TWO METHODS FOR EXAMINING THE STABILITY OF REGRESSION COEFFICIENTS

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Abstract: This paper investigates the power of two methodologies, the tests of Brown, Durbin and Evans (1968) and variable parameter regression, to detect several varieties of instability in the coefficients of a linear regression model. The stability study reported by Khan (1974) is replicated with variable parameter regression, and his results are in part rejected and in part sharpened.

In pursuing quantitative research, econometricians are frequently able to appeal to theoretical arguments to justify a choice for the functional form of an economic model. Rarely, however, are they able to justify a maintained hypothesis that the coefficients of a model are stable over a sample interval, for example, that the marginal propensity to consume was the same in the Great Depression as in the 1960s. Stability of estimated coefficients is an empirical question that has frequently been ignored. Among the early works in this field are Quandt (1958) on estimating the location of a shift from one regression scheme to another, and Quandt (1960) and Chow (1960) on testing the null hypothesis of stability in regression coefficients against the alternative hypothesis of a shift at a particular point in time. Recently Brown, Durbin and Evans (1968) (hereafter BDE) proposed two more general methods of testing for the statistical significance of structural variation of regression coefficients, and Khan (1974) applied one of those methods to test the stability of the demand for money in the United States.

The purpose of this paper is to investigate, using Monte Carlo simulation, the power of the BDE tests to reject a false null hypothesis of stable coefficients under a variety of regimes of coefficient variation. We compare

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the results of the BDE tests to those obtained from an application of variable parameter regression (VPR) (Athans, 1974; Sarris, 1974; and Rosenberg, 1973), which can be used to measure the magnitude, as well as to test the significance, of coefficient variation. The first section describes variable parameter regression as it is applied in this paper. The second section reviews the BDE tests and shows how they are related to a special case of the VPR model. The third section reports on Monte Carlo simulation experiments with the two techniques. We find that in some circumstances, which seem likely to arise in economic studies, the BDE tests are unable to detect coefficient variation, but that VPR does quite well. In the last section we replicate Khan's (1974) results using VPR. We are led to reject the hypothesis of stability in the demand for money in some cases, which Khan could not, and to sharpen his evidence for stability in others.

1. Variable Parameter Regression

Variable parameter regression assumes the researcher has observed T ordered scalar observations $(y_1,\ldots, y_T)$ generated by the model:

$$y_t = x_t\beta_t + u_t \qquad (1a)$$

$$u_t \sim N(0, \sigma^2) \qquad (1b)$$

where x_t is an n-dimensional row vector of known exogenous variables and β_t is an n-dimensional vector of unknown coefficients. The stochastic scalar u_t is drawn from a Gaussian distribution. The dynamic evolution of the β vector is assumed to follow a random walk with zero drift through time:

$$\beta_t = \beta_{t-1} + p_t \tag{2a}$$

$$p_t \sim N(0, \sigma^2 P) \tag{2b}$$

where $\sigma^2 P$ is the stationary covariance matrix of the innovation p_t. The normalizing variance term σ^2 of (2b) is the same as that which appears in (1b) and is shown explicitly for expositional convenience. The innovations u_t and p_t are mutually and serially uncorrelated. Although more complex models of the dynamic evolution of the β's are possible (Rosenberg, 1973; Sarris, 1974) we retain the simple form of (2) here. We are not concerned with estimating a particular stochastic process generating the β's, but rather with the stability of the β's and with the pattern of any indicated variation. If a researcher is led to reject the hypothesis of stable coefficients he or she might wish to consider structural explanations of the β's as functions of additional exogenous variables, for example, the marginal propensity to consume could be a function of wealth or consumer sentiment, rather than develop more complicated models of stochastically evolving coefficients. Our results in Section 3 indicate that the simple model of (2) will be adequate in most cases.

Table 1 summarizes the algorithm for estimating the coefficients of model (1,2). Let $\hat{\beta}_t(s)$ be the estimate of β_t based on the observations $(y_1, \ldots, y_s)$. We exhibit in Table 1 the one-step-ahead predicted coefficient estimate $\hat{\beta}_t(t-1)$, with estimation covariance matrix $\sigma^2 R_t$, the updated estimate $\hat{\beta}_t(t)$, with covariance matrix $\sigma^2 S_t$, and the smoothed estimate $\hat{\beta}_t(T)$, with covariance matrix $\sigma^2 V_t$. These results are developed at length in Sage and Melsa (1971). The scalar stochastic innovation e_t driving the updating algorithm is a zero mean, serially uncorrelated Gaussian process

TABLE 1

VARIABLE PARAMETER REGRESSION ALGORITHMS

1. Prediction

$$\hat{\beta}_t(t-1) = \hat{\beta}_{t-1}(t-1)$$

$$R_t = S_{t-1} + P$$

2. Updating

$$\hat{\beta}_t(t) = \hat{\beta}_t(t-1) + K_t(y_t - x_t\hat{\beta}_t(t-1))$$

$$\text{where } K_t = R_t x_t'[x_t R_t x_t' + 1]^{-1}$$

$$S_t = R_t - K_t x_t R_t$$

3. Smoothing

$$\hat{\beta}_t(T) = \hat{\beta}_t(t) + G_t(\hat{\beta}_{t+1}(T) - \hat{\beta}_t(t))$$

$$\text{where } G_t = S_t[S_t + P]^{-1}$$

$$V_t = S_t + G_t[V_{t+1} - R_{t+1}]G_t'$$

$$V_T = S_T$$

4. Innovations

$$e_t = y_t - x_t\hat{\beta}_t(t-1)$$

$$E_t = x_t R_t x_t' + 1$$

5. Initialization

$$\hat{\beta}_{n+1}(n) = M\ \{\sum_{t=1}^{n} H_t' x_t' e_t^* / E_t^*\}$$

$$R_{n+1} = S_n^* + P + H_{n+1} M H_{n+1}'$$

TABLE 1
(continued)

where $M = \{ \sum_{t=1}^{n} H_t' x_t' x_t H_t / E_t^* \}^{-1}$

$$H_t = (I_n - K_{t-1} x_{t-1}) H_{t-1}$$

$$H_1 = I_n$$

and e_t^*, E_t^* and S_t^* are computed as in 1, 2 and 4 for $\hat{\beta}_1(0) = 0$, $R_1 = 0$.

6. Log-likelihood Function

$$L = -\tfrac{1}{2} T \ln(2\pi) - \tfrac{1}{2} \sum_{t=n+1}^{T} \ln(\sigma^2 E_t) - \tfrac{1}{2} \sum_{t=n+1}^{T} (e^2/\sigma^2 E_t)$$

with variance $\sigma^2 E_t$ whose properties are derived in Mehra (1970). The first n observations on y_t are used to compute the prior estimate $\hat{\beta}_{n+1}(n)$ and estimation covariance matrix $\sigma^2 R_{n+1}$ that initializes the recursive algorithm of steps 1 and 2. This method of solving the initialization problem is described in Garbade (1975).

All the elements of the algorithm in Table 1 are known to the researcher except the variance σ^2 and the relative covariance matrix P of the coefficient dynamics, equation (2). These terms are estimated by maximizing the log-likelihood function exhibited in Table 1. The expression of the likelihood function is due to Schweppe (1965) and is also considered in Mehra (1971). An estimate of σ^2, for given P, is available by analytic maximization:

$$\hat{\sigma}^2 = \sum_{t=n+1}^{T} \{e_t^2/(T-n)E_t\} \tag{3}$$

The concentrated log-likelihood function, to within a constant, is then:

$$L^*(P) = -(T-n)\ln(\hat{\sigma}) - \tfrac{1}{2} \sum_{t=n+1}^{T} \ln(E_t) \tag{4}$$

where $\hat{\sigma}$ and E_t are both implicit functions of P. L* is thus a complicated non-linear function of P. Numerical methods must be used to maximize L* as a function of the unknown elements in the covariance matrix P.

Testing the null hypothesis of no coefficient variation is accomplished by computing the test statistic:

$$-2[\ln(\lambda)] = -2[L^*(0) - L^*(\hat{P})] \tag{5}$$

where $\hat{P}$ is the maximum likelihood estimate of P. When P = 0 the smoothed estimates $\beta_t(T)$ of Table 1 can be shown to be identical for all t and equal to the ordinary least squares estimates $(X'X)^{-1}X'y$ where $X' = (x_1', \ldots, x_T')$ and $y' = (y_1, \ldots, y_T)$. In this paper we consistently maintain the hypothesis that P is a diagonal matrix, so the statistic of (5) is compared to a chi-square distribution with n degrees of freedom.

One of the advantages of using VPR to investigate structural stability in a regression model is that the significance of variation in individual coefficients as well as the significance of total variation may be examined. Suppose the space of admissible estimates of P is restricted to the positive half-line, emanating from the origin, corresponding to the variance of the k^{th} coefficient. The test statistic $-2[L^*(0) - L^*(\hat{P}_{restricted})]$ could then be compared to a chi-square distribution with one degree of freedom to test whether the null hypothesis of stability for that coefficient can be rejected in the presence of a maintained hypothesis of stability for all the other coefficients. Additional hypotheses may be tested by suitably restricting the space of admissible estimates of P.

