} \geq X_2 - X_1$$

has a solution, which, by Theorem 1, can be extended to a solution of $\bar{S}(a)$. Therefore $\bar{S}(a)$ has a solution, and is thus consistent.

THEOREM 3: Any open system which satisfies the triangle inequality and has positive intervals has the extension property, and is consistent.

The lines of proof follow those of Theorems 1 and 2.

A system is defined to satisfy the triangle equality if

$$a_{rs} + a_{st} = a_{rt}$$

THEOREM 4: If a system satisfies the triangle inequality and has null intervals then it also satisfies the triangle equality and has null cycles.

For, from

$$a_{rs} + a_{sr} = 0, a_{rs} + a_{st} \geq a_{rt}$$

follows also

$$a_{rs} + a_{st} < a_{rt}$$

so that

$$a_{rs} + a_{st} = a_{rt}$$

By induction,

$$a_{r1} + a_{1m} + \dots + a_{qp} = a_{rp}$$

and then

$$a_{r1} + a_{1m} + \dots + a_{pr} = 0,$$

that is, the cycles are null.

7. Consistency

THEOREM 1: A necessary and sufficient condition that an open system be consistent is that its cycles be positive.

If $S(a)$ is consistent, let X be a solution. Then, for any cycle r, l, m, \dots, p, r there are the relations

$$a_{rl} > X_r - X_l, a_{lm} > X_l - X_m, \dots, a_{pr} > X_p - X_r,$$

from which it follows, by addition, that

$$a_{rlm\dots pr} > 0.$$

Therefore, if $S(a)$ is consistent, all its cycles must be positive.

Conversely, let the cycles of $S(a)$ be positive. Then the derived system $S(A)$ is defined, satisfies the triangle inequality, and has positive intervals. Hence, by Theorem 6.3, $S(A)$ is consistent. But, by Theorem 4.1, $S(A)$ is equivalent to $S(a)$. Therefore, $S(a)$ is consistent.

Similarly:

THEOREM 2: A necessary and sufficient condition that a closed system be consistent is that its cycles be non-negative.

8. Cycle reversibility

A cycle is defined to be reversible in a system if the reverse cycle has the same coefficient, thus:

$$a_{rl\dots pr} = a_{rp\dots lr}$$

The condition of k-cycle reversibility for a system is that all cycles of k element be reversible with regard to it; and the general condition of cycle reversibility is the reversibility condition taken unrestrictedly, in respect to all cycles of any number of elements.

THEOREM 1: For the reversibility of cycles in a system, the reversibility of 3-cycles is necessary and sufficient.

The proof is by induction, by showing that, given 3-cycle reversibility, the k-cycle condition is implied by that for (k-1)-cycles .

Thus, from

$$a_{1\dots k} + a_{kl} = a_{k\dots 1} + a_{lk}$$

with

$$a_{ol} + a_{lk} + a_{ko} = a_{ok} + a_{kl} + a_{lo} ,$$

by addition, there follows

$$a_{ol} + a_{1\dots k} + a_{ko} = a_{ok} + a_{k\dots 1} + a_{lo} .$$

THEOREM 2: If a system has positive intervals and reversible cycles, then it is consistent.

Thus, if

$$a_{rs\dots pr} = a_{rp\dots sr}$$

and

$$a_{rs} + a_{sr} > 0 ,$$

then

$$\begin{aligned} 2a_{rs\dots pr} &= a_{rs\dots pr} + a_{rp\dots sr} \\ &= (a_{rs} + a_{sr}) + \dots + (a_{pr} + a_{rp}) > 0 , \end{aligned}$$

so the cycles are positive, and hence, by Theorem 7.1, the system is consistent.

