RESTRICTED REDUCED FORMS FROM SINGLE EQUATION METHODS: ASYMPTOTIC RELATIVE EFFICIENCY

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I. Introduction

Dhrymes (1973) has shown that the two-stage least-squares (TSLS) induced restricted reduced form estimators are not necessarily asymptotically efficient relative to the unrestricted reduced form estimators. Since the method of limited information maximum likelihood (LIML) yields estimators of the parameters of a single structural equation that are asymptotically equivalent to the TSLS estimators, one would expect that this would also be true of the LIML induced restricted reduced form estimators. However there is an appropriately defined restricted reduced form estimator of a single endogenous variable using the LIML estimator that has an asymptotic covariance matrix that differs from the covariance matrix of the unrestricted reduced form estimator by a positive semi-definite matrix. Furthermore it is shown that whether the TSLS restricted reduced form estimator is asymptotically better or worse than the unrestricted reduced form estimator of a single endogenous variable in a simplified model depends on the sign of an unknown scalar. So one either does better or worse on all the parameters of a single, reduced form equation by using the unrestricted reduced form estimator as opposed to the TSLS restricted reduced form estimator.

II. The Model

Consider the single structural equation,

1) $$y = Z\delta + \varepsilon = [Y \quad X_1] \begin{bmatrix} \gamma \\ \beta \end{bmatrix} + \varepsilon$$

where $\delta = \begin{pmatrix} \gamma \\ \beta \end{pmatrix}$ is an unknown parameter vector, and Y is correlated with ε. This single equation is part of a larger system of equations which has the reduced form

2) $$[y \quad Y] = X[\pi \quad \Pi] + [v \quad V]$$

where $X = [X_1 \quad X_2]$, y is $n\times 1$, Y is $n\times G$, X_1 is $n\times K_1$, X_2 is $n\times K_2$, π is $K\times 1$, $K_1+K_2 = K$, Π is $K\times G$, γ is $G\times 1$, β is $K_1\times 1$, and δ is $(G+K_1)\times 1$.

If we define

3) $$A = \begin{bmatrix} \Pi & \begin{matrix} I_{K_1\times K_1} \\ O_{K_2\times K_1} \end{matrix} \end{bmatrix}$$

the relationship between structural coefficients and reduced form coefficients is

4) $$A\delta = \pi .$$

It is also assumed that the error vector ε has a scalar covariance matrix $\sigma^2 I$, and the rows of the reduced form error matrix $[v \quad V]$ are independently distributed, each row having mean zero and covariance matrix given by

5)
$$\bar{\Omega} = \begin{bmatrix} \omega^2 & \rho \\ \rho' & \Omega \end{bmatrix}$$

where ω^2 is the variance of each element of v, Ω is the covariance matrix for each row of V, and ρ is the row vector of covariances between an element of v and the corresponding row of V. In deriving the reduced form for y,

6)
$$y = X\Pi\gamma + X_1\beta + \varepsilon + V\gamma$$

one obtains $v = \varepsilon + V\gamma$, which gives us

7)
$$\omega^2 = \sigma^2 + \gamma'\Omega\gamma + 2\sigma_1\gamma$$

where σ_1 is the row vector of covariances between an element of ε and the corresponding row of V.

If we estimate π by ordinary least squares,

8)
$$p = (X'X)^{-1}X'y$$

and if $\lim_{n\to\infty} \frac{X'X}{n} = M$ nonsingular, then

9)
$$\sqrt{n}\,(p-\pi) \overset{\ell}{\to} N(0,\omega^2 M^{-1})$$

where $\overset{\ell}{\to}$ means converges in distribution.