In addition to formal hypothesis testing, VPR also provides the researcher with a graphical presentation of the estimated evolving coefficients based on the full set of observations, i.e. with the $\hat{\beta}_t(T)$'s of Table 1, and with standard errors for those estimates, the square roots of the diagonal elements of $\sigma^2 V_t$. These are useful in deciding whether the variation of the coefficients is quantitatively important as well as statistically significant. They may also provide useful guidance if the coefficients were shifting over part of the sample interval but were stable over the remainder. One of the simulations reported in Section 3 will address the question of the usefulness of VPR when the coefficients shift infrequently.

2. The Brown, Durbin and Evans Tests

In this section we review the BDE tests. Both tests begin with the computation of a series of recursive residuals. These residuals are closely related to the e_t innovations in the VPR algorithm of Table 1. Consider again model (1). Let X_t' be the n by t matrix $(x_1',\ldots,x_t')$ and let Y_t' be the n-dimensional row vector $(y_1,\ldots,y_t)$ for $n \leq t \leq T$. Under the null hypothesis that the coefficients in model (1) are stable the least-squares estimate of β based on observations up to time t is $b_t = (X_t'X_t)^{-1}X_t'Y_t$ with estimation covariance matrix $\sigma^2(X_t'X_t)^{-1}$. BDE define the recursive scalar-valued residual w_t:

$$w_t = \frac{y_t - x_t b_{t-1}}{[1 + x_t(X_{t-1}'X_{t-1})^{-1}x_t']^{\frac{1}{2}}} \qquad t=n+1,\ldots,T \tag{6}$$

and show that w_t is a serially independent Gaussian process with mean zero and variance σ^2.

Actually, the recursive residuals of BDE are exactly the e_t innovations of Table 1 (divided by a scaling factor to make their variance equal to σ^2) when the hypothesis of P=0, i.e., stable coefficients, is maintained. Moreover, the recursive algorithm suggested by BDE to compute efficiently the b_t's (exhibited on page 1209 of Khan (1974)) is the prediction and updating components of VPR when P is set to zero. Thus the BDE tests and the VPR algorithms are closely related under a true null hypothesis of P=0.

Brown, Durbin and Evans suggest two tests of the stability of the regression coefficients, both of which use the recursive residuals. In the "cusum" test the cumulative sum series:

$$W_t = \sum_{s=n+1}^{t} w_s/\hat{\sigma} \qquad t = n+1,\ldots,T \qquad (7)$$

$$\hat{\sigma}^2 = (Y_T - X_T b_T)'(Y_T - X_T b_T)/(T-n)$$

is computed. Under the null hypothesis of stability W_t is approximately normally distributed with mean zero and variance t-n. Thus if $|W_t| > 2(t-n)^{\frac{1}{2}}$ one is led to reject the null hypothesis at about a 5% confidence level. In the "cusum of squares" test the cumulative sum of squares series:

$$S_t = \sum_{s=n+1}^{t} w_s^2 \Big/ \sum_{s=n+1}^{T} w_s^2 \qquad t = n+1,\ldots,T \qquad (8)$$

is computed. S_t is a monotonically increasing sequence of positive numbers with $S_T=1$. Under the null hypothesis of stability $1-S_t$ has a beta distribution[1] with parameters $\alpha = -1 + (T-n)/2$ and $\beta = -1 + (t-n)/2$ and therefore S_t has mean value $(t-n)/(T-n)$. BDE suggest constructing a confidence interval for S_t as $[(t-n)/(T-n)] \pm c_o$, where c_o is chosen from Table 1 of Durbin (1969) using $\alpha = .5$ times the confidence level desired and $n = \frac{1}{2}(T-n)-1$ in that table. If $|S_t - (t-n)/(T-n)| > c_o$ the null hypothesis is rejected.

Brown, Durbin and Evans have observed that their methodology does not provide a parametric test of stability, as is the case with VPR, but rather a graphical presentation of departures from constancy. In borderline cases this can lead to problems of interpretation. For example, is the hypothesis of stability rejected if the 5% confidence interval is violated in only one period for a sample set of 20 observations? Is it rejected if the test statistics lie outside the interval for three successive observations, bearing in mind that W_t and S_t are serially correlated processes? We will see below that possibly only in extreme cases of coefficient shifts are the BDE tests unambiguous.

3. Simulation Results

In this section we present Monte Carlo simulation results obtained from applying VPR and the BDE tests to four different patterns of variation in the coefficients of a linear regression model. The patterns were chosen to provide one where the BDE tests are expected to be quite powerful, one where VPR is appropriate, one where neither is appropriate, and one where the null hypothesis of stability is true. The objective of these simulations is to get some idea of the relative merits of the two methodologies under various regimes of coefficient variation. We find both methods are quite powerful in rejecting the hypothesis of stability when there is a discrete shift in the coefficients. If, however, the coefficients are changing gradually from period to period, the BDE tests are extremely weak in detecting those changes. Two classes of gradual variation may be distinguished. When the effect of incremental changes in the coefficients exhibit persistence (defined explicitly below) we find VPR is capable of detecting and estimating those changes. When the effects of the incremental changes are not persistent, however, VPR does not do very well. The power of the BDE tests is extremely weak in either case. These results are significant to econometric investigations since a scheme of gradually evolving coefficients is more likely to characterize the pattern of instability in many models than a pattern of a discrete shift in the coefficients at a point in time.

In each simulation described in this section we set $T = 103$ and $n = 2$.

The model is:

$$y_t = \beta_{1,t} + \beta_{2,t}x_t + u_t \tag{9a}$$

$$u_t \sim N(0,1) \tag{9b}$$

where x_t is a scalar. In each replication each of the 103 values of x_t were chosen randomly from a uniform distribution over the range (-5., 5.). Because the x's are reselected in each replication our results do not depend on any particular drawing or pattern, and may be interpreted as pertaining to a well-dispersed sequence of observations on the exogenous data series. The cases of coefficient behavior studied were:

Case I: Discrete Jump

$$\beta_{1,t} = \begin{cases} 1. & 1 \leq t \leq 53 \\ 3. & 54 \leq t \leq 103 \end{cases}$$

$$\beta_{2,t} = \begin{cases} 1. & 1 \leq t \leq 53 \\ -1. & 54 \leq t \leq 103 \end{cases}$$

Case II: Random Walk with Zero Drift

$$\beta_{1,t} = \beta_{1,t-1} + p_{1,t}$$

$$p_{1,t} \sim N(0., .01), \qquad \beta_{1,1} = 1.$$

$$\beta_{2,t} = \beta_{2,t-1} + p_{2,t}$$

$$p_{2,t} \sim N(0., .0025), \qquad \beta_{2,1} = 1.$$

Case III: Convergent Markov Process

$$\beta_{1,t} = .7 + .3\beta_{1,t-1} + p_{1,t}$$

$$p_{1,t} \sim N(0., .01), \qquad \beta_{1,1} = 1.$$

$$\beta_{2,t} = .7 + .3\beta_{2,t-1} + p_{2,t}$$

$$p_{2,t} \sim N(0., .0025), \qquad \beta_{2,1} = 1.$$

Case IV: Constant Coefficients

$$\beta_{1,t} = 1. \qquad \text{for all } t$$

$$\beta_{2,t} = 1. \qquad \text{for all } t$$

In cases II and III the values of the p's were drawn once and the resulting β's were identical across replications.

Case I, where both coefficients change by a discrete amount at the same time, should favor the BDE tests. The discrete jump case seriously violates the assumptions of the VPR model, since that model assumes identically distirbuted increments to the coefficients in every period. Case II is the VPR model of equation (2). Case III is similar to II but the coefficients tend toward unity instead of following a random walk. Cases II and III are both special cases of the more general model:

$$\Delta\beta_t = M(\beta^* - \beta_{t-1}) + p_t \tag{10}$$

where M is an adjustment matrix whose eigenvalues are positive and not greater than two. If M equals the identity matrix then $\beta_t = \beta^* + p_t$ and we have a stochastic coefficient regime. Knowledge of β_{t-1} is of no value in predicting β_t, and the stochastic innovation p_t has no persistent effect on subsequent β's. If M equals the zero matrix we have Case II, where the effect of the p_t innovations are fully persistent. Case III is between these two extremes. There is some persistence in the effect of p_t on subsequent values of the β's, but the magnitude of the effect diminishes as time goes on. Rosenberg (1973) treats model (10) in some detail. We remind the reader that our purpose here is not the estimation of the parameters of model (10), but rather the use of the simpler model (2) in detecting coefficient instability.

We now turn to consider the simulation results for our four cases of coefficient variation. Because estimation of the diagonal elements of P, the covariance matrix for the coefficient innovations, requires numerical methods (cnf. the discussion following equation (4)) it was infeasible to estimate those variances in every replication. To reduce computer expenses we estimated them for an initial set of five replications. The squared average standard deviations were then assumed as the diagonal entries of P in 49 additional replications. Let $\overline{P}$ be this assumed covariance matrix in the 49 replications. The graphical results which follow are from the 49 replications.