For any system $S(a)$, define

$$C_{rs\dots t} = a_{rs\dots tr} - a_{rt\dots sr}$$

then $C_{rs\dots t}$ is an antisymmetric cyclic function of the indices r, s, \dots, t , depending just on the cyclic order of the indices and changing its sign when the cyclic order is reversed. The cycle reversibility

condition for the system now has the statement

$$C_{rs\dots t} = 0$$

and it has been shown to be necessary and sufficient just that

$$C_{rst} = 0 .$$

Thus the reversibility conditions are not all independent, but are implied by those for the 3-cycles. Moreover, not all the 3-cycle reversibility conditions are independent; but, as appears in the following theorem, the reversibility of three of the four 3-cycles in any four elements implies that for the fourth.

THEOREM 3: For any four elements $\alpha, \beta, \gamma, \delta$ there is the identity

$$C_{\beta\gamma\delta} + C_{\alpha\delta\gamma} + C_{\delta\alpha\beta} + C_{\gamma\beta\alpha} = 0$$

This can be verified directly.

By the dependencies shown in this Theorem, the $\frac{1}{6} n(n-1)(n-2)$ conditions for 3-cycle reversibility, contained in and implying a much larger set of general reversibility conditions, reduce to a set of $\frac{1}{2} (n-1)(n-2)$ independent conditions.

THEOREM 4: There are $\frac{1}{2} (n-1)(n-2)$ independent cycle reversibility conditions in a system of order n .

9. Median solutions

Any solution X of a system $S(a)$ must satisfy the condition

$$a_{rs} > X_r - X_s > -a_{sr} ,$$

that is, the differences $X_r - X_s$ must lie in the intervals $[-a_{sr}, a_{rs}]$, which are non-empty provided $a_{rs} + a_{sr} > 0$. In particular, a solution X such that these differences lie at the mid-points of these intervals

will be called a median of the system. Thus, if X is a median of $S(a)$ then $X_r - X_s = \frac{1}{2} (a_{rs} - a_{sr})$. The condition that a system admit a median is decidedly stronger than that of consistency alone.

THEOREM 1: A necessary and sufficient condition that any system $S(a)$ admit a median is that

$$a_{rs} + a_{st} + a_{tr} = a_{ts} + a_{sr} + a_{rt}, \quad a_{rs} + a_{sr} > 0.$$

The condition is necessary, since a median is a particular solution of the system, the existence of which implies that the intervals $a_{rs} + a_{sr}$ of the system are positive. Moreover, addition of the relations

$$X_r - X_s = \frac{1}{2} (a_{rs} - a_{sr})$$

$$X_s - X_t = \frac{1}{2} (a_{st} - a_{ts})$$

$$X_t - X_r = \frac{1}{2} (a_{tr} - a_{rt})$$

gives

$$0 = a_{rs} - a_{sr} + a_{st} - a_{ts} + a_{tr} - a_{rt}.$$

Also it is sufficient. For it provides that, for any k , and all r, s

$$a_{rs} - a_{sr} = (a_{rk} - a_{kr}) - (a_{sk} - a_{kr})$$

from which, it follows that the numbers

$$X_r = \frac{1}{2} (a_{rk} - a_{kr})$$

satisfy

$$X_r - X_s = \frac{1}{2} (a_{rs} - a_{sr});$$

and, then since

$$a_{rs} + a_{sr} > 0$$

they must be a solution of the system, which is, moreover, a median.

Now, combining with Theorem 8.1:

THEOREM 2: A necessary and sufficient condition that a system admit a median solution is that its intervals be positive and its cycles reversible.

9. Simple systems

A system $S(a)$ which is such that, for some k, s

$$a_{rk} + a_{kr} > 0, a_{rk} + a_{ks} \leq a_{rs}$$

for all r, s , will be called simple, with respect to the index k .

THEOREM : If $S(a)$ is simple, with respect to k , then it is consistent, and admits as solution all sets of number X_r such that

$$a_{rk} > X_r > -a_{kr} .$$

For then, from relations

$$a_{rk} > X_r, a_{ks} > -X_s ,$$

there follows

$$a_{rs} \geq a_{rk} + a_{ks} > X_r - X_s .$$