III. TSLS induced restricted reduced form estimator

If we are only interested in estimating the reduced form for y another way to estimate π would be to estimate δ by TSLS, estimate A by ordinary least squares,

$$\hat{A} = \begin{bmatrix} P & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix}$$

where

10) $$P = (X'X)^{-1}X'Y \quad \text{and}$$

11) $$\hat{\delta} = (Z'X(X'X)^{-1}X'Z)^{-1}Z'X(X'X)^{-1}X'y$$

and then estimate π by

12) $$\hat{\pi} = \hat{A}\hat{\delta} \ .$$

This estimator $\hat{\pi}$ would be the Dhrymes (1973) TSLS induced restricted reduced form estimator for the system defined only by equations 1 and 2. The TSLS estimator of Π when Y is already given in reduced form with no restrictions is the ordinary least squares estimator. If we let Π_i be the i^{th} row of Π, $i = 1,K$, and let P_i be the i^{th} row of P then

13) $$\hat{\pi} - \pi = (\hat{A} \qquad I_{K\times K} \otimes \gamma') \begin{bmatrix} \hat{\delta} - \delta \\ P_1' - \Pi_1' \\ \cdot \\ \cdot \\ \cdot \\ P_K' - \Pi_K' \end{bmatrix}$$

and $\sqrt{n}\,(\hat{\pi} - \pi) \overset{\ell}{\rightarrow} (A \qquad I \otimes \gamma')Z$

where Z is the limiting distribution of

14) $$\sqrt{n} \begin{bmatrix} \hat{\delta} - \delta \\ P_1' - \Pi_1' \\ \\ P_K' - \Pi_K' \end{bmatrix} .$$

Under the usual assumptions, Z is distributed as a normal random variable with mean 0 and covariance matrix given by (Theil p. 499)

15) $$\begin{bmatrix} \sigma^2 (A'MA)^{-1} & (A'MA)^{-1}A' \otimes \sigma_1 \\ A(A'MA)^{-1} \otimes \sigma_1' & M^{-1} \otimes \Omega \end{bmatrix} .$$

The covariance matrix of the limiting distribution of

16) $\sqrt{n}\,(\hat{\pi} - \pi)$ is then

17) $$\sigma^2 A(A'MA)^{-1}A' + 2A(A'MA)^{-1}A'\sigma_1\gamma + \gamma'\Omega\gamma M^{-1} \quad .$$

Subtracting 17 from the covariance matrix in expression 9,

$$\sigma^2 M^{-1} + \gamma'\Omega\gamma M^{-1} + 2\sigma_1\gamma M^{-1} - \sigma^2 A(A'MA)^{-1}A' - 2\sigma_1\gamma A(A'MA)^{-1}A'$$

$$- \gamma'\Omega\gamma M^{-1}$$

$$= \sigma^2 (M^{-1} - A(A'MA)^{-1}A') + 2\sigma_1\gamma(M^{-1} - A(A'MA)^{-1}A') \quad .$$

The matrix $M^{-1} - A(A'MA)^{-1}A'$ is positive semi-definite so the whole expression is positive semi-definite if

18) $$\sigma^2 + 2\sigma_1\gamma \geq 0$$

or if by 7

19) $$\omega^2 - \gamma'\Omega\gamma \geq 0 \quad .$$

Since $\sigma^2 + 2\sigma_1\gamma$ can be either positive or negative, the TSLS induced restricted reduced form estimators are not necessarily asymptotically efficient relative to the unrestricted reduced form. However, for estimating π either $\hat{\pi}$ or p is asymptotically more efficient (depending on the sign of expression 18).

Even though the parameters of the simplified model are related by $A\delta = \pi$, the method of TSLS does not use the prior identifying restrictions to obtain better estimates of Π. This is not true of the method of LIML as will be shown in the next section.

IV. LIML induced restricted reduced form estimator

If $\tilde{\delta}$ is the LIML estimator of δ, then we could also estimate π by

$$20) \qquad \tilde{\pi} = \hat{A}\tilde{\delta} .$$

Since again for the simple model defined only by equations 1 and 2, the LIML estimator of A is $\hat{A}$, this would be the Dhrymes (1973) LIML induced restricted reduced form estimator of π. This estimator would have the same asymptotic distribution as $\hat{\pi}$. However when calculating $\tilde{\delta}$, one also estimates π by an estimator π^* which is not equal to $\tilde{\pi}$. This estimator π^* could also be considered an induced restricted reduced form estimator of π. The estimator π^* is asymptotically efficient relative to p.