Table 2 reports the test statistic $-2[L^*(0) - L^*(\hat{P})]$ for the five

TABLE 2

TEST STATISTICS FOR VARIABLE PARAMETER REGRESSION

$-2[L^*(0) - L^*(\hat{P})]$ for first five replications

Case:	I	II	III	IV
Replication 1	185.10	12.57	.42	.12
2	193.54	10.65	.00	.00
3	174.16	12.92	1.60	3.62
4	164.75	13.96	3.86	.01
5	168.74	5.01	.33	1.33

$-2[L^*(0) - L^*(\bar{P})]$ for first five replications

Case:	I	II	III	IV
Replication 1	92.55	6.28	.50	-.08
2	96.77	5.33	-.64	-.31
3	87.08	6.46	-.19	.11
4	82.37	6.98	.46	-.22
5	84.37	2.51	.19	-.05

Frequency of $-2[L^*(0) - L^*(\bar{P})]$ for forty-nine additional replications

$2[L^*(\bar{P}) - L^*(0)]$	χ^2(2 d.f.)	I	II	III	IV
> .58	.75	1.00	.86	.02	.12
> 1.39	.50	1.00	.78	.0	.0
> 2.77	.25	1.00	.55	.0	.0
> 4.61	.10	1.00	.31	.0	.0
> 5.99	.05	1.00	.20	.0	.0
> 7.38	.025	1.00	.08	.0	.0
> 9.31	.01	1.00	.0	.0	.0
> 10.60	.005	1.00	.0	.0	.0

replications in which P was estimated by maximizing the likelihood function. Comparing these statistics to a chi-square distribution with two degrees of freedom we find that VPR never failed to reject the hypothesis of stable coefficients at a confidence level well in excess of .5% in the first case, failed to reject it only once at a confidence level of .5% in the second case and never rejected that hypothesis in the last case where the hypothesis was true. In the third case, where persistence is present but weak, VPR was unable to detect the coefficient variation. This suggests the VPR model of (2) is more powerful in detecting instabilities when there is significant persistence in the dynamic evolution of the coefficients than when persistence is weak.

The lower panel of Table 2 shows the frequency of occurence of the statistic $-2[L^*(0) - L^*(\overline{P})]$ for the 49 additional replications. Since $L^*(\hat{P}) \geq L^*(\overline{P})$ by definition of $\hat{P}$ these frequencies are relatively greater at lower values of the statistic than the frequencies of $-2[L^*(0) - L^*(\hat{P})]$. This observation is confirmed by comparing the middle panel of Table 2 with the top panel. With this caveat of relatively lower values of $-2[L^*(0) - L^*(\overline{P})]$ in mind, the lower panel lends additional support to the conjecture that VPR is capable of detecting the instabilities of cases I and II, but is quite weak in the third case.

Figure 1 exhibits the smoothed estimated values of the coefficients (the $\hat{\beta}_t(T)$'s of Table 1) for the case of a discrete jump, averaged over the 49 replications. These mean estimates are bordered by plus and minus one sample standard deviation of the sampled $\hat{\beta}_t(T)$'s. The estimates are quite stable in the beginning and end of the sample period, when the

FIGURE 1

VPR ESTIMATION FOR DISCRETE JUMP

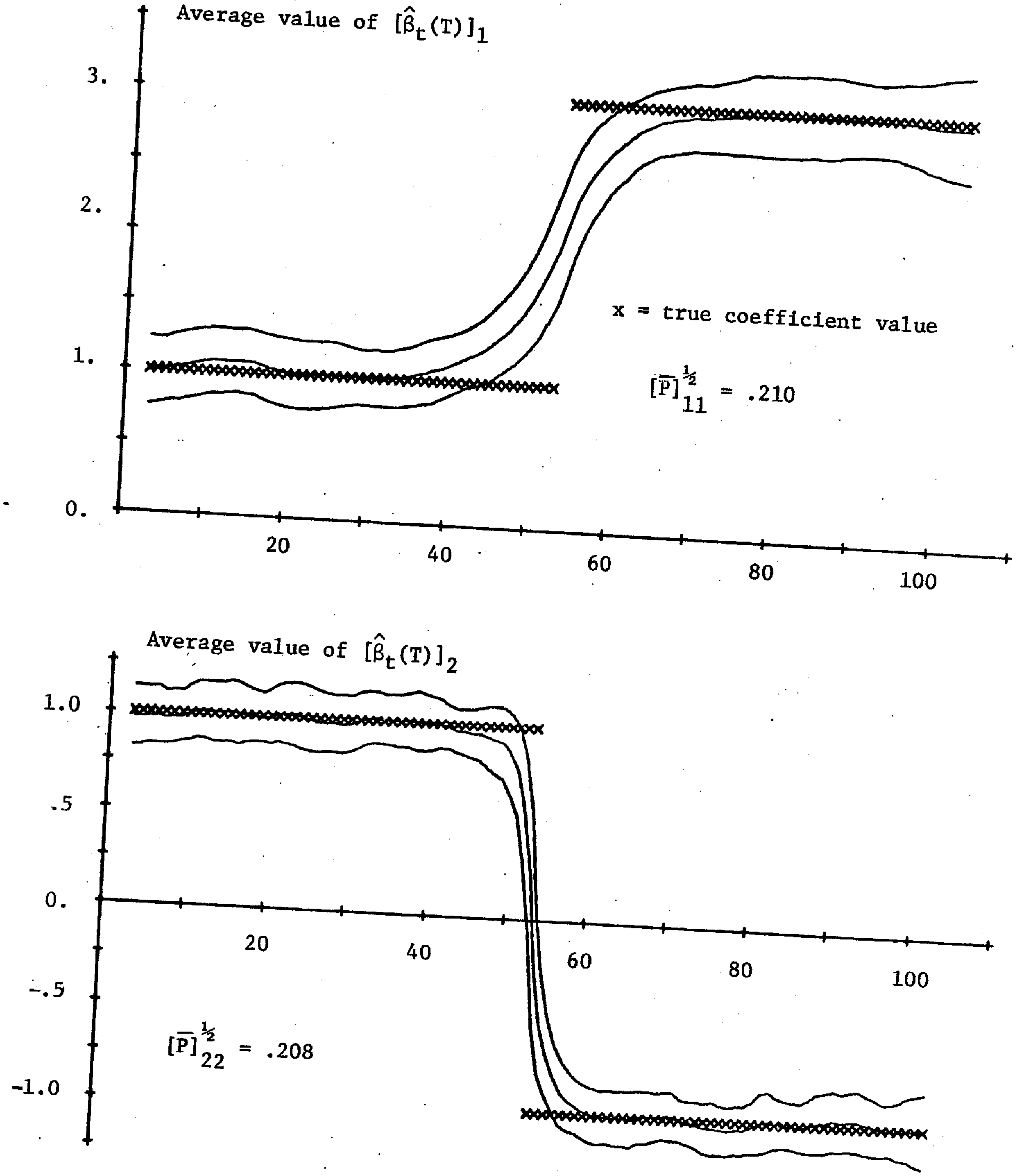
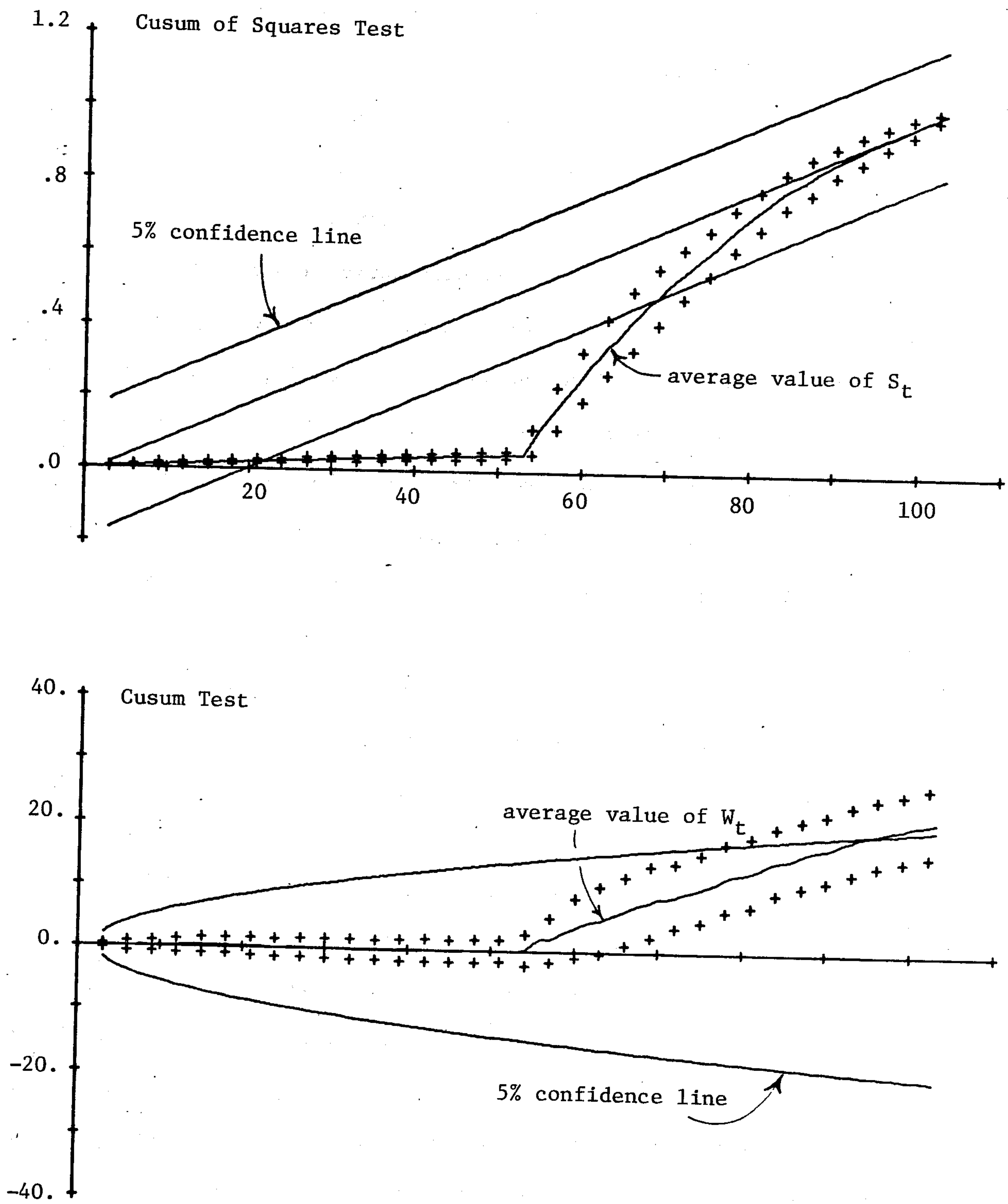


FIGURE 2

BDE STATISTICS FOR DISCRETE JUMP



coefficients are stable, and track the discrete jump surprisingly well. About 20 to 30 periods appear to be required for the smoothed estimates to move completely from the first regime to the second. The estimates begin to move away from the true coefficient values before the jump actually takes place. This, of course, is because we are examining the smoothed estimates, each of which makes use of all 103 observations on y_t. The updated estimates, the $\beta_t(t)$'s in Table 1, use only information observed up to time t, so those estimates would not show such seemingly anticipatory behavior. The results of Figure 1 indicate VPR is useful in estimating the magnitude and timing of the shift in the coefficients as well as in testing for the presence of the instability. The timing and magnitude of the shift could doubtless be better estimated by the maximum likelihood method of Quandt (1958) for this case of a discrete shift in regimes, but VPR provides a reliable guide here as well as in regimes where Quandt's method would be less useful, for example, case II.

Figure 2 exhibits the average values of W_t, equation (7), and S_t, equation (8), for the BDE tests, bordered by plus and minus one sample standard deviation, from the 49 replications. These graphs provide the distributions of the test statistics for the variable coefficient regime of case I. The cusum of squares test is quite reliable in rejecting the null hypothesis of stability. The cusum test is not as reliable, and failed to reject the hypothesis in about half of the trials. Khan (1974, pg. 1215) has claimed that the location of a shift in regression parameters is indicated when the S_t series crosses the significance line. (In a footnote he also observes that the shift could have occurred before the crossing of the significance line.) Examination of Graph 2 shows that both of these conjectures are incorrect generalizations. The average S_t series becomes statistically significant long before the shift actually occurs (about period 23 compared to the shift in period 53). The reason for this is that the S_t test statistics are based on the full set of sample observations, due to the sum to T in equation (8). Information which arrives after period t is incorporated in the construction of S_t, so that statistic depends on future as well as past observations. In the present case of a jump in the coefficients the expectation of the denominator in equation (8) will be greater than $(T-n)\sigma^2$ since the expectation of w_t^2 for periods after the jump is greater than σ^2. This depresses the value of S_t in periods prior to the jump, as is plainly evident in Figure 2.

Figure 3 reports the results of applying VPR to case II. We have already seen (Table 2) that VPR is able to detect the presence of coefficient instability in this case. Figure 3 indicates the method can also track the time varying coefficients.

Figure 4 shows the distribution of the BDE statistics for case II. Although the cusum test appears to do well this is an artifact of the downward drift in the constant term $\beta_{1,t}$ of model (9) which is shown in Figure 3. This drift implies a consistent over-prediction of y_t by a least squares regression and consistent negative values in the recursive residuals of (6). If the coefficients are following a random walk y_t is generated from the model:

$$y_t = \beta_{1,1} + \beta_{2,1}x_t + u_t + \sum_{s=2}^{t} p_{1,s} + x_t \sum_{s=2}^{t} p_{2,s} \qquad (11)$$

so w_t, and hence W_t, has an expected value of zero, where the expectation is taken over the p_t innovations on the coefficients as well as over the residuals u_t. Thus randomly walking coefficients do not bias a cusum statistic

FIGURE 3

VPR ESTIMATION FOR RANDOM WALK

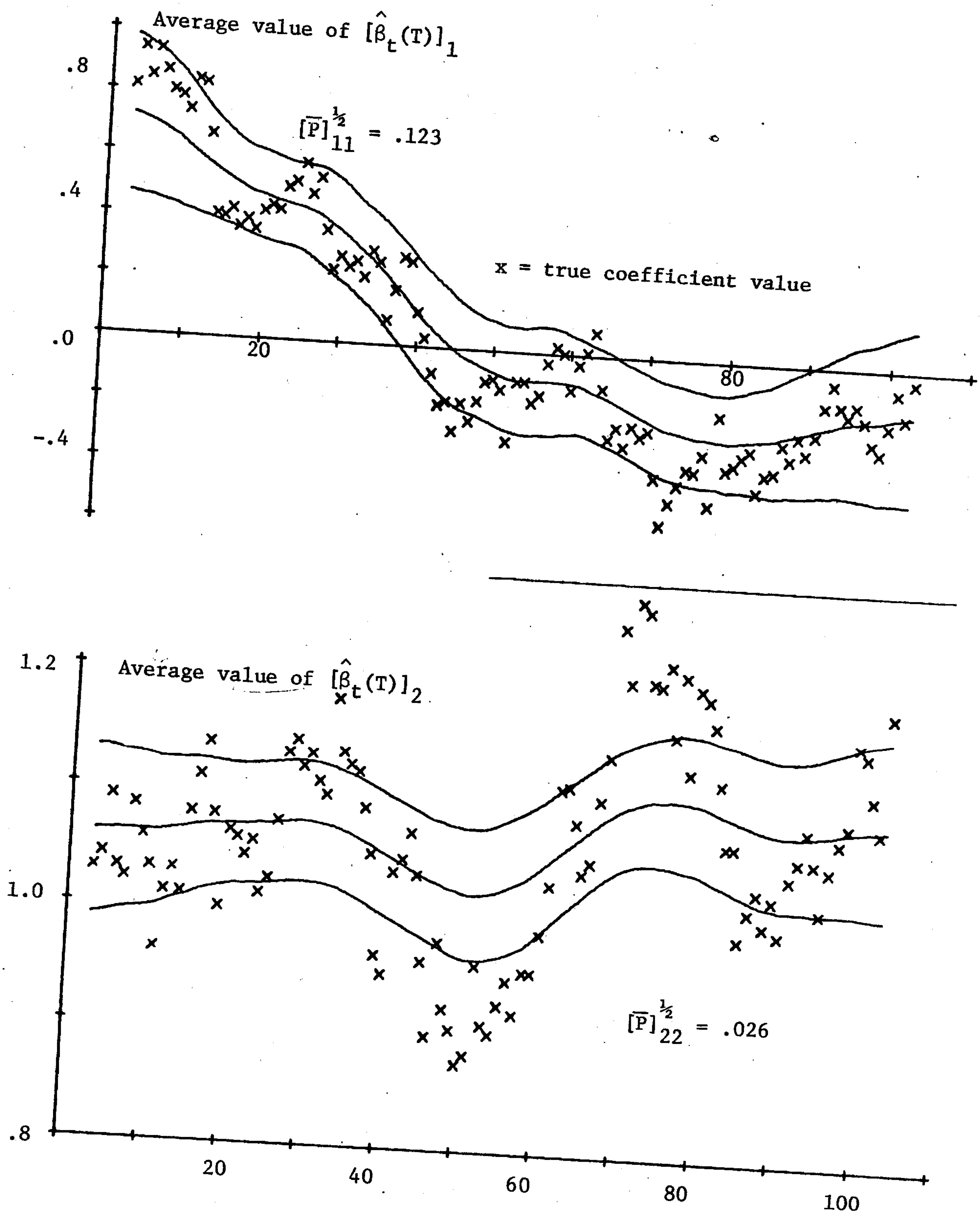
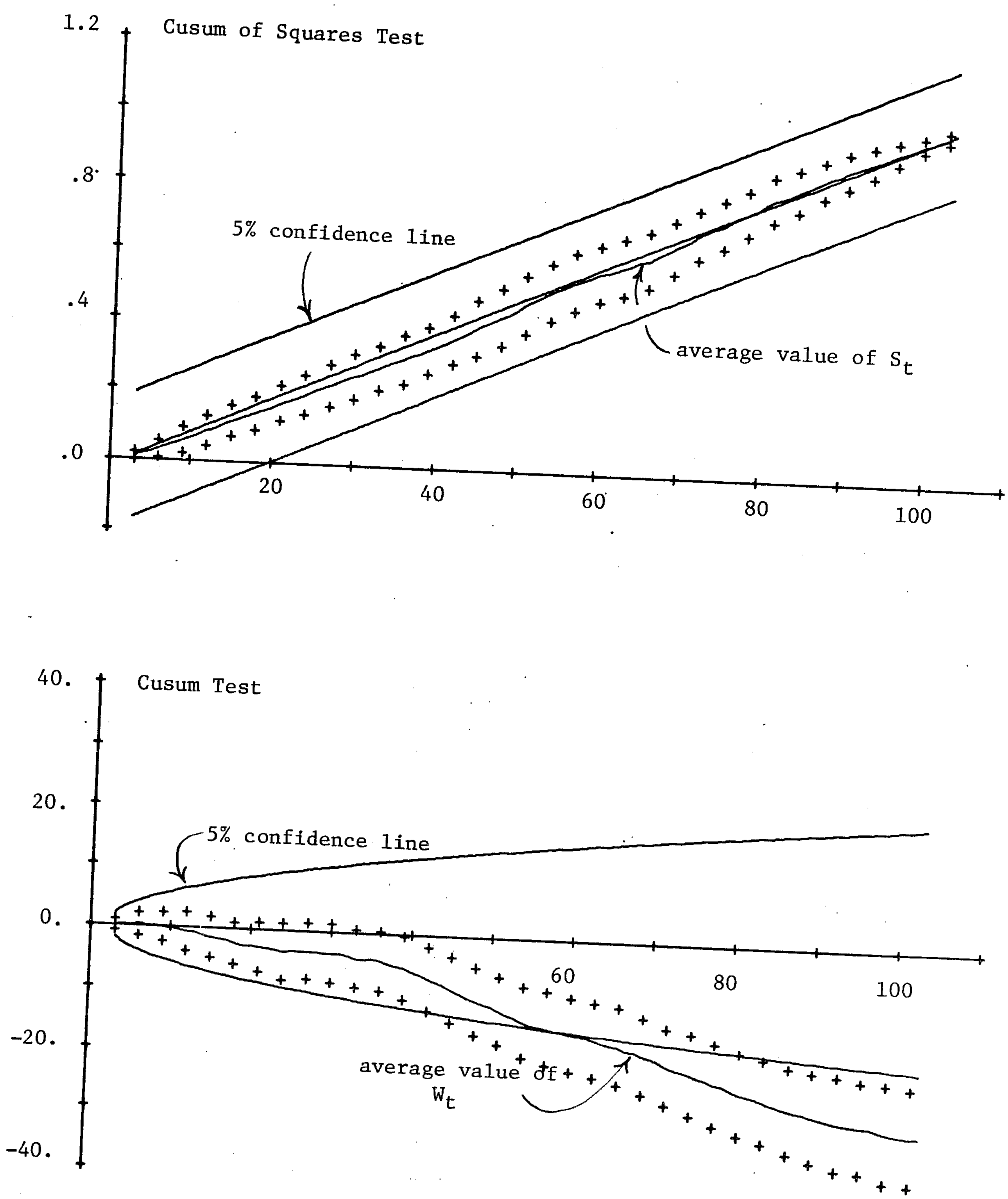