The LIML estimator $\tilde{\delta}$ can be considered as the solution to the extremal problem (Sant 1975)

$$21) \qquad \min_{h} (a - h)(M^{-1} \otimes S)(a - h)'$$

where

22) $a = (p' \quad P_1 \cdot \cdot \cdot P_K)$

23) $S = n^{-1}[(y \quad Y) - (Xp \quad XP)]'[(y \quad Y) - (Xp \quad XP)]$ and

24) $h = (\delta' A' \quad \Pi_1 \cdot \cdot \cdot \Pi_K)$.

The estimator π^* is then

25) $\pi^* = \tilde{A}\tilde{\delta}$.

In estimating δ, LIML also estimates π and Π in such a way that $\tilde{A} \neq \hat{A}$ where

$$\tilde{A} = \begin{bmatrix} \tilde{\Pi} & \begin{matrix} I \\ 0 \end{matrix} \end{bmatrix}$$

and $\tilde{\Pi}$ is formed by using the minimizing values $\tilde{\Pi}_i$ of expression 21.

The asymptotic distribution of

26) $\sqrt{n}\ (\pi^* - \pi)$

is then normal with mean 0 and covariance matrix

27) $\omega^2 A(A'MA)^{-1}A'$

The asymptotic normality follows from Ferguson (1958).[1/] The asymptotic covariance matrix of

28)
$$\sqrt{n}\begin{pmatrix}\pi^* - \pi \\ \tilde{\Pi}_1' - \Pi_1' \\ \\ \tilde{\Pi}_K' - \Pi_K'\end{pmatrix}$$

is given by

29)
$$H(H'(M \otimes \bar{\Omega}^{-1})H)^{-1}H'$$

where

30)
$$H = \left[\begin{pmatrix}1 & \gamma' \\ & \\ 0_{G\times 1} & I_{G\times G}\end{pmatrix} \otimes I_K\right]\begin{bmatrix}A & 0 \\ & \\ 0 & I_{G\times G} \otimes I_{K\times K}\end{bmatrix}$$

Evaluating 29 for the covariance matrix of $\sqrt{n}\ (\pi^* - \pi)$ gives equation 27.

Subtracting 27 from the covariance matrix in expression 9 gives

31)
$$\omega^2(M^{-1} - A(A'MA)^{-1}A')$$

which is a positive semi-definite matrix. Even if we are given a complete system of structural equations the estimator π^* for each reduced form equation would be asymptotically better equation by equation than the unrestricted reduced form estimator. We would not want to estimate each structural equation by a limited information technique and then solve for the reduced form, but using only the identifying restrictions of a single equation as π^* does, is enough information to yield asymptotically more efficient estimators of the reduced form of a single endogenous variable than the unrestricted reduced form estimator.

V. Conclusion

When one is interested in the reduced form of a single variable, the use of the identifying restrictions by utilizing the TSLS estimator does not necessarily lead to an estimator that is asymptotically efficient relative to the unrestricted reduced form estimator. However, the LIML estimator uses both sample information and prior information in such a way that it obtains an estimator of the reduced form that is asymptotically more efficient than the unrestricted reduced form.

The LIML induced restricted reduced form estimator presented in this paper is different than the one implied by Dhrymes but it is a natural estimator to consider. It has the desirable asymptotic properties. Even when the structural parameters of more than one equation has been estimated, π^* will be a better estimator of a single reduced form equation than will those estimators which incorporate the other structural parameters as suggested by Dhrymes.

Footnote

[1]/See also Rothenberg (1973 p. 69) for a derivation of the asymptotic covariance matrix of the minimum Chi-square estimator.

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