FIGURE 4

BDE STATISTICS FOR RANDOM WALK



from the expected value of zero obtained under the null hypothesis of stability. The variance of W_t will depend on the covariance structure of the compound disturbance term in (11). If the variance of W_t is substantially greater than t-n there would be a good probability of observing values of W_t greater than $2(t-n)^{\frac{1}{2}}$ and hence of rejecting the hypothesis of stability. The power of the test, however, is not likely to be very great since the statistic is not biased by the presence of coefficients following a random walk.

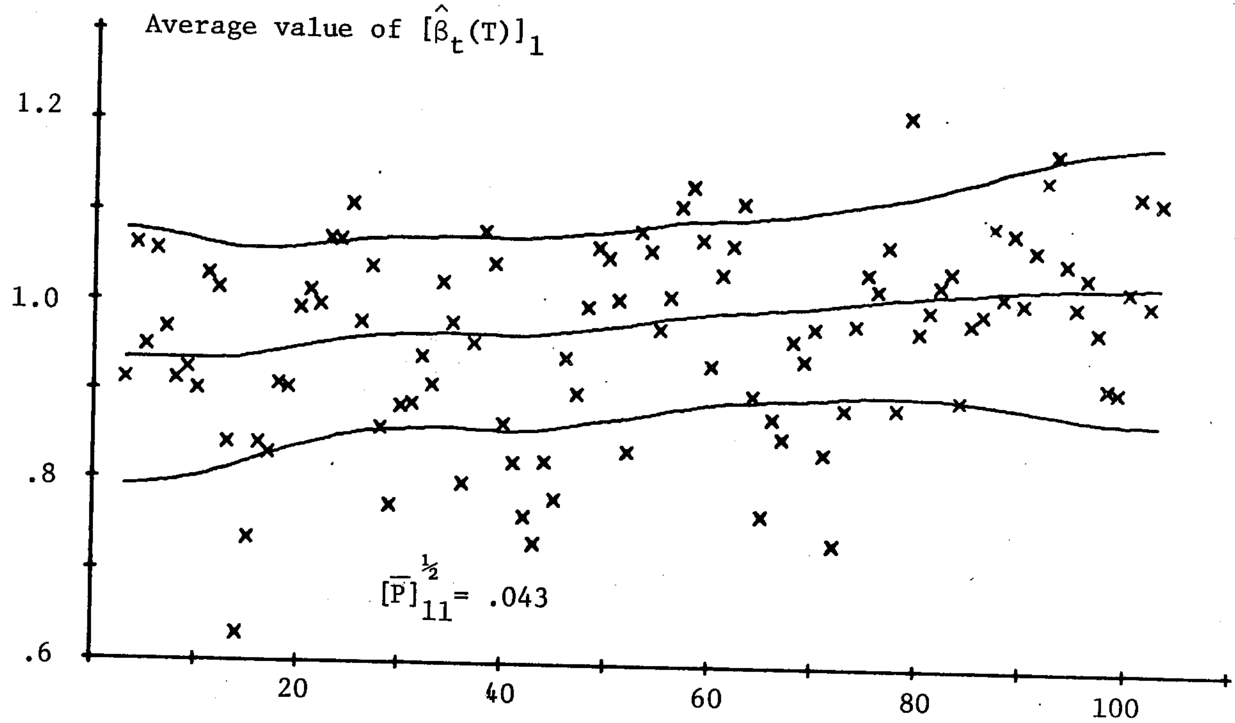
Figure 4 shows clearly the failure of the cusum of squares test when the coefficients follow a random walk. Comparing Figure 4 with the cusum of squares graph in Figure 8, where the coefficients are stable, suggests the distribution of the S_t statistic is not changed substantially when the coefficients obey a random walk, making the cusum of squares test quite weak in such cases.

Figure 5 shows the average estimated coefficients obtained from applying VPR to case III. Although the likelihood ratio tests did not justify rejecting the hypothesis of stable coefficients, i.e., $P = 0$, the estimates in Figure 5 are based on the assumption that $P = \overline{P}$ rather than the null matrix. The smoothed estimates do a poor job of tracking the actual coefficients through time. Figure 6 shows the BDE tests do an equally poor job of detecting the coefficient variation in case III.

Figure 7 reports the smoothed estimates from VPR when the coefficients are actually stable. There is a mild positive bias in the constant term of model (9) that can be attributed to sampling error, but the mean estimates are clearly satisfactory. In these simulations we again assumed $P = \overline{P}$ although the likelihood tests did not allow rejection of the null hypothesis

FIGURE 5

VPR ESTIMATION FOR CONVERGENT MARKOV PROCESS



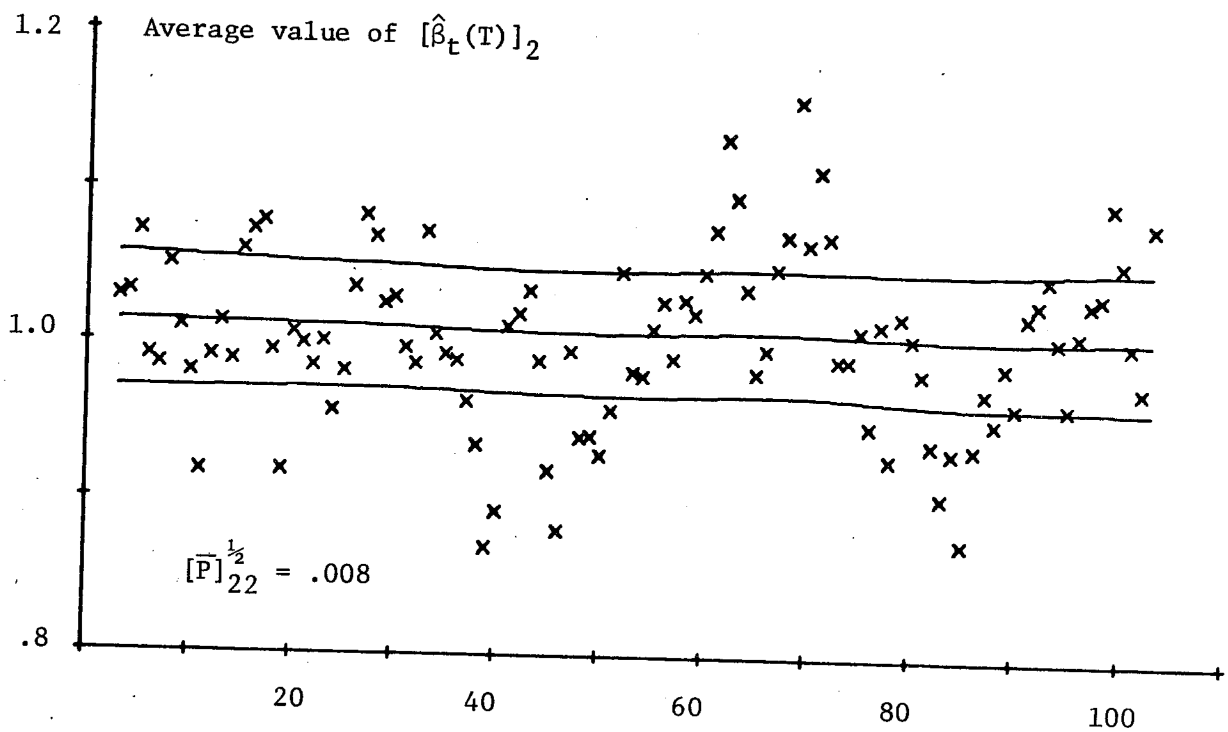


FIGURE 6

BDE STATISTICS FOR CONVERGENT MARKOV PROCESS

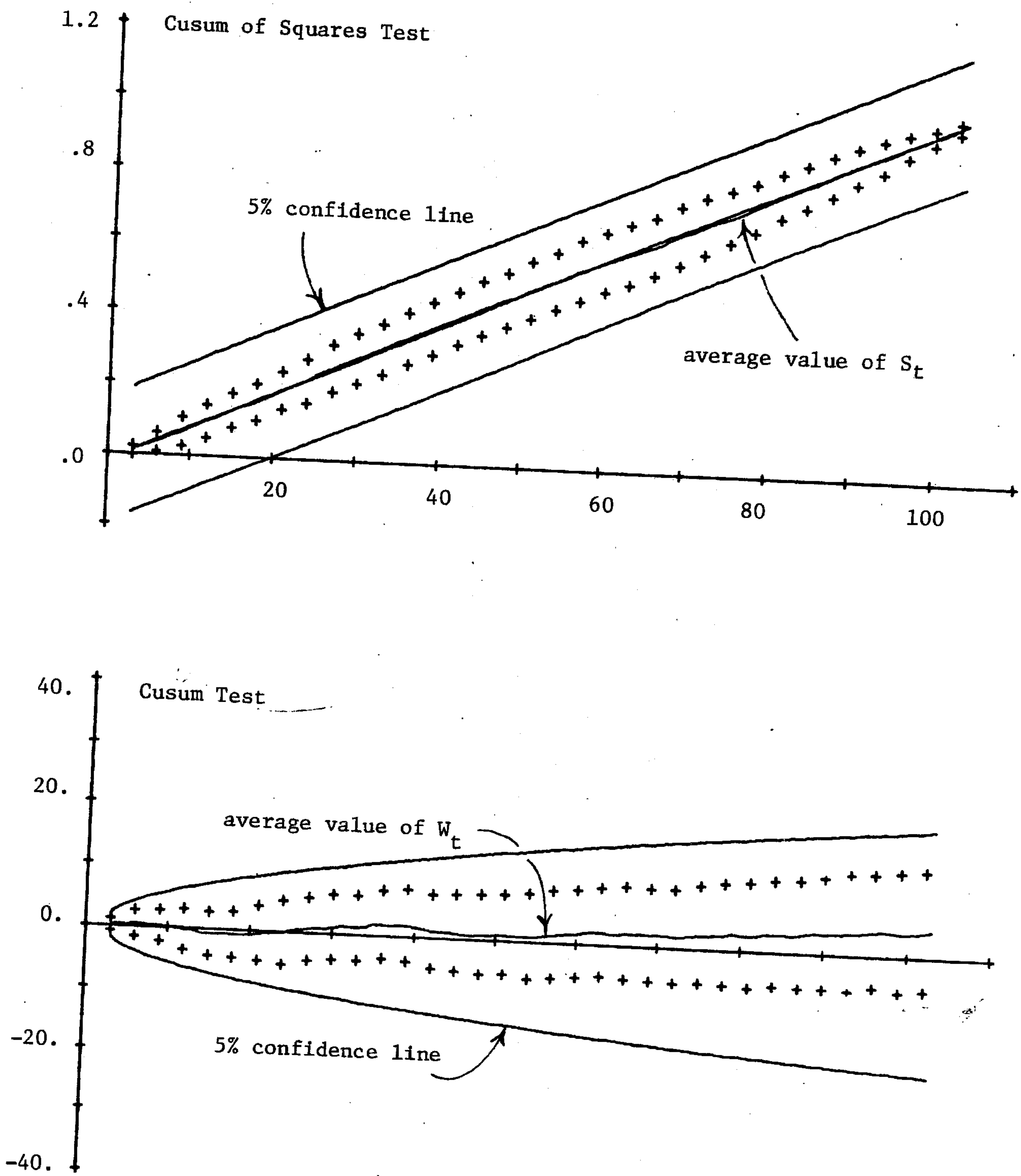
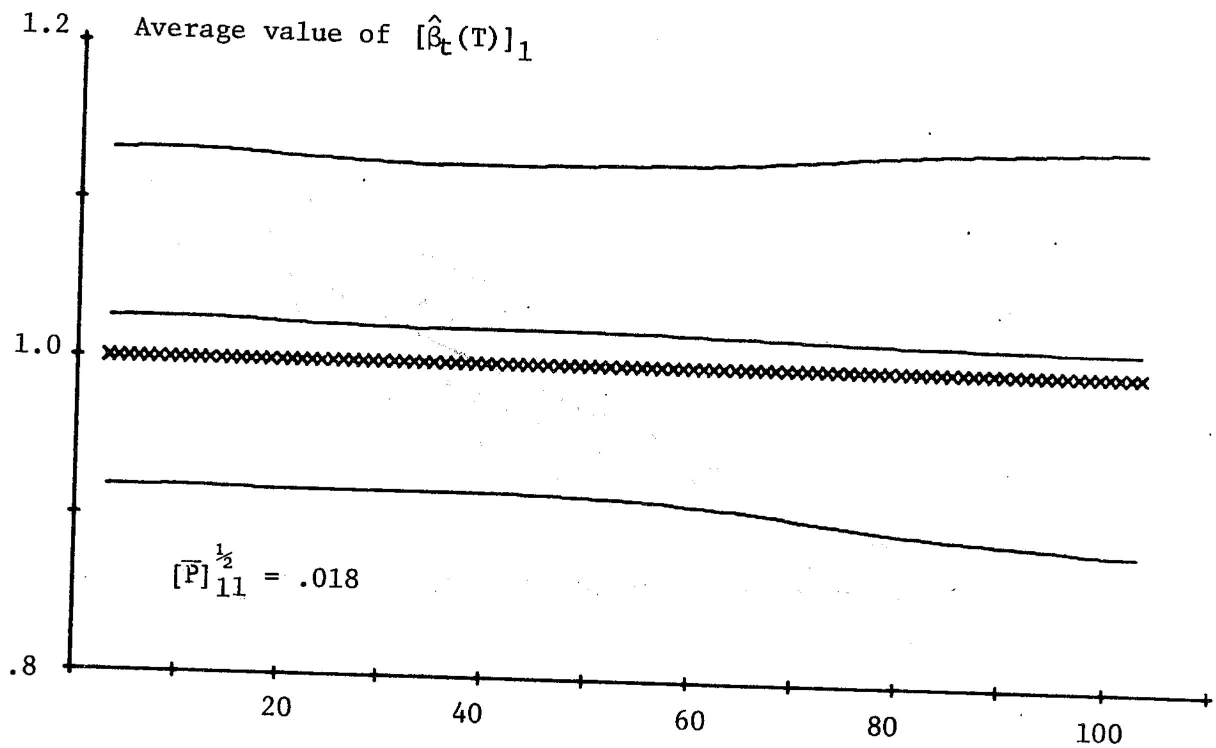


FIGURE 7

VPR ESTIMATION FOR STABLE COEFFICIENTS



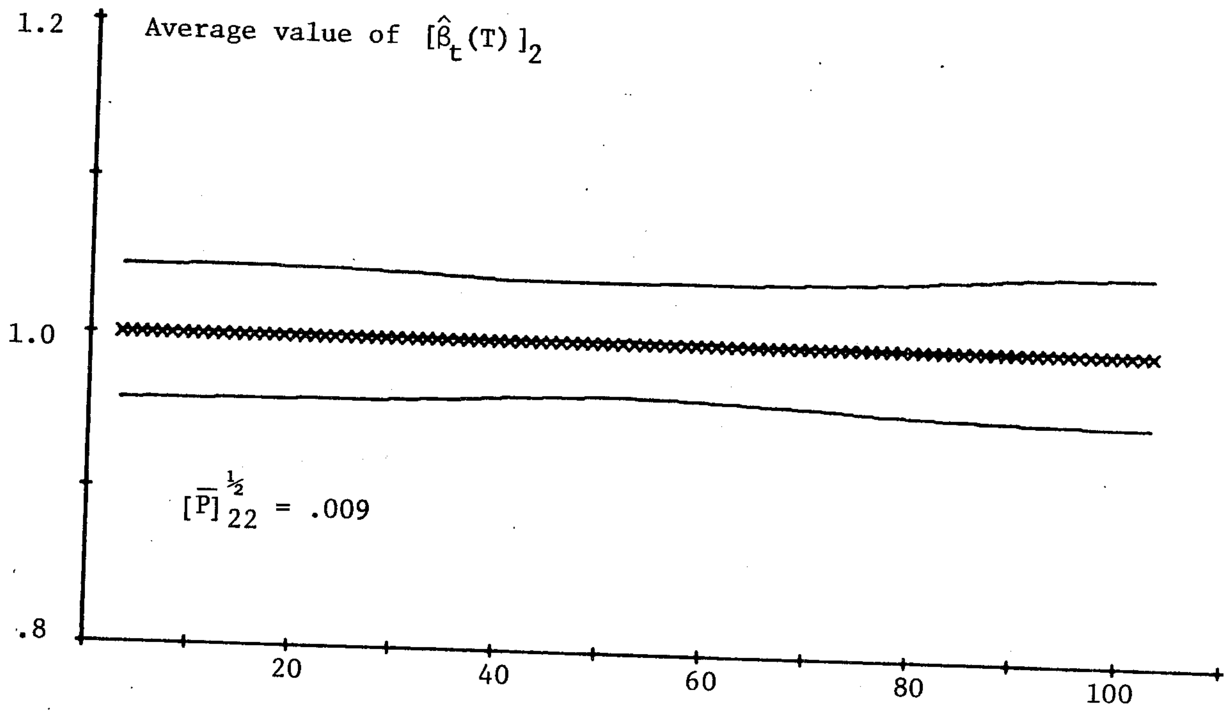
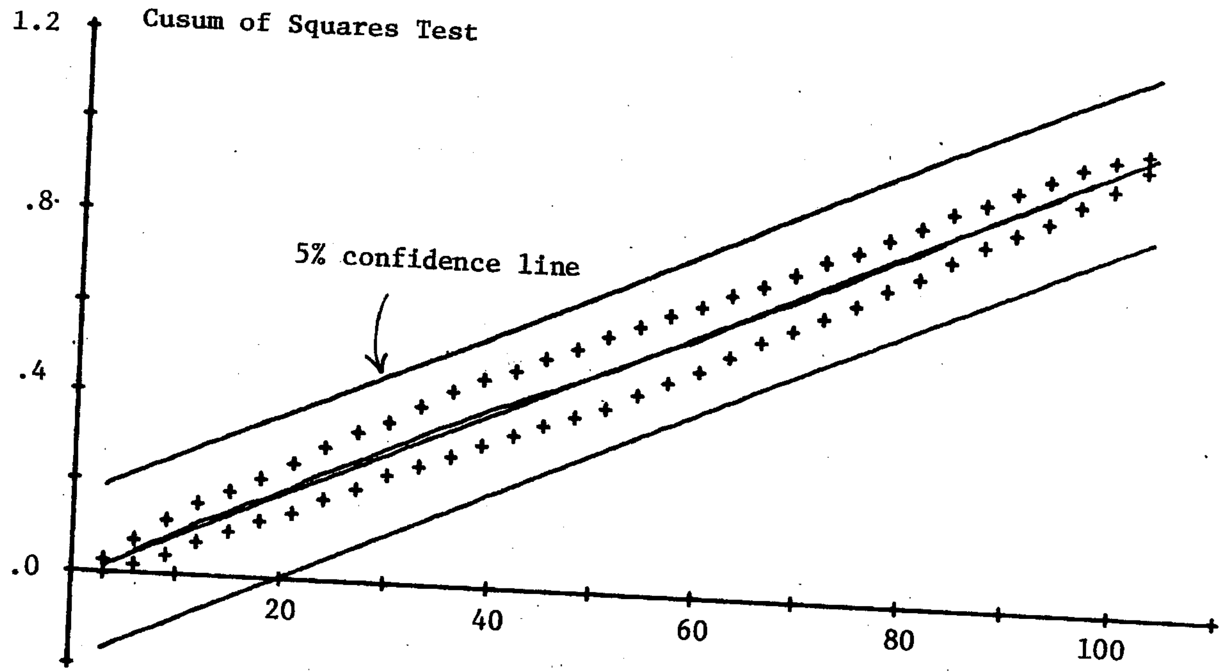
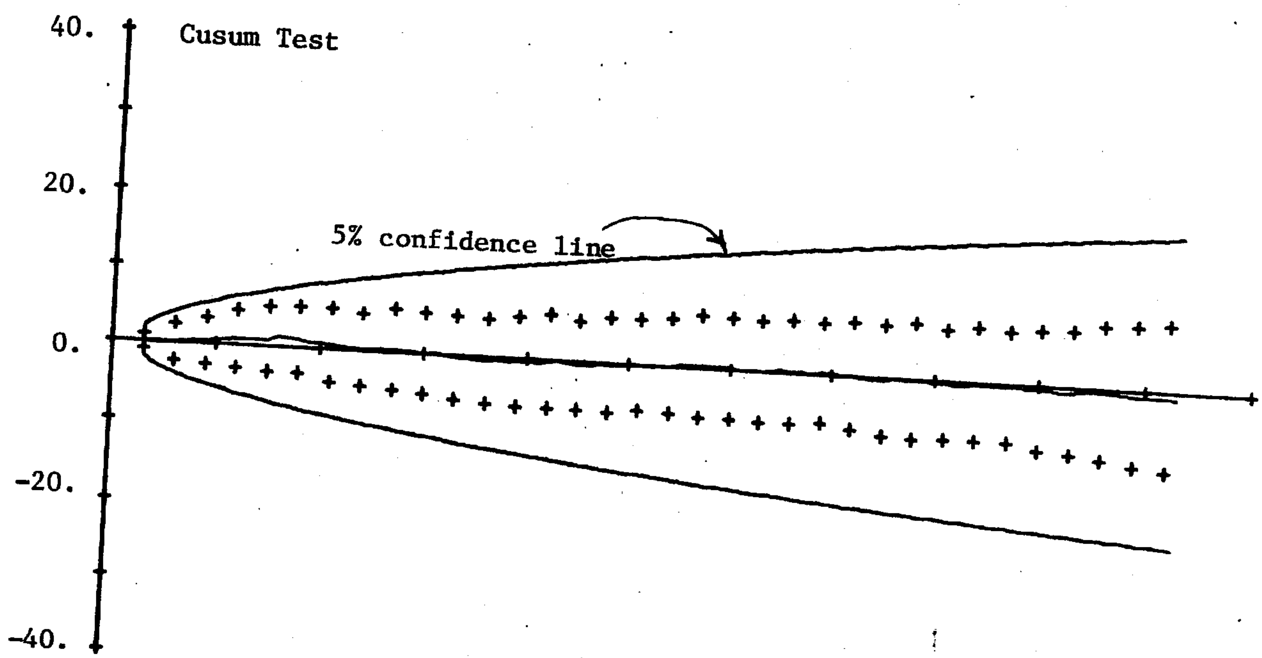


FIGURE 8

BDE STATISTICS FOR STABLE COEFFICIENTS





of stability.

To conclude the results of these simulation experiments, we have found that variable parameter regression is a robust method for detecting coefficient instabilities when there is substantial persistence in the coefficients from period to period, whether the persistence is because the coefficients shift only occassionally or whether it is because they follow something like a random walk. When there is only weak persistence VPR may fail to reject a false null hypothesis of stability. The tests of Brown, Durbin and Evans, on the other hand, are powerful only when there is a discrete shift. Those tests are quite weak under any regime of incrementally varying coefficients, whether the effects of the increments are persistent or not. It seems reasonable to conjecture that variation similar to case II is likely to characterize the nature of the coefficient shifts in many economic models. Application of the BDE tests to these models may fail to reject a false null hypothesis. This leads us to reconsider the findings of Khan (1974) in the next section.

4. The Stability of the Demand for Money

In a recent article Khan (1974) applied the BDE cusum of squares test to the problem of whether the coefficients in a money demand function were stable during the Twentieth century. He was unable to reject the hypothe of stability in any of eight versions of the demand function. However, as we saw in the previous section, the cusum of squares test may not be able to reject such an hypothesis if the coefficients are undergoing gradual evolution. Since this type of change seems likely for a money demand function, Khan's conclusions may not be as powerful as they seem. In this section we apply variable parameter regression to Khan's model to see whether his conclusions can be sustained.

Khan's model is:

$$\Delta \ln M_t = \beta_{1,t} + \beta_{2,t}\Delta \ln R_t + \beta_{3,t}\Delta \ln Y_t + u_t \tag{12}$$

where M_t is real per capita money balances, R_t is an interest rate and Y_t is a real per capita income variable. While we make no criticism of this model here, we do note that, because it is expressed in first differences, it implies an asymptotically unbounded variance on the demand for money as time goes on. Note that β_1 is a time rate of growth coefficient because of the differencing. Our data is taken from the volume of historical data published by the U.S. Department of Commerce (1973). The aggregate quantity of money is either the narrow money supply M_1 (series B109 and B110 in the foregoing volume) or the braoder M_2 (B111 and B112). Aggregate money is deflated by the population (A114) and the GNP deflator (B61 and B62). The interest rate is either the commercial paper yield, Rcp, (B80) or the corporate bond rate, Rcb, (B74). Income is measured by net national product (A6). Table 3 compares the stationary estimation results of model (12) with the corresponding results reported by Khan for his data set.

Application of the cusum of squares test to the four versions of the money demand model gave results quite similar to Khan's. The test statistic was never significant at the 5% level of confidence. In each version, however, the statistic moved away from the mean value line (t-n)/(T-n) in mid-1940 and almost became significant in 1950. This feature was also present in Khan's figures.

Table 4 reports the estimated values of $[\hat{P}]^{\frac{1}{2}}_{ii}$ for each version of model (12) and the values of the log-likelihood ratio statistics. These statistics

TABLE 3

STATIONARY ESTIMATES OF DEMAND FOR MONEY

Definition of Money		Equation	Constant	ΔlnRcp	ΔlnRcb	ΔlnY	R^2	d.w.	s.e.
ΔlnM_1	1916-1966	(1)	.00296 (.40)	-.0833 (2.86)		.4947 (4.71)	.360	1.43	.051
		(1K)	.009 (1.47)	-.085 (2.86)		.332 (3.69)	.280	1.60	.049
	1916-1966	(2)	.00499 (.68)		-.3198 (3.17)	.4147 (4.04)	.381	1.53	.050
		(2K)	.012 (2.03)		-.303 (4.61)	.208 (2.49)	.332	1.66	.046
ΔlnM_2	1891-1966	(3)	.01477 (2.73)	-.0697 (3.07)		.3850 (4.93)	.294	1.46	.045
		(3K)	.017 (2.99)	-.070 (2.65)		.279 (3.46)	.183	1.63	.045
	1901-1966	(4)	.01614 (2.75)		-.2688 (2.98)	.2942 (3.50)	.284	1.43	.046
		(4K)	.019 (3.63)		-.256 (4.30)	.175 (2.32)	.300	1.50	.041

t values shown in parentheses.

Estimates from Khan (1974) denoted by "K" following equation number.

TABLE 4

PARAMETERS OF VPR ESTIMATION OF EQUATION (12)

Money	Interest	$[\hat{P}]^{\frac{1}{2}}_{11}$	$[\hat{P}]^{\frac{1}{2}}_{22}$	$[\hat{P}]^{\frac{1}{2}}_{33}$	$\hat{\sigma}$	$2[L^*(\hat{P}) - L^*(0)]$
M1	Rcp	.4452	1.6368	2.3361	.0337	6.42
M1	Rcb	.1425	.0	1.3705	.0461	1.68
M2	Rcp	.0075	.8712	.0	.0418	.82
M2	Rcb	.0073	.0	.0	.0458	.02

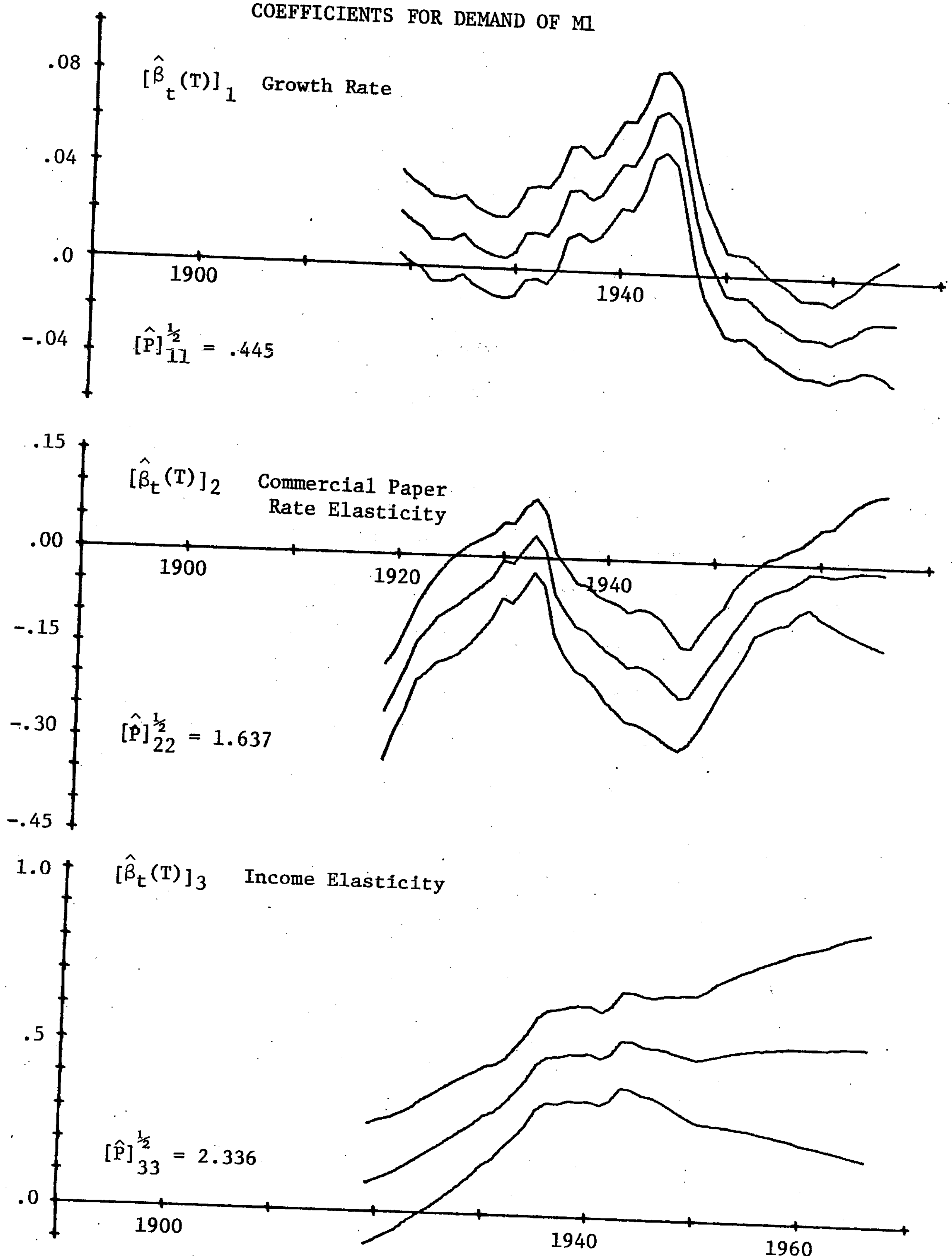
imply we can reject the hypothesis of stability for the case of narrowly defined money and the commercial paper rate. There is virtually no instability when the broader money supply coupled with the bond rate is examined. There is some evidence of instability with M_2 and the commercial paper rate (primarily in the interest elasticity of demand) and a bit more with M_1 and the bond rate (in the growth rate term and particularly in the income elasticity of demand).

Besides the likelihood ratio statistics, VPR provides graphical evidence on the estimated evolution of the coefficients. Figure 9 shows the smoothed estimates of the coefficients of model (12) for M_1 and the commercial paper rate, bordered by plus and minus one standard error of estimate. The fluctuations are substantial and well-defined for the growth rate and the interest elasticity, but the income elasticity appears to have stabilized mid-way through the Great Depression at about .5. The estimated interest elasticity of demand for narrowly defined money has fallen to zero in the post-war period, a phenomenon examined recently by Cagan and Schwartz (1975). The trend growth rate of demand has turned (insignificantly) negative from its rather high value during the Second World War. Although we do not exhibit it here, the pattern of variation in the interest elasticity of demand for M_2 using the commercial paper rate is almost identical to that shown in Figure 9. When the narrowly defined money supply M_1 is coupled with the bond rate rather than the commercial paper rate variation in the interest elasticity vanishes, but the income elasticity and the growth rate behave as in Figure 9.

We find that a demand for money function expressed in terms of the broadly defined money supply and the bond rate is quite stable. When the

FIGURE 9

COEFFICIENTS FOR DEMAND OF M1



commercial paper rate is introduced the interest elasticity fluctuates substantially for either definition of money, and when M_1 is used the trend rate of growth and the income elasticity of demand fluctuate. These findings are sharper than those of Khan. Khan claimed, as we do, that the demand function was more stable when the bond rate was used, but he could not distinguish between the two alternative definitions of the money stock. Our results indicate that the demand function for broadly defined money is more stable.

Conclusions

This paper investigated the problem of detecting and estimating coefficient variation with two alternative methodologies, the tests of Brown, Durbin and Evans and variable parameter regression. We found the BDE tests are powerful when there is a discrete break in the regression regime, but that they fail to detect the presence of incrementally varying coefficients. We found that VPR is powerful when there is a discrete break and when there is persistence in a regime of incrementally varying coefficients. When persistence is reduced VPR as applied here becomes less powerful. We note, however, that the power of VPR may be increased in the latter case if the researcher is willing to estimate values for M and β^* in equation (10), which is more difficult and expensive than using the simple structure of (2). Finally, neither methodology gave misleading results when the coefficients were in fact stable.

We also considered Khan's results on the stability of the demand for money, and found the stability of different functions differ sharply. Fluctuations in the elasticity of demand with respect to the commercial paper rate have been substantial in this century, as have fluctuations in the trend growth rate and the income elasticity of demand for M_1. Demand for M_2 using the corporate bond rate has been quite stable.

FOOTNOTES

1. The distribution of S_t may be derived by noting that:

$$S_t^{-1} -1 = + \sum_{s=t+1}^{T} w_s^2 / \sum_{s=n+1}^{t} w_s^2$$

Since the numerator and denominator of $S_t^{-1} -1$. are sums of squares of independently distributed normal variates with zero means and equal variances it follows that $(S_t^{-1} -1)(t-n)/(T-t)$ is distributed as an F statistic with (T-t, t-n) degrees of freedom (Mood and Graybill, 1963, pg. 232). The result for S_t is then immediate since if the statistic v has an F distribution with (m,n) degrees of freedom then $w = mv/(n(1 + mv/n))$ has a beta distribution with parameters $\alpha = (m-2)/2$ and $\beta = (n-2)/2$ (Mood and Graybill, 1963, pg. 232).